Mathematical Reasoning

Basics

- In mathematical language, there are two kinds of reasoning – inductive and deductive

Logic

- Logic is the subject that deals with the method of reasoning.
- It provides us rules for determining the validity of a given argument in proving theorem.

Statement (Proposition)

- The basic unit involved in mathematical reasoning is a mathematical statement
- A statement is an assertive sentence which is either true or false but not both a true statement is called valid statement. Otherwise it is called invalid statement.
- Statements are denoted by the small letters
- i.e., p, q, r … etc.
- Example
    - In 2003, the president of India was a woman.
    - An elephant weighs more than a human being
        - First sentence is false and second sentence is true
    - Mathematics is fun.
        - This sentence is subjective in the sense that for those who like mathematics, it may be fun but for others it may not be. This means that this sentence is not always true. Hence it is not a statement.
- Statement should be a "mathematically acceptable" statement
- Ambiguous sentence is not acceptable as a statement in mathematics.

Open and Compound Statement

- A sentence which contains one or more variable such that when certain values are given to the variable it becomes a statement, is called an **open statement**.
- If two or more simple statements are combined by the use of words such as 'and', 'or', 'not', 'if', 'then', 'if and only if ', then the resulting statement is called a **compound statement.**
    - Example
        - There is something wrong with the bulb or with the wiring
        - The sky is blue and the grass is green.
            - The component statements are
                - p: The sky is blue.
                - q: The grass is green.
                - The connecting word is 'and'.
    - The compound statement with 'And' is true if all its component statements are true.
    - The component statement with 'And' is false if any of its component statements is false (this includes the case that some of its component statements are false or all of its component statements are false).
    - A compound statement with an 'Or' is true when one component statement is true or both the component statements are true.
    - A compound statement with an 'Or' is false when both the component statements are false.
    - In Or statement there are two types
        - **Exclusive "Or"**
            - Example
                - Two lines intersect at a point or are parallel.
        - **Inclusive "Or"**

- Example
  - To enter a country, you need a passport or a voter registration card.

Elementary Operation of Logic

- **Conjunction**
  - A compound sentence formed by two simple sentences p and q using connective ‘and’ is called the conjunction of p and q and it is represented by $p \wedge q$.
- **Disjunction**
  - A compound sentence formed by two simple sentences p and q using connectives ‘or’ is called the disjunction of p and q and it is represented by $p \vee q$.
- **Negation**
  - A statement which is formed by changing the truth value of a given statement by using the word like ‘no’, ‘not’ is called negation of given statement.
  - If p is a statement, the negation of p is denoted by $\sim p$.
  - Example
    - p: New Delhi is a city
    - The negation of this statement is
      - It is not the case that New Delhi is a city
      - It is false that New Delhi is a city.
      - New Delhi is not a city
- **Conditional Sentence (Implication)**
  - Two simple sentences p and q connected by the phase, if and then, is called conditional sentence of p and q arid it is denoted by $p \Rightarrow q$.
- **Biconditional Sentence (Bi-implication)**
  - The two simple sentences connected by the phrase, ‘if and only if this is called biconditional sentences. It is denoted by the symbol ‘<=>’.
- **Table for Basic Logical Connections**

p	q	$\sim p$	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Truth Value and Truth Table

- A statement can be either ‘true’ or ‘false’ which are called truth values of a .statement and it is represented by the symbols T and F, respectively.
- A truth table is a summary of truth values of the resulting statements for all possible assignment of values to the variables appearing in a compound statement.
- Number of rows depends on their number of statements.
- **Tautology and Contradiction**
  - The compound statement which are true for every value of their components are called tautology.
  - The compound statements which are false for every value of their components are called contradiction (fallacy).
- **Truth Table**

p	q	$p \Rightarrow q$	$q \Rightarrow p$	Tautology $(p \Rightarrow q) \vee (q \Rightarrow p)$	Contradiction $\sim\{(p \Rightarrow q) \vee (q \Rightarrow p)\}$
T	T	T	T	T	F
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

Quantifiers and Quantified Statements

- In this statements, there are two important symbols are used.
  - The symbol '∀' stands for 'all values of '.This is known as universal quantifier.
  - The symbol '∃' stands for 'there exists'. The symbol ∃ is known as existential quantifier.
- Example
  - p: For every prime number p, p is an irrational number.
    - This means that if S denotes the set of all prime numbers, then for all the members' p of the set S, p is an irrational number.
- **Quantified Statement**
  - An open sentence with a quantifier becomes a statement, called a quantified statement.
- **Negation of a Quantified Statement**
  - $\sim\{\forall\ x \in A : p(x) \text{ is true}\} = \{\exists\ x \in A \text{ such that (s.t.) } \sim p(x) \text{ is true}\}$
  - $\sim\{\exists x \in A : p(x) \text{ is true}\} = \{\forall\ x \in A : \sim p(x) \text{ is true}\}$

Implications

- These are statements with words "if-then", "only if" and "if and only if "
- Example
  - r: If a number is a multiple of 9, then it is a multiple of 3.
  - Let p and q denote the statements
    - p : a number is a multiple of 9.
    - q: a number is a multiple of 3.
- Then, if p then q is the same as the following:
  - p implies q is denoted by $p \Rightarrow q$. The symbol ⇒ stands for implies.
  - p is a sufficient condition for q.
  - p only if q.
  - q is a necessary condition for p.
  - ~q implies ~p.

Contrapositive and converse

- Contrapositive and converse are certain other statements which can be formed from a given statement with "if-then".
- 'If and only if', represented by the symbol '⇔'means the following equivalent forms for the given statements p and q.
  - p if and only if q
  - q if and only if p
  - p is necessary and sufficient condition for q and vice-versa
  - $p \Leftrightarrow q$
- Example
  - If the physical environment changes, then the biological environment changes.

- Then the contrapositive of this statement is If the biological environment does not change, then the physical environment does not change

Validating Statements

There are some general rules for checking whether a statement is true or not

- **Rule 1**
  - If p and q are mathematical statements, then in order to show that the statement "p and q" is true, the following steps are followed.
    - Step-1 Show that the statement p is true.
    - Step-2 Show that the statement q is true.
- **Rule 2**
  - Statements with "Or"
  - If p and q are mathematical statements, then in order to show that the statement "p or q" is true, one must consider the following.
    - Case 1 By assuming that p is false, show that q must be true.
    - Case 2 By assuming that q is false, show that p must be true.
- **Rule 3**
  - Statements with "If-then"
  - In order to prove the statement "if p then q" we need to show that any one of the following case is true.
    - Case 1 By assuming that p is true, prove that q must be true.(Direct method)
    - Case 2 By assuming that q is false, prove that p must be false.(Contrapositive method)
- **Rule 4**
  - Statements with "if and only if "
  - In order to prove the statement "p if and only if q", we need to show.
  - If p is true, then q is true and If q is true, then p is true
- **By Contradiction**
  - To check whether a statement p is true, we assume that p is not true i.e. ~p is true.
  - Then, we arrive at some result which contradicts our assumption. Therefore, we conclude that p is true.
  - The method involves giving an example of a situation where the statement is not valid. Such an example is called a counter example