

PERMUTATION AND COMBINATION

1. FUNDAMENTAL PRINCIPLES OF COUNTING

1.1 Fundamental Principle of Multiplication

If an event can occur in m different ways following which another event can occur in n different ways following which another event can occur in p different ways. Then the total number of ways of simultaneous happening of all these events in a definite order is $m \times n \times p$.

1.2 Fundamental Principle of Addition

If there are two jobs such that they can be performed independently in m and n ways respectively, then either of the two jobs can be performed in $(m + n)$ ways.

2. SOME BASIC ARRANGEMENTS AND SELECTIONS

2.1 Combinations

Each of the different selections made by taking some or all of a number of distinct objects or items, irrespective of their arrangements or order in which they are placed, is called a combination.

2.2 Permutations

Each of the different arrangements which can be made by taking some or all of a number of distinct objects is called a permutation.



- Let r and n be positive integers such that $1 \leq r \leq n$. Then, the number of all permutations of n distinct items or objects taken r at a time, is

$${}^n P_r = {}^n C_r \times r!$$

$$\text{Proof : Total ways} = n(n-1)(n-2) \dots (n-r+1)$$

$$= \frac{n(n-1)(n-2) \dots (n-r+1)(n-r)!}{(n-r)!}$$

$$= \frac{n!}{(n-r)!}$$

$$= {}^n P_r.$$

So, the total no. of arrangements (permutations) of n -distinct items, taking r at a time is ${}^n P_r$ or $P(n, r)$.

- The number of all permutations (arrangements) of n distinct objects taken all at a time is $n!$.
- The number of ways of selecting r items or objects from a group of n distinct items or objects, is

$$\frac{n!}{(n-r)!r!} = {}^n C_r.$$

3. GEOMETRIC APPLICATIONS OF ${}^n C_r$

- (i) Out of n non-concurrent and non-parallel straight lines, points of intersection are ${}^n C_2$.
- (ii) Out of ' n ' points the number of straight lines are (when no three are collinear) ${}^n C_2$.
- (iii) If out of n points m are collinear, then No. of straight lines = ${}^n C_2 - {}^m C_2 + 1$
- (iv) In a polygon total number of diagonals out of n points (no three are collinear) = ${}^n C_2 - n = \frac{n(n-3)}{2}$.
- (v) Number of triangles formed from n points is ${}^n C_3$. (when no three points are collinear)
- (vi) Number of triangles out of n points in which m are collinear, is ${}^n C_3 - {}^m C_3$.
- (vii) Number of triangles that can be formed out of n points (when none of the side is common to the sides of polygon), is ${}^n C_3 - {}^n C_1 - {}^n C_1 \cdot {}^{n-4} C_1$
- (viii) Number of parallelograms in two systems of parallel lines (when 1st set contains m parallel lines and 2nd set contains n parallel lines), is = ${}^n C_2 \times {}^m C_2$
- (ix) Number of squares in two system of perpendicular parallel lines (when 1st set contains m equally spaced parallel lines and 2nd set contains n same spaced parallel lines)

$$= \sum_{r=1}^{m-1} (m-r)(n-r); (m < n)$$

4. PERMUTATIONS UNDER CERTAIN CONDITIONS

The number of all permutations (arrangements) of n different objects taken r at a time :

- (i) When a particular object is to be always included in each arrangement, is ${}^{n-1} C_{r-1} \times r!$.
- (ii) When a particular object is never taken in each arrangement, is ${}^{n-1} C_r \times r!$.

5. DIVISION OF OBJECTS INTO GROUPS

5.1 Division of items into groups of unequal sizes

1. The number of ways in which $(m + n)$ distinct items can be divided into two unequal groups containing

$$m \text{ and } n \text{ items, is } \frac{(m+n)!}{m!n!}.$$

2. The number of ways in which $(m+ n+ p)$ items can be divided into unequal groups containing m, n, p items, is

$${}^{m+n+p} P_m \cdot {}^{n+p} P_m = \frac{(m+n+p)!}{m!n!p!}.$$

3. The number of ways to distribute $(m+n+p)$ items among 3 persons in the groups containing m, n and p items

$$= (\text{No. of ways to divide}) \times (\text{No. of groups})!$$

$$= \frac{(m+n+p)!}{m!n!p!} \times 3!$$

5.2 Division of Objects into groups of equal size

The number of ways in which mn different objects can be divided equally into m groups, each containing n objects and the order of the groups is not important, is

$$\left(\frac{(mn)!}{(n!)^m} \right) \frac{1}{m!}$$

The number of ways in which mn different items can be divided equally into m groups, each containing n objects and the order of groups is important, is

$$\left(\frac{(mn)!}{(n!)^m} \times \frac{1}{m!} \right) m! = \frac{(mn)!}{(n!)^m}$$

6. PERMUTATIONS OF ALIKE OBJECTS

1. The number of mutually distinguishable permutations of n things, taken all at a time, of which p are alike of one kind, q alike of second kind such that $p + q = n$, is

$$\frac{n!}{p!q!}$$

2. The number of permutations of n things, of which p are alike of one kind, q are alike of second kind and

remaining all are distinct, is $\frac{n!}{p!q!}$. Here $p + q \neq n$

3. The number of permutations of n things, of which p_1 are alike of one kind; p_2 are alike of second kind; p_3 are alike of third kind; ; p_r are alike of r^{th} kind such that

$$p_1 + p_2 + \dots + p_r = n, \text{ is } \frac{n!}{p_1!p_2!p_3!\dots p_r!}$$

4. Suppose there are r things to be arranged, allowing repetitions. Let further p_1, p_2, \dots, p_r be the integers such that the first object occurs exactly p_1 times, the second occurs exactly p_2 times subject, etc. Then the total number of permutations of these r objects to the above condition, is

$$\frac{(p_1 + p_2 + \dots + p_r)!}{p_1!p_2!p_3!\dots p_r!}$$

7. DISTRIBUTION OF ALIKE OBJECTS

- (i) The total number of ways of dividing n identical items among r persons, each one of whom, can receive 0, 1, 2, or more items ($\leq n$), is ${}^{n+r-1}C_{r-1}$.

OR

The total number of ways of dividing n identical objects into r groups, if blank groups are allowed, is ${}^{n+r-1}C_{r-1}$.

- (ii) The total number of ways of dividing n identical items among r persons, each of whom, receives at least one item is ${}^{n-1}C_{r-1}$.

OR

The number of ways in which n identical items can be divided into r groups such that blank groups are not allowed, is ${}^{n-1}C_{r-1}$.

- (iii) The number of ways in which n identical items can be divided into r groups so that no group contains less than k items and more than m ($m < k$) is

The coefficient of x^n in the expansion of $(x^m + x^{m+1} + \dots + x^k)^r$

8. NO. OF INTEGRAL SOLUTIONS OF LINEAR EQUATIONS AND INEQUATIONS

Consider the eqn. $x_1 + x_2 + x_3 + x_4 + \dots + x_r = n$... (i)

where x_1, x_2, \dots, x_r and n are non-negative integers.

This equation may be interpreted as that n identical objects are to be divided into r groups.

1. The total no. of non-negative integral solutions of the equation $x_1 + x_2 + \dots + x_r = n$ is ${}^{n+r-1}C_{r-1}$.
2. The total number of solutions of the same equation in the set N of natural numbers is ${}^{n-1}C_{r-1}$.

3. In order to solve inequations of the form

$$x_1 + x_2 + \dots + x_m \leq n$$

we introduce a dummy (artificial) variable x_{m+1} such that $x_1 + x_2 + \dots + x_m + x_{m+1} = n$, where $x_{m+1} \geq 0$.

The no. of solutions of this equation are same as the no. of solutions of in Eq. (i).

9. CIRCULAR PERMUTATIONS

1. The number of circular permutations of n distinct objects is $(n - 1)!$.
2. If anti-clockwise and clockwise order of arrangements are not distinct then the number of circular permutations of n distinct items is $1/2 \{(n - 1)!\}$
e.g., arrangements of beads in a necklace, arrangements of flowers in a garland etc.

10. SELECTION OF ONE OR MORE OBJECTS

1. The number of ways of selecting one or more items from a group of n distinct items is $2^n - 1$.

Proof : Out of n items, 1 item can be selected in ${}^n C_1$ ways; 2 items can be selected in ${}^n C_2$ ways; 3 items can be selected in ${}^n C_3$ ways and so on.....

Hence, the required number of ways

$$\begin{aligned} &= {}^n C_1 + {}^n C_2 + {}^n C_3 + \dots + {}^n C_n \\ &= ({}^n C_0 + {}^n C_1 + {}^n C_2 + \dots + {}^n C_n) - {}^n C_0 \\ &= 2^n - 1. \end{aligned}$$

2. The number of ways of selecting r items out of n identical items is 1 .
3. The total number of ways of selecting zero or more items from a group of n identical items is $(n + 1)$.
4. The total number of selections of some or all out of $p + q + r$ items where p are alike of one kind, q are alike of second kind and rest are alike of third kind, is $[(p + 1)(q + 1)(r + 1)] - 1$.
5. The total number of ways of selecting one or more items from p identical items of one kind; q identical items of second kind; r identical items of third kind and n different items, is $(p + 1)(q + 1)(r + 1)2^n - 1$

11. THE NUMBER OF DIVISORS AND THE SUM OF THE DIVISORS OF A GIVEN NATURAL NUMBER

$$\text{Let } N = p_1^{n_1} \cdot p_2^{n_2} \cdot p_3^{n_3} \dots p_k^{n_k} \quad \dots(1)$$

where p_1, p_2, \dots, p_k are distinct prime numbers and n_1, n_2, \dots, n_k are positive integers.

1. Total number of divisors of $N = (n_1 + 1)(n_2 + 1) \dots (n_k + 1)$.
2. This includes 1 and n as divisors. Therefore, number of divisors other than 1 and n , is

$$(n_1 + 1)(n_2 + 1)(n_3 + 1) \dots (n_k + 1) - 2.$$

3. The sum of all divisors of (1) is given by

$$= \left\{ \frac{p_1^{n_1+1} - 1}{p_1 - 1} \right\} \left\{ \frac{p_2^{n_2+1} - 1}{p_2 - 1} \right\} \left\{ \frac{p_3^{n_3+1} - 1}{p_3 - 1} \right\} \dots \left\{ \frac{p_k^{n_k+1} - 1}{p_k - 1} \right\}.$$

12. DEARRANGEMENTS

If n distinct objects are arranged in a row, then the no. of ways in which they can be dearranged so that none of them occupies its original place, is

$$n! \left\{ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots + (-1)^n \frac{1}{n!} \right\}$$

and it is denoted by $D(n)$.

If r ($0 \leq r \leq n$) objects occupy the places assigned to them i.e., their original places and none of the remaining $(n - r)$ objects occupies its original places, then the no. of such ways, is

$$\begin{aligned} D(n - r) &= {}^n C_r \cdot D(n - r) \\ &= {}^n C_r \cdot (n - r)! \left\{ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^{n-r} \frac{1}{(n - r)!} \right\}. \end{aligned}$$