

Limits and Derivatives

Introduction

- Calculus is that branch of mathematics which mainly deals with the study of change in the value of a function as the points in the domain change.

Limits

- In general as $x \rightarrow a$, $f(x) \rightarrow l$, then l is called limit of the function $f(x)$
- Symbolically written as $\lim_{x \rightarrow a} f(x) = l$.
- For all the limits, function should assume at a given point $x = a$
- The two ways x could approach a number are either from left or from right, i.e., all the values of x near a could be less than a or could be greater than a .
- The two types of limits
 - Right hand limit
 - Value of $f(x)$ which is dictated by values of $f(x)$ when x tends to from the right.
 - Left hand limit.
 - Value of $f(x)$ which is dictated by values of $f(x)$ when x tends to from the left.
- In this case the right and left hand limits are different, and hence we say that the limit of $f(x)$ as x tends to zero does not exist (even though the function is defined at 0).

Algebra of limits

Theorem 1

Let f and g be two functions such that both $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist, then

- Limit of sum of two functions is sum of the limits of the function s,i.e

$$\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x).$$

- Limit of difference of two functions is difference of the limits of the functions, i.e.

$$\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$$

- Limit of product of two functions is product of the limits of the functions, i.e.,

$$\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x).$$

- Limit of quotient of two functions is quotient of the limits of the functions (whenever the denominator is non zero), i.e.,

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$$

- In particular as a special case of (iii), when g is the constant function such that $g(x) = \lambda$, for some real number λ , we have

$$\lim_{x \rightarrow a} [(\lambda \cdot f)(x)] = \lambda \cdot \lim_{x \rightarrow a} f(x).$$

Limits of polynomials and rational functions

- A function f is said to be a polynomial function if $f(x)$ is zero function or if $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$, where a_i 's are real numbers such that $a_n \neq 0$ for some natural number n .

- We know that $\lim_{x \rightarrow a} x = a$.
- $\lim_{x \rightarrow a} x^2 = \lim_{x \rightarrow a} (x \cdot x) = \lim_{x \rightarrow a} x \cdot \lim_{x \rightarrow a} x = a \cdot a = a^2$

Hence,

$$\lim_{x \rightarrow a} x^n = a^n$$

- Let $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ be a polynomial function

$$\begin{aligned} \lim_{x \rightarrow a} f(x) &= \lim_{x \rightarrow a} [a_0 + a_1x + a_2x^2 + \dots + a_nx^n] \\ &= \lim_{x \rightarrow a} a_0 + \lim_{x \rightarrow a} a_1x + \lim_{x \rightarrow a} a_2x^2 + \dots + \lim_{x \rightarrow a} a_nx^n \\ &= a_0 + a_1 \lim_{x \rightarrow a} x + a_2 \lim_{x \rightarrow a} x^2 + \dots + a_n \lim_{x \rightarrow a} x^n \\ &= a_0 + a_1a + a_2a^2 + \dots + a_na^n \\ &= f(a) \end{aligned}$$

- A function f is said to be a rational function, if $f(x) = \frac{g(x)}{h(x)}$ where $g(x)$ and $h(x)$ are polynomials such that $h(x) \neq 0$.

Then

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} \frac{g(x)}{h(x)} = \frac{\lim_{x \rightarrow a} g(x)}{\lim_{x \rightarrow a} h(x)} = \frac{g(a)}{h(a)}$$

- However, if $h(a) = 0$, there are two scenarios –
 - when $g(a) \neq 0$
 - limit does not exist
 - When $g(a) = 0$.
 - $g(x) = (x - a)^k g_1(x)$, where k is the maximum of powers of $(x - a)$ in $g(x)$
 - Similarly, $h(x) = (x - a)^l h_1(x)$ as $h(a) = 0$. Now, if $k \geq l$, we have

$$\begin{aligned} \lim_{x \rightarrow a} f(x) &= \frac{\lim_{x \rightarrow a} g(x)}{\lim_{x \rightarrow a} h(x)} = \frac{\lim_{x \rightarrow a} (x - a)^k g_1(x)}{\lim_{x \rightarrow a} (x - a)^l h_1(x)} \\ &= \frac{\lim_{x \rightarrow a} (x - a)^{(k-l)} g_1(x)}{\lim_{x \rightarrow a} h_1(x)} = \frac{0 \cdot g_1(a)}{h_1(a)} = 0 \end{aligned}$$

If $k < l$, the limit is not defined.

Theorem 2

For any positive integer n

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

Proof

Dividing $(x^n - a^n)$ by $(x - a)$, we see that

$$x^n - a^n = (x-a) (x^{n-1} + x^{n-2} a + x^{n-3} a^2 + \dots + x a^{n-2} + a^{n-1})$$

$$\begin{aligned} \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} &= \lim_{x \rightarrow a} (x^{n-1} + x^{n-2} a + x^{n-3} a^2 + \dots + x a^{n-2} + a^{n-1}) \\ &= a^{n-1} + a a^{n-2} + \dots + a^{n-2} (a) + a^{n-1} \\ &= a^{n-1} + a^{n-1} + \dots + a^{n-1} + a^{n-1} \text{ (n terms)} \\ &= na^{n-1} \end{aligned}$$

Note:

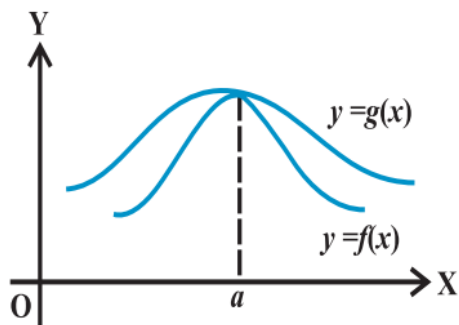
The expression in the above theorem for the limit is true even if n is any rational number and a is positive.

Limits of Trigonometric Functions

Theorem 3

Let f and g be two real valued functions with the same domain such that $f(x) \leq g(x)$ for all x in the domain of definition,

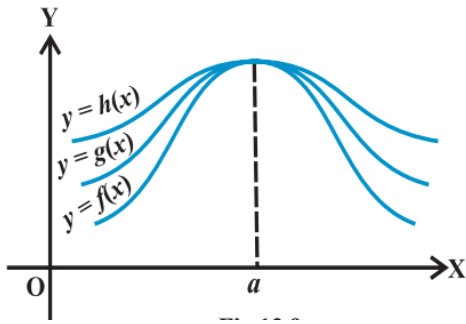
For some a, if both $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist, then $\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$



Theorem 4 (Sandwich Theorem)

Let f, g and h be real functions such that $f(x) \leq g(x) \leq h(x)$ for all x in the common domain of definition.

For some real number a, if $\lim_{x \rightarrow a} f(x) = l = \lim_{x \rightarrow a} h(x)$, then $\lim_{x \rightarrow a} g(x) = l$.

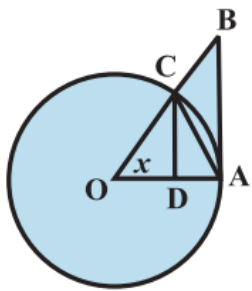


To Prove:

$$\cos x < \frac{\sin x}{x} < 1 \quad \text{for } 0 < |x| < \frac{\pi}{2}$$

Proof:

We know that $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$. Hence, it is sufficient to prove the inequality for $0 < x < \frac{\pi}{2}$



- O is the centre of the unit circle such that the angle AOC is x radians and $0 < x < \frac{\pi}{2}$
- Line segments BA and CD are perpendiculars to OA . Further, join AC . Then
- Area of $\Delta OAC < \text{Area of sector } OAC < \text{Area of } \Delta OAB$

$$\text{i.e., } \frac{1}{2} \text{OA} \cdot \text{CD} < \frac{x}{2\pi} \cdot \pi \cdot (\text{OA})^2 < \frac{1}{2} \text{OA} \cdot \text{AB}.$$

$$\text{i.e., } \text{CD} < x \cdot \text{OA} < \text{AB}.$$

From ΔOCD ,

$$\sin x = \frac{\text{CD}}{\text{OA}} \text{ (since } \text{OC} = \text{OA}) \text{ and hence } \text{CD} = \text{OA} \sin x. \text{ Also } \tan x = \frac{\text{AB}}{\text{OA}} \text{ and}$$

hence $\text{AB} = \text{OA} \cdot \tan x$. Thus

$$\text{OA} \sin x < \text{OA} \cdot x < \text{OA} \cdot \tan x.$$

Since length OA is positive, we have

$$\sin x < x < \tan x.$$

Since $0 < x < \frac{\pi}{2}$, $\sin x$ is positive and thus by dividing throughout by $\sin x$, we have

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}. \text{ Taking reciprocals throughout, we have}$$

$$\sin x < x < \tan x.$$

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$$\cos x < \frac{\sin x}{x} < 1$$

Hence Proved

The following are two important limits

$$(i) \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1. \quad (ii) \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$$

Proof:

The function $\frac{\sin x}{x}$ is sandwiched between the function $\cos x$ and the constant function which takes value 1.

Since $\lim_{x \rightarrow 0} \cos x = 1$, also we know that $1 - \cos x = 2 \sin^2\left(\frac{x}{2}\right)$.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} &= \lim_{x \rightarrow 0} \frac{2 \sin^2\left(\frac{x}{2}\right)}{x} = \lim_{x \rightarrow 0} \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}} \cdot \sin\left(\frac{x}{2}\right) \\ &= \lim_{x \rightarrow 0} \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}} \cdot \lim_{x \rightarrow 0} \sin\left(\frac{x}{2}\right) = 1 \cdot 0 = 0 \end{aligned}$$

Using the fact that $x \rightarrow 0$ is equivalent to $\frac{x}{2} \rightarrow 0$. This may be justified by putting $y = \frac{x}{2}$

Derivatives

• Some Real time Applications

- People maintaining a reservoir need to know when will a reservoir overflow knowing the depth of the water at several instances of time
- Rocket Scientists need to compute the precise velocity with which the satellite needs to be shot out from the rocket knowing the height of the rocket at various times.
- Financial institutions need to predict the changes in the value of a particular stock knowing its present value.
- Helpful to know how a particular parameter is changing with respect to some other parameter.

• Derivative of a function at a given point in its domain of definition.

• Definition 1

- Suppose f is a real valued function and a is a point in its domain of definition.
- The derivative of f at a is defined by

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

Provided this limit exists.

- Derivative of $f(x)$ at a is denoted by $f'(a)$

• Definition 2

- Suppose f is a real valued function, the function defined by

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Wherever limit exists is defined to be derivative of f at x

- Denoted by $f'(x)$.
- This definition of derivative is also called the **first principle of derivative**.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Thus

- $f'(x)$ is denoted by $\frac{d}{dx}(f(x))$ or if $y = f(x)$, it is denoted by dy/dx .
- This is referred to as derivative of $f(x)$ or y with respect to x .
- It is also denoted by $D(f(x))$.
- Further, derivative of f at $x = a$

is also denoted by $\frac{d}{dx}f(x) \Big|_a$ or $\frac{df}{dx} \Big|_a$ or even $\left(\frac{df}{dx}\right)_{x=a}$.

Theorem 5

- Let f and g be two functions such that their derivatives are defined in a common domain. Then
 - Derivative of sum of two functions is sum of the derivatives of the functions.

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$

- Derivative of difference of two functions is difference of the derivatives of the functions.

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x).$$

- Derivative of product of two functions is given by following product rule.

$$\frac{d}{dx}[f(x) \cdot g(x)] = \frac{d}{dx}f(x) \cdot g(x) + f(x) \cdot \frac{d}{dx}g(x)$$

- Derivative of quotient of two functions is given by the following quotient rule (whenever the denominator is non-zero).

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}f(x) \cdot g(x) - f(x) \frac{d}{dx}g(x)}{(g(x))^2}$$

- Let $u = f(x)$ and $v = g(x)$.

- **Product Rule:**

- $(uv)' = u'v + uv'$.
 - Also referred as Leibnitz rule for differentiating product of functions

- **Quotient rule**

- $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

- Derivative of the function $f(x) = x$ is the constant

Theorem 6

- Derivative of $f(x) = x^n$ is nx^{n-1} for any positive integer n .

- **Proof**

- By definition of the derivative function, we have

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h}$$

Binomial theorem tells that $(x+h)^n = \binom{n}{C_0}x^n + \binom{n}{C_1}x^{n-1}h + \dots + \binom{n}{C_n}h^n$ and hence $(x+h)^n - x^n = h(nx^{n-1} + \dots + h^{n-1})$. Thus

$$\begin{aligned} \frac{df(x)}{dx} &= \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(nx^{n-1} + \dots + h^{n-1})}{h} \\ &= \lim_{h \rightarrow 0} (nx^{n-1} + \dots + h^{n-1}), = nx^{n-1}. \end{aligned}$$

- This can be proved as below alternatively

$$\begin{aligned}
\frac{d}{dx}(x^n) &= \frac{d}{dx}(x \cdot x^{n-1}) \\
&= \frac{d}{dx}(x) \cdot (x^{n-1}) + x \cdot \frac{d}{dx}(x^{n-1}) \text{ (by product rule)} \\
&= 1 \cdot x^{n-1} + x \cdot ((n-1)x^{n-2}) \text{ (by induction hypothesis)} \\
&= x^{n-1} + (n-1)x^{n-1} = nx^{n-1}.
\end{aligned}$$

Theorem 7

- Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ be a polynomial function, where a_i s are all real numbers and $a_n \neq 0$. Then, the derivative function is given by

$$\frac{df(x)}{dx} = na_n x^{n-1} + (n-1)a_{n-1} x^{n-2} + \dots + 2a_2 x + a_1.$$

Quick Reference:

- For functions f and g the following holds:**

$$\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$$

$$\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$$

$$\lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$$

- Following are some of the standard limits**

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1, \lim_{x \rightarrow a} \frac{\sin(x-a)}{x-a} = 1$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$$

$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1, \lim_{x \rightarrow a} \frac{\tan(x-a)}{x-a} = 1$$

$$\lim_{x \rightarrow 0} \frac{\sin^{-1} x}{x} = 1, \lim_{x \rightarrow 0} \frac{\tan^{-1} x}{x} = 1$$

$$\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a, a > 0, a \neq 1$$

- Derivatives**

- The derivative of a function f at a is defined by

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

- Derivative of a function f at any point x is defined by

$$f'(x) = \frac{df(x)}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

- **For functions u and v the following holds:**

$$(u \pm v)' = u' \pm v'$$

$$(uv)' = u'v + uv' \quad \Rightarrow \quad \frac{d}{dx}(uv) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} \quad \Rightarrow \quad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$$

- **Following are some of the standard derivatives**

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$$