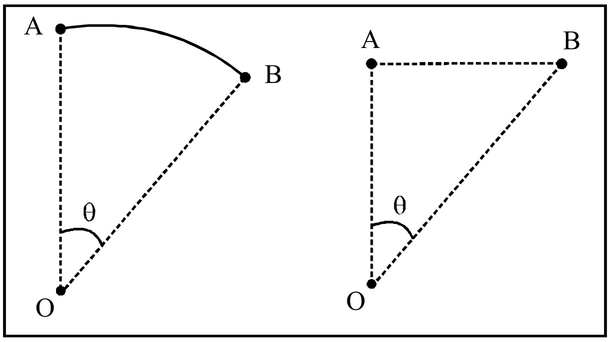
ROTATIONAL MOTION

1. KINEMATICS OF SYSTEM OF PARTICLES

1.1 System of particles can move in different ways as observed by us in daily life. To understand that we need to understand few new parameters.

(a) Angular Displacement

Consider a particle moves from A to B in the following figures.

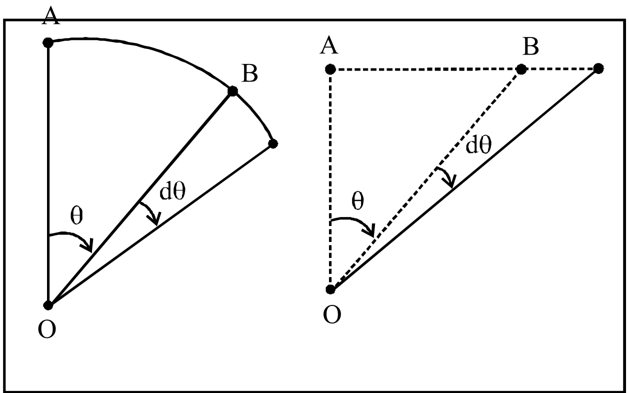


Angle is the angular displacement of particle about O.

Units → radian

(b) Angular Velocity

The rate of change of angular displacement is called as angular velocity.



$\omega = \frac{d\theta}{dt}$ $\qquad$ $\omega = \frac{d\theta}{dt}$

Units → Rad/s

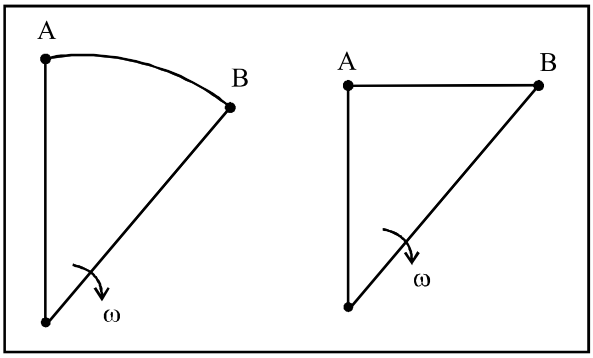
It is a vector quantity whose direction is given by right hand thumb rule.

According to right hand thumb rule, if we curl the fingers of right hand along with the body, then right hand thumb gives us the direction of angular velocity.

It is always along the axis of the motion.

(c) Angular Acceleration

Angular acceleration of an object about any point is rate of change of angular velocity about that point.



$\alpha = \frac{d\omega}{dt}$ $\qquad$ $\alpha = \frac{d\omega}{dt}$

Units → Rad/s^2

It is a vector quantity. If α is constant then similarly to equation of motion (i.e.)

ω, α θ, t are related $\omega = \omega_0 + \alpha t$

$$\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

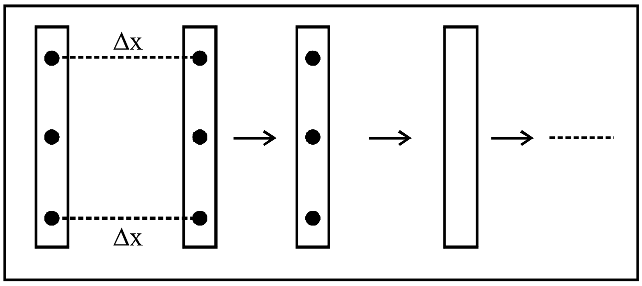
$$\omega_f^2 - \omega_0^2 = 2\alpha\theta$$

1.2 Various types of motion

(a) Translational Motion

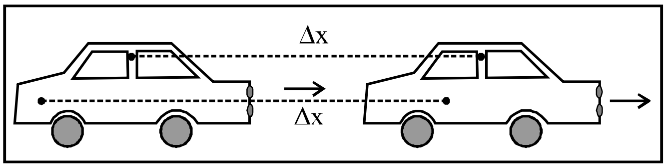
System is said to be in translational motion, if all the particles lying in the system have same linear velocity.

Example



Motion of a rod as shown.

Example

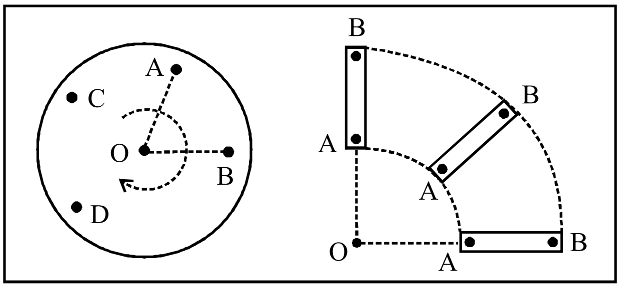


Motion of body of car on a straight rod.

In both the above examples, velocity of all the particles is same as they all have equal displacements in equal intervals of time.

(b) Rotational Motion

A system is said to be in pure rotational motion, when all the points lying on the system are in circular motion about one common fixed axis.



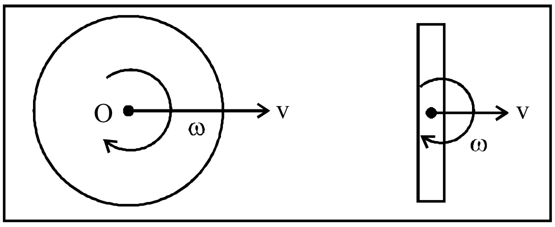
In pure rotational motion.

Angular velocity of all the points is same about the fixed axis.

(c) Rotational + Translational

A system is said to be in rotational + translational motion, when the particle is rotating with some angular velocity about a movable axis.

For example :



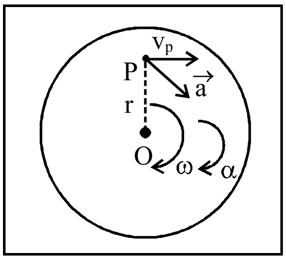
v = velocity of axis.

ω = Angular velocity of system about O.

1.3 Inter Relationship between kinematics variable

In general if a body is rotating about any axis (fixed or movable), with angular velocity ω and angular acceleration α then velocity of any point p with respect to axis is $\vec{v} = \vec{\omega} \times \vec{r}$ and $\vec{a} = \vec{\alpha} \times \vec{r} - \omega^2 \vec{r}$.

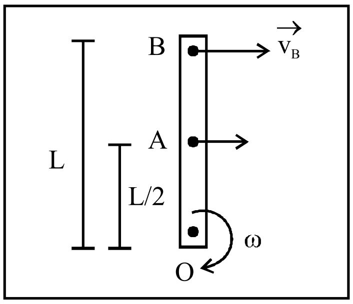
i.e.,



$$\vec{v}_p = \vec{\omega} \times \vec{r}$$

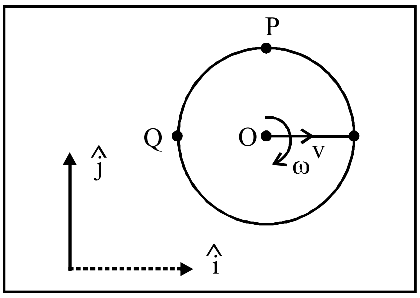
$$\vec{a} = \vec{\alpha} \times \vec{r} - \omega^2 \vec{r}$$

Example



$v_B = \omega L$ and $v_A = \dfrac{\omega L}{2}$, with directions as shown in figure.

Now in rotational + translational motion, we just superimpose velocity and acceleration of axis on the velocity and acceleration of any point about the axis. (i.e.)



$$\vec{v}_{PO} = \omega R \hat{i}$$

$$\vec{v}_0 = v\hat{i}$$

$$\because \quad \vec{v}_P - \vec{v}_0 = \vec{v}_{PO}$$

$$\Rightarrow \quad \vec{v}_P = \vec{v}_{PO} + \vec{v}_O$$

$$\omega R + v\hat{i}$$

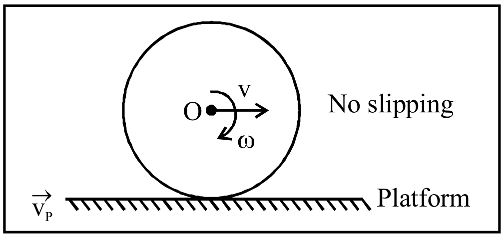
Similarly $\vec{v}_{QO} = \omega R\hat{j}$

$$\vec{v}_0 = v\hat{i}$$

$$\therefore \quad \vec{v}_Q = v\hat{i} + \omega R\,\hat{j}$$

Inter-relation between v of axis and ω or a of axis and α depends on certain constraints.

General we deal with the case of no slipping or pure rolling.



The constraint in the above case is that velocity of points of contact should be equal for both rolling body and playfrom.

(i.e.) $v - r\omega = v_P$

If platform is fixed then

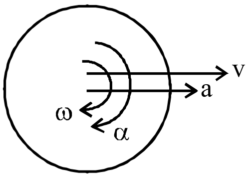
$v_P = 0 \Rightarrow \boxed{v = r\omega}$

An differentiating the above term we get

$$\frac{dv}{dt} = \frac{r\,d\omega}{dt}.$$

Now if $\frac{dv}{dt} = a$

$\frac{d\omega}{dt} = \alpha$



then $\boxed{a = r\alpha}$

Remember if acceleration is assumed opposite to velocity then $a = -\frac{dv}{dt}$ instead of $a = \frac{dv}{dt}$.

Similary : If α and ω are in opposite direction the $\alpha = -\frac{d\omega}{dt}$.

Accordingly the constraints can change depending upon the assumptions.

2. ROTATIONAL DYNAMICS

2.1 Torque

Similar to force, the cause of rotational motion is a physical quantity called a torque.

Torque incorporates the following factors.

→ Amount of force.

→ Point of application of force.

→ Direction of application of force.

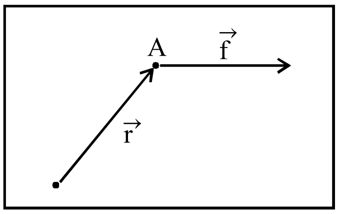
Combining all of the above.

Torque $\tau = rf \sin\theta$ about a point O.

Where r = distance from the point O to point of application of force.

f= force

θ = angle between $\vec{r}$ and $\vec{f}$



→ Torque about O.

→ A is point of application of force.

Magnitude of torque can also be rewritten as

$\tau = rf_{\perp}$ or $\tau = r_{\perp}f$ where

$f_{\perp}$ = component of force in the direction $\perp$ to $\vec{r}$.

$r_{\perp}$ = component of force in the direction $\perp$ to $\vec{f}$.

Direction :

Direction of torque is given by right hand thumb rule. If we curl the fingers of right hand from first vector $(\vec{r})$ to second vector $(\vec{f})$ then right hand thumb gives us direction of their cross product.

→ Torque is always defined about a point or about an axis.

→ When there are multiple forces, the net torque needs to be calculated, (i.e.)

$\vec{\tau}_{net} = \vec{\tau}_{F_1} + \vec{\tau}_{F_2} + \ldots\ldots\ldots\tau_{F_n}$

All torque about same point/axis.

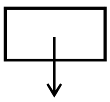
If $\sum\tau = 0$, then the body is in rotational equilibrium.

→ If $\sum F = 0$ along with $\sum\tau = 0$, then body is in mechanical equilibrium.

→ If equal and opp. force act to produce same torque then they constitutes a couple.

→ For calculating torque, it is very important to find the eff. point of application of force.

→ Mg → Acts at com/centre of gravity.



→ N → Point of application depends upon situation to situation.

2.2 Newtwon's Laws

$\sum \tau = I\alpha$.

→ I = moment of Inertia

→ α = Angular Acceleration.

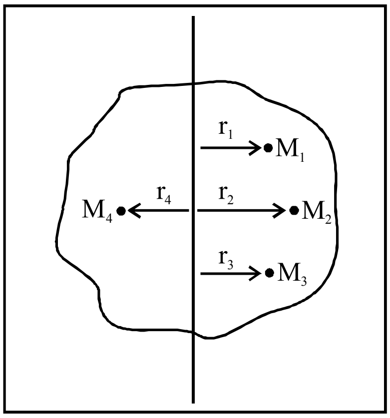
2.3 Moment of Inertia

→ Gives the measure of mass distribution about on axis.

→ $I = \sum m_i r_i^2$

$r_i = \perp$ distance of the i^{th} mass from axis.

→ Always defined about an axis.

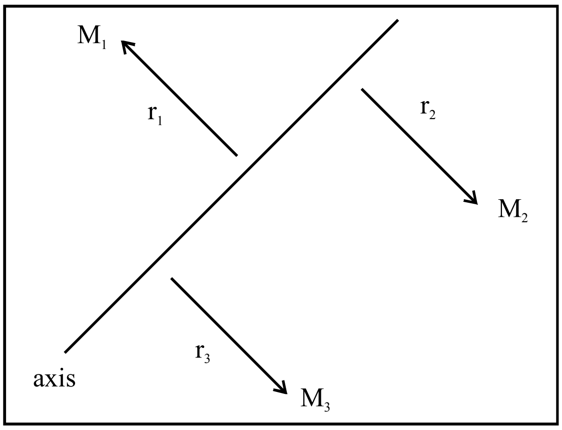


$I = M_1 r_1^2 + M_2 r_2^2 + M_3 r_3^2 + M_4 r_4^2$

→ SI units → kgm^2

→ Gives the measure of rotational inertia and is equavelent to mass.

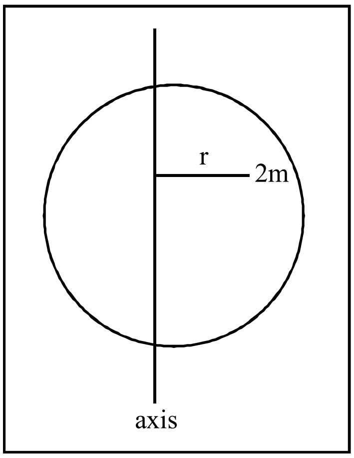
(a) Moment of Inertia of a discreet particle system :



$I = M_1 r_1^2 + M_2 r_2^2 + M_3 r_3^2$

(b) Continuous Mass Distribution

For continuous mass distribution, we need to take help of integration :



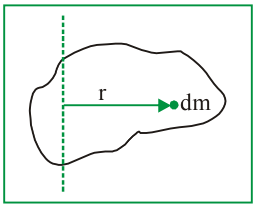
$I_{axis} = \int r^2 dm$

3. MOMENT OF INERTIA

3.1 Moment of inertia of Continuous Bodies

When the distribution of mass of a system of particle is continuous, the discrete sum $I = \sum m_i r_i^2$ is replaced by an integral. The moment of inertia of the whole body takes the form

$$I = \int r^2 \, dm$$



Keep in mind that here the quantity r is the perpendicular distance to an axis, not the distance to an origin. To evaluate this integral, we must express m in terms of r.

Note..

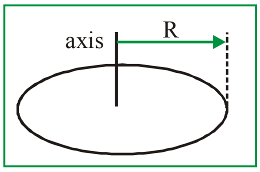
Comparing the expression of rotational kinetic energy with 1/2 mv^2, we can say that the role of moment of inertia (I) is same in rotational motion as that of mass in linear motion. It is a measure of the resistance offered by a body to a change in its rotational motion.

3.2 Moment of Inertia of some important bodies

1. Circular Ring

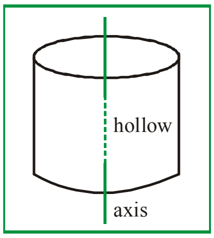
Axis passing through the centre and perpendicular to the plane of ring.

$I = MR^2$



2. Hollow Cylinder

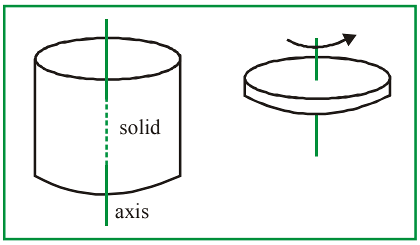
$I = MR^2$



3. Solid Cylinder and a Disc

About its geometrical axis :

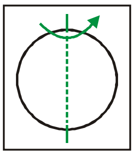
$$I = \frac{1}{2}MR^2$$



4. (a) Solid Sphere

Axis passing through the centre :

$I = 2/5\ MR^2$



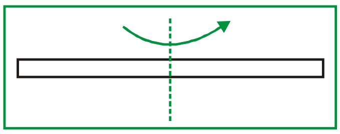
(b) Hollow Sphere

Axis passing through the centre :

$I = 2/3\ MR^2$

5. Thin Rod of length *l* :

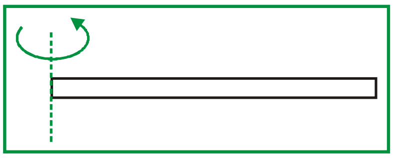
(a) Axis passing through mid point and perpendicular to the length :



$$I = \frac{M\ell^2}{12}$$

(b) Axis passing through an end and perpendicular to the rod:

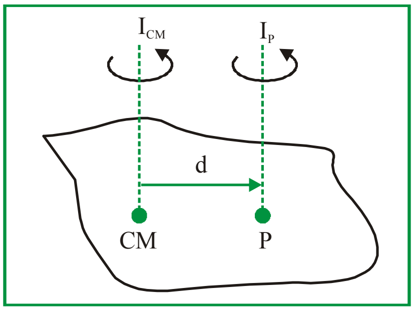
$$I = \frac{M\ell^2}{3}$$



3.3 Theorems on Moment of Inertia

1. Parallel Axis Theorem : Let I_{cm} be the moment of inertia of a body about an axis through its centre of mass and Let I_p be the moment of inertia of the same body about another axis which is parallel to the original one.

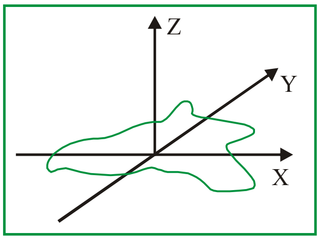
If d is the distance between these two parallel axes and M is the mass of the body then according to the parallel axis theorem :



$$I_p = I_{cm} + Md^2$$

2. Perpendicular Axis Theorem :

Consider a plane body (i.e., a plate of zero thickness) of mass M. Let X and Y axes be two mutually perpendicular lines in the plane of the body. The axes intersect at origin O.



Let I_x = moment of inertia of the body about X–axis.

Let I_y = moment of inertia of the body about Y–axis.

The moment of inertia of the body about Z–axis (passing through O and perpendicular to the plane of the body) is given by :

$$I_z = I_x + I_y$$

The above result is known as the perpendicular axis theorem.

3.4 Radius of Gyration

If M is the mass and I is the moment of inertia of a rigid body, then the radius of gyration (k) of a body is given by :

$$k = \sqrt{\frac{I}{M}}$$

4. ANGULAR MOMENTUM (L) AND IMPULSE

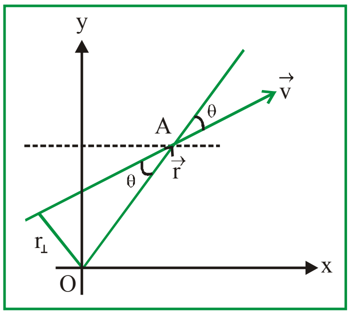
4.1 Angular Momentum

(a) For a particle

Angular momentum about origin (O) is given as :

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times (m\vec{v})$$

where $\vec{r}$ = position vector of the particle ; $\vec{v}$ = velocity



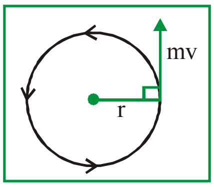
$\Rightarrow \quad L = mv\, r \sin\theta = mv\,(OA) \sin\theta = mvr_\perp$

where $r_\perp$ = perpendicular distance of velocity vector from O.

(b) For a particle moving in a circle

For a particle moving in a circle of radius r with a speed v, its linear momentum is mv, its angular momentum (L) is given as :

$$L = mvr_\perp = mvr$$



(c) For a rigid body (about a fixed axis)

L = sum of angular momentum of all particles

$$= m_1v_1r_1 + m_2v_2r_2 + m_3v_3r_3 + ...$$

$$= m_1r_1^2\omega + m_2r_2^2\omega + m_3r_3^2\omega + ... \qquad (v = r\omega)$$

$$= (m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + ...)\omega \Rightarrow L = I\omega$$

(compare with linear momentum p = mv in linear motion)

L is also a vector and its direction is same as that of ω (i.e. clockwise or anticlockwise)

We knows,

$$\vec{L} = I\vec{\omega}$$

$$\frac{d\vec{L}}{dt} = I\frac{d\vec{\omega}}{dt} = I\vec{\alpha} = \vec{\tau}_{net}$$

4.2 Conservation of angular momentum

If $\vec{\tau}_{net} = 0$

$\Rightarrow \quad \frac{d\vec{L}}{dt} = 0$

$\Rightarrow \quad \vec{L}$ = constant

$\Rightarrow \quad \vec{L}_f = \vec{L}_i$

4.3 Angular Impulse

$$\vec{J} = \int \vec{\tau}\, dt = \Delta\vec{L}$$

5. WORK AND ENERGY

5.1 Work done by a Torque

Consider a rigid body acted upon by a force F at perpendicular distance r from the axis of rotation. Suppose that under this force, the body rotates through an angle Δθ.

Work done = force × displacement

$$W = F\, r.\, \Delta\theta$$

$$W = \tau\, \Delta\theta$$

Work done = (torque) × (angular displacement)

$$\text{Power} = \frac{dW}{dt} = \tau\frac{d\theta}{dt} = \tau\,\omega$$

5.2 Kinetic Energy

Rotational kinetic energy of the system

$$= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + ...$$

$$= \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2 + ...$$

$$= \frac{1}{2}\left(m_1 r_2^2 + m r_2^2 + m_3 r_3^2 + ...\right)\omega^2$$

Hence rotational kinetic energy of the system = $\frac{1}{2}\ I\omega^2$

The total kinetic energy of a body which is moving through space as well as rotating is given by :

$$K = K_{translational} + K_{rotational}$$

$$K = \frac{1}{2} MV_{CM}^2 + \frac{1}{2}\ I_{CM}\omega^2$$

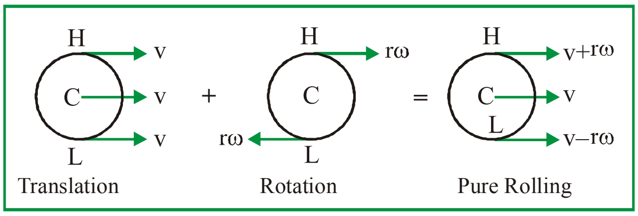
where V_{CM} = velocity of the centre of mass

I_{CM} = moment of inertia about CM

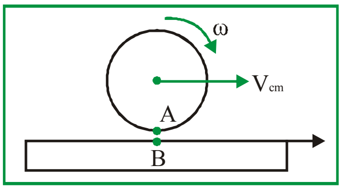
ω = angular velocity of rotation

6. ROLLING

1. Friction is responsible for the motion but work done or dissipation of energy against friction is zero as there is no relative motion between body and surface at the point of contact.
2. In case of rolling all point of a rigid body have same angular speed but different linear speed. The linear speed is maximum for the point H while minimum for the point L.



3. Condition for pure rolling : (without slipping)

(i) 

general (when surface is moving)

in terms of velocity : $V_{cm} - \omega R = V_B$

in terms of rotation : $a_{cm} - \alpha R = a_B$

special case (when $V_B = 0$)

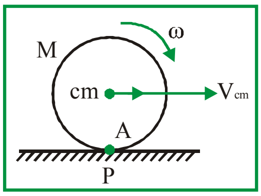
in terms of velocity : $V_{cm} = \omega R$

in terms of acceleration : $a_{cm} = \alpha\ R$

(ii) Total KE of Rolling body :

(i) $K = \frac{1}{2} I_P \omega^2$ **OR**

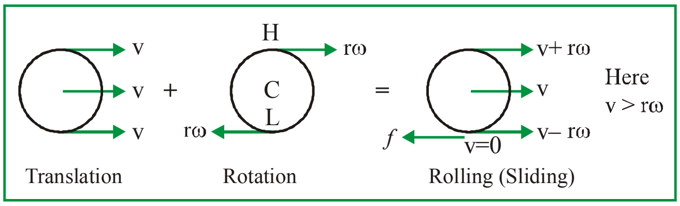
(ii) $K = \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} MV_{cm}^2$



where (a) $I_P = I_{cm} + MR^2$ (parallel axes theorem)

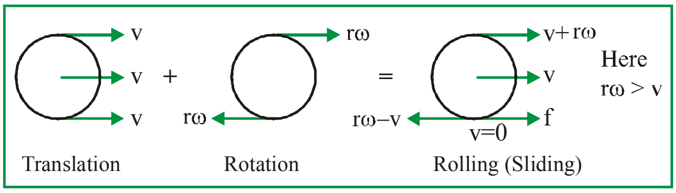
(b) $V_{cm} = \omega R$ [pure] rolling condition.

4. Forward Slipping



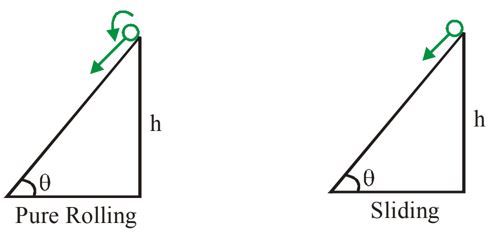
The bottom most point slides in the forward direction w.r.t. ground, so friction force acts opposite to velocity at lowest point i.e. opposite to direction of motion e.g. When sudden brakes are applied to car its 'v' remain same while 'ωr' decreases so its slides on the ground.

5. Backward Slipping



The bottom most point slides in the backward direction w.r.t. ground, so friction force acts opposite to velocity i.e. friction will act in the direction of motion e.g. When car starts on a slippery ground, its wheels has small 'v' but large 'ωr' so wheels slips on the ground and friction acts against slipping.

6.1 Rolling and sliding motion on an inclined plane



Physical Quantity	Rolling	Sliding	Falling
Velocity	$V_R = \sqrt{(2gh)/\beta}$	$V_S = \sqrt{2gh}$	$V_F = \sqrt{2gh}$
Acceleration	$a_R = g \sin\theta/\beta$	$a_S = g \sin\theta$	$a_F = g$
Time of descend	$t_R = 1/\sin\theta\sqrt{\beta(2h/g)}$	$t_S = (1/\sin\theta)\sqrt{2h/g}$	$t_F = \sqrt{2h/g}$

(where $\beta = [1 + I/Mr^2]$)

- Velocity of falling and sliding bodies are equal and is more than rollings.
- Acceleration is maximum in case of falling and minimum in case of rolling.
- Falling body reaches the bottom first while rolling last.