# Three Dimensional Geometry

## 1. CENTRAL IDEA OF 3D

There are infinite number of points in space. We want to identify each and every point of space with the help of three mutually perpendicular coordinate axes OX, OY and OZ.

## 2. AXES

Three mutually perpendicular lines OX, OY, OZ are considered as three axes.

## 3. COORDINATE PLANES

Planes formed with the help of x and y axes is known as x–y plane similarly y and z axes y – z plane and with z and x axis z – x plane.

## 4. COORDINATE OF A POINT

Consider any point P on the space drop a perpendicular form that point to x – y plane then the algebraic length of this perpendicular is considered as z–coordinate and from foot of the perpendicular drop perpendiculars to x and y axes these algebric length of perpendiculars are considered as y and x coordinates respectively.

## 5. VECTOR REPRESENTATION OF A POINT IN SPACE

If coordinate of a point P in space is (x, y, z) then the position vector of the point P with respect to the same origin is $x\hat{i} + y\hat{j} + z\hat{k}$.

## 6. DISTANCE FORMULA

Distance between any two points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ is given as $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

**Vector method**

We know that if position vector of two points A and B are given as $\overrightarrow{OA}$ and $\overrightarrow{OB}$ then

$$|AB| = |\overrightarrow{OB} - \overrightarrow{OA}|$$

$$\Rightarrow \quad |AB| = (x_2 i + y_2 j + z_2 k) - (x_1 i + y_1 j + z_1 k)|$$

$$\Rightarrow \quad |AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

## 7. DISTANCE OF A POINT P FROM COORDINATE AXES

Let PA, PB and PC are distances of the point P(x, y, z) from the coordinate axes OX, OY and OZ respectively then

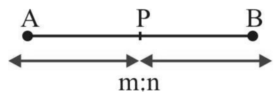
$$PA = \sqrt{y^2 + z^2},\ PB = \sqrt{z^2 + x^2},\ PC = \sqrt{x^2 + y^2}$$

## 8. SECTION FORMULA

**(i) Internal Division :**

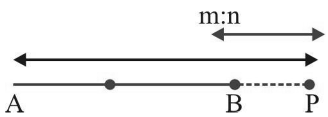
If point P divides the distance between the points A $(x_1, y_1, z_1)$ and B $(x_2, y_2, z_2)$ in the ratio of m : n (internally). The coordinate of P is given as

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$



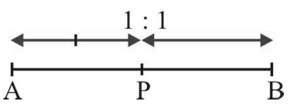

**(ii) External division**

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$$




**(iii) Mid point**

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

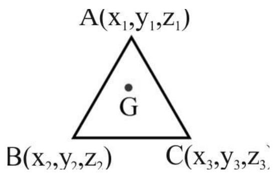



*Note...*

All these formulae are very much similar to two dimension coordinate geometry.

## 9. CENTROID OF A TRIANGLE

$$G\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3},\frac{z_1+z_2+z_3}{3}\right)$$




## 10. IN CENTRE OF TRIANGLE ABC

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c},\frac{ay_1+by_2+cy_3}{a+b+c},\frac{az_1+bz_2+cz_3}{a+b+c}\right)$$

Where | AB | = a, | BC | = b, | CA | = c

## 11. CENTROID OF A TETRAHEDRON

A $(x_1, y_1, z_1)$ B $(x_2, y_2, z_2)$ C $(x_3, y_3, z_3)$ and D $(x_4, y_4, z_4)$ are the vertices of a tetrahedron then coordinate of its centroid (G) is given as

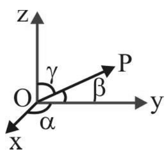
$$\left(\frac{\sum xi}{4},\frac{\sum yi}{4},\frac{\sum zi}{4}\right)$$

## 12. RELATION BETWEEN TWO LINES

Two lines in the space may be coplanar and may be none coplanar. Non coplanar lines are called skew lines if they never intersect each other. Two parallel lines are also non intersecting lines but they are coplanar. Two lines whether intersecting or non intersecting, the angle between them can be obtained.

## 13. DIRECTION COSINES AND DIRECTION RATIOS

**(i)** Direction cosines : Let α, β, γ be the angles which a directed line makes with the positive directions of the axes of x, y and z respectively, the cos α, cosβ, cosγ are called the direction cosines of the line. The direction cosines are usually denoted by (*l*, m, n).




Thus $l = \cos\alpha$, $m = \cos\beta$, $N = \cos\gamma$.

**(ii)** If *l*, m, n, be the direction cosines of a lines, then

$l^2 + m^2 + n^2 = 1$

**(iii)** Direction ratios : Let a, b, c be proportional to the direction cosines, *l*, m, n, then a, b, c are called the direction ratios.

If a, b, c are the direction ratio of any line L then $a\hat{i}+b\hat{j}+c\hat{k}$ will be a vector parallel to the line L.

If *l*, m, n are direction cosine of line L then $\ell\hat{i}+m\hat{j}+n\hat{k}$ is a unit vector parallel to the line L.

**(iv)** If *l*, m, n be the direction cosines and a, b, c be the direction ratios of a vector, then

$$\left(\ell=\frac{a}{\sqrt{a^2+b^2+c^2}},m=\frac{b}{\sqrt{a^2+b^2+c^2}},n\frac{c}{\sqrt{a^2+b^2+c^2}}\right)$$

or $$\ell=\frac{-a}{\sqrt{a^2+b^2+c^2}},m=\frac{-b}{\sqrt{a^2+b^2+c^2}},n=\frac{-c}{\sqrt{a^2+b^2+c^2}}$$

**(v)** If OP = r, when O is the origin and the direction cosines of OP are *l*, m, n then the coordinates of P are (*l*r, mr, nr).

If direction cosine of the line AB are *l*, m, n, | AB | = r, and the coordinate of A is $(x_1, y_1, z_1)$ then the coordinate of B is given as $(x_1 + rl, y_1 + rm, z_1 + rn)$

**(vi)** If the coordinates P and Q are $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ then the direction ratios of line PQ are, $a = x_2 - x_1$, $b = y_2 - y_1$ and $c = z_2 - z_1$ and the direction cosines of line PQ are $l = \frac{x_2-x_1}{|PQ|}$, $m = \frac{y_2-y_1}{|PQ|}$ and $n = \frac{z_2-z_1}{|PQ|}$

**(vii)** Direction cosines of axes : Since the positive x–axis makes angles 0°, 90°, 90° with axes of x, y and z respectively. Therefore

Direction cosines of x–axis are (1, 0, 0)

Directio cosines of y–axis are (0, 1, 0)

Direction cosines of z–axis are (0, 0, 1)

## 14. ANGLE BETWEEN TWO LINE SEGMENTS

If two lines having direction ratios $a_1$, $b_1$, $c_1$ and $a_2$, $b_2$, $c_2$ respectively then we can consider two vector parallel to the lines as $a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ and angle between them can be given as.

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

**(i)** The line will be perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

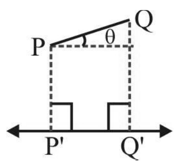
**(ii)** The lines will be parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

**(iii)** Two parallel lines have same direction cosines i.e. $l_1 = l_2$, $m_1 = m_2$, $n_1 = n_2$

## 15. PROJECTION OF A LINE SEGMENT ON A LINE

**(i)** If the coordinates P and Q are $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ then the projection of the line segments PQ on a line having direction cosines $l$, m, n is

$| l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1) |$




**(ii)** Vector form : projection of a vector $\vec{a}$ on another vector $\vec{b}$ is $\vec{a}\cdot\hat{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$ In the above case we can consider $\overrightarrow{PQ}$ as $(x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j} + (z_2 - z_1)\,\hat{k}$ in place of $\vec{a}$ and $l\,\hat{i} + m\hat{j} + n\hat{k}$ in place of $\vec{b}$.

**(iii)** $l|\vec{r}|$, $m|\vec{r}|$, and $n|\vec{r}|$ are the projection of $\vec{r}$ in OX, OY and OZ axes.

**(iv)** $\vec{r} = |\vec{r}|(l\hat{i} + m\hat{j} + n\hat{k})$

**A PLANE**

If line joining any two points on a surface lies completely on it then the surface is a plane.

**OR**

If line joining any two points on a surface is perpendicular to some fixed straight line. Then this surface is called a plane. This fixed line is called the normal to the plane.

## 16. EQUATION OF A PLANE

**(i)** Normal form of the equation of a plane is $l x + my + nz = p$, where, $l$, m n are the direction cosines of the normal to the plane and p is the distance of the plane from the origin.

**(ii)** General form : $ax + by + cz + d = 0$ is the equation of a plane, where a, b, c are the direction ratios of the normal to the plane.

**(iii)** The equation of a plane passing through the point $(x_1, y_1, z_1)$ is given by $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ where a, b, c are the direction ratios of the normal to the plane.

**(iv)** Plane through three points : The equation of the plane through three non–collinear points $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, $(x_3, y_3, z_3)$ is $\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$

**(v)** Intercept Form : The equation of a plane cutting intercept a, b, c on the axes is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

**(vi)** Vector form : The equation of a plane passing through a point having position vector $\vec{a}$ and normal to vector $\vec{n}$ is

$(\vec{r} - \vec{a})\cdot\vec{n} = 0$ or $\vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$

*Note...*

**(a)** Vector equation of a plane normal to unitvector $\hat{n}$ and at a distance d from the origin is $\vec{r}\cdot\hat{n} = d$

**(b)** **Planes parallel to the coordinate planes**

**(i)** Equation of yz–plane is $x = 0$

**(ii)** Equation of xz–plane is $y = 0$

**(iii)** Equation of xy–plane is $z = 0$

**(c)** **Planes parallel to the axes :**

If $a = 0$, the plane is parallel to x–axis i.e. equation of the plane parallel to the x–axis is $by + cz + d = 0$.

Similarly, equation of planes parallel to y–axis and parallel to z–axis are $ax + cz + d = 0$ and $ax + by + d = 0$ respectively.

**(d)** **Plane through origin :** Equation of plane passing through origin is $ax + by + cz = 0$.

**(e)** **Transformation of the equation of a plane to the normal form :** To reduce any equation $ax + by + cz - d = 0$ to the normal form, first write the constant term on the right hand side and make it positive, then divided each term by $\sqrt{a^2+b^2+c^2}$, where a, b, c are coefficients of x, y and z respectively e.g.

$$\frac{ax}{\pm\sqrt{a^2+b^2+c^2}}+\frac{by}{\pm\sqrt{a^2+b^2+c^2}}+\frac{cz}{\pm\sqrt{a^2+b^2+c^2}}$$

$$=\frac{d}{\pm\sqrt{a^2+b^2+c^2}}$$

Where (+) sign is to be taken if $d > 0$ an (–) sign is to be taken if $d > 0$.

**(f)** Any plane parallel to the given plane $ax + by + cz + d = 0$ is $ax + by + cz + \lambda = 0$ distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + dy + xz + d_2 = 0$ is given as $\frac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}}$

**(g)** **Equation of a plane passing through a given point and parallel to the given vectors :** The equation of a plane passing through a point having position vector $\vec{a}$ and parallel to $\vec{b}$ and $\vec{c}$ is $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ parametric form (where $\lambda$ and $\mu$ are scalars).

or $\vec{r}\cdot(\vec{b}\times\vec{c}) = \vec{a}\cdot(\vec{b}\times\vec{c})$ (non parametric form)

**(h)** A plane $ax + by + cz + d = 0$ divides the line segment joining $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$. in the ratio $\left(-\frac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d}\right)$

**(i)** The xy–plane divides the line segment joining the point $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ in the ratio $-\frac{z_1}{z_2}$. Similarly yz – plane in $-\frac{x_1}{x_2}$ and zx–plane in $-\frac{y_1}{y_2}$

## 17. ANGLE BETWEEN TWO PLANES

**(i)** Consider two planes $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$. Angle between these planes is the angle between their normals. Since direction ratios of their normals are (a, b, c) and (a′, b′, c′) respectively, hence θ the angle between them is given by

$$\cos\theta = \frac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}}$$

Planes are perpendicular if $aa' + bb' + cc' = 0$ and planes are parallel if $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$

**(ii)** The angle θ between the plane $\vec{r}\cdot\vec{n} = d_1$, $\vec{r}\cdot\vec{n}_2 = d_2$ to given by, $\cos\theta = \frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$ Planes are perpendicular if $\vec{n}_1\cdot\vec{n}_2 = 0$ and Planes are parallel if $\vec{n}_1 = \lambda\vec{n}_2$.

## 18. A PLANE AND A POINT

**(i)** Distance of the point (x′, y′, z′) from the plane $ax + by + az + d = 0$ is given by $\frac{ax'+by'+cz'+d}{\sqrt{a^2+b^2+c^2}}$

**(ii)** The length of the perpendicular from a point having position vector $\vec{a}$ to plane $\vec{r}\cdot\vec{n} = d$ to given by

$$p = \frac{|\vec{a}\cdot\vec{n} - d|}{|\vec{n}|}.$$

## 19. ANGLE BISECTORS

**(i)** The equations of the planes bisecting the angle between two given planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

**(ii)** Equation of bisector of the angle containing origin : First make both the constant terms positive. Then the positive sign in $\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$ gives the bisector of the angle wich contains the origin.

**(iii)** Bisector of acute/obtuse angle : First make both the constant terms positive. Then

$a_1a_2 + b_1b_2 + c_1c_2 > 0$

$\Rightarrow$ origin lies on obtuse angle

$a_1a_2 + b_1b_2 + c_1c_2 < 0$

$\Rightarrow$ origin lies in acute angle

## 20. FAMILY OF PLANES

**(i)** Any plane passing through the line of intersection of non–parallel planes or equation of the plane through the given line in non symmetrical form.

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is

$a_1x + b_1y + c_1z + d_1 + \lambda\,(a_2x + b_2y + c_2z + d_2) = 0$

**(ii)** The equation of plane passing through the intersection of the planes $\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}\cdot\vec{n}_2 = d_2$ is $\vec{r}\cdot(\vec{n}_1 + \lambda\vec{n}_2) = d_1 + \lambda d_2$ where $\lambda$ is arbitrary scalar

**(iii)** **Plane through a given line :** Equation of any plane through the line in symmetrical form.

$\dfrac{x-x_1}{\ell} = \dfrac{y-y_1}{m} = \dfrac{z-z_1}{n}$ is $A\,(x - x_1) + B\,(y - y_1) + C\,(z - z_1) = 0$ where $Al + Bm + Cn = 0$

## 21. AREA OF A TRIANGLE

Let $A\,(x_1, y_1, z_1)$, $B\,(x_2, y_2, z_2)$, $C\,(x_3, y_3, z_3)$ be the vertices of a triangle, then $\Delta = \sqrt{(\Delta_x^2 + \Delta_y^2 + \Delta_z^2)}$

where $\Delta_x = \dfrac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}$, $\Delta_y = \dfrac{1}{2}\begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}$ and

$$\Delta_z = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

**Vector Method –** From two vector $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Then area is given by

$$\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix}$$

## 22. VOLUME OF A TETRAHEDRON

Volume of a tetrahedron with vertices $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ is given by

$$V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

## 23. EQUATION OF A LINE

**(i)** A straight line in space is characterised by the intersection of two planes which are not parallel and therefore, the equation of a straight line is a solution of the system constituted by the equations of the two planes, $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$. This form is also known as non–symmetrical form.

**(ii)** The equation of a line passing through the point $(x_1, y_1, z_1)$ and having direction ratios a, b, c is

$\dfrac{x-x_1}{a} = \dfrac{y-y_1}{b} = \dfrac{z-z_1}{c} = r$. This form is called symmetric form. A general point on the line is given by $(x_1 + ar, y_1 + br, z_1 + cr)$.

**(iii)** Vector equation : Vector equation of a straight line passing through a fixed point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\,\vec{b}$ where $\lambda$ is a scalar.

**(iv)** The equation of the line passing through the points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

**(v)** Vector equation of a straight line passing through two points with position vectors $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$.

**(vi)** Reduction of cartesion form of equation of a line to vector form and vice versa

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

$$\Leftrightarrow \vec{r} = (x_1\hat{i} + y\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k}).$$

*Note...*

Straight lines parallel to co–ordinate axes :

| | Straight lines | Equation |
|---|---|---|
| (i) | Through origin | $y = mx, z = nx$ |
| (ii) | x–axis | $y = 0, z = 0$ |
| (iii) | y–axis | $x = 0, z = 0$ |
| (iv) | z–axis | $x = 0, y = 0$ |
| (v) | Parallel to x–axis | $y = p, z = q$ |
| (vi) | Parallel to y–axis | $x = h, z = q$ |
| (vii) | Parallel to z–axis | $x = h, y = p$ |

## 24. ANGLE BETWEEN A PLANE AND A LIN

**(i)** If θ is the angle between line $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and the plane $ax + by + cz + d = 0$, then

$$\sin\theta = \left[\frac{a\ell + bm + cn}{\sqrt{(a^2+b^2+c^2)}\sqrt{\ell^2+m^2+n^2}}\right]$$

**(ii)** Vector form : If θ is the angle between a line $\vec{r} = (\vec{a} + \lambda\vec{b})$ and $\vec{r}\cdot\vec{n} = d$ then $\sin\theta = \left[\frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right]$.

**(iii)** Condition for perpendicularity $\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$, $\vec{b}\times\vec{n} = 0$

**(iv)** Condition for parallel $al + bm + cn = 0$, $\vec{b}\cdot\vec{n} = 0$

## 25. CONDITION FOR A LINE TO LIE IN A PLANI

**(i)** Cartesian form : Line $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ would lie in a plane

$ax + by + cz + d = 0$, if $ax_1 + by_1 + cz_1 + d = 0$ and

$al + bm + cn = 0$.

**(ii)** Vector form : Line $\vec{r} = \vec{a} + \lambda\vec{b}$ would lie in the plane $\vec{r}\cdot\vec{n} = d$ if $\vec{b}\cdot\vec{n} = 0$ and $\vec{a}\cdot\vec{n} = d$

## 26. COPLANER LINES

**(i)** If the given lines are $\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ and $\frac{x-\alpha'}{\ell'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'}$, then condition for intersection/ coplanarity is $\begin{vmatrix} \alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} = 0$ and plane containing the above two lines is $\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} = 0$

**(ii)** Condition of coplanarity if both the lines are in general assymetric form :-

$ax + by + cz + d = 0 = a'x + b'y + c'z + d'$ and

$\alpha x + \beta y + \gamma z + \delta = 0 = \alpha' x + \beta' y + \gamma' z + \delta'$

They are coplanar if $\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \delta' \end{vmatrix} = 0$

## 27. SKEW LINES

**(i)** The straight lines which are not parallel and non–coplanar i.e. non–intersecting are called skew lines.

If $\Delta = \begin{vmatrix} \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} \neq 0$, then lines are skew.

**(ii)** Vector Form : For lines $\vec{a}_1 + \lambda\vec{b}_1$ and $\vec{a}_2 + \lambda\vec{b}_2$ to be skew

$(\vec{b}_1\times\vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1) \neq 0$ or $[\vec{b}_1\ \vec{b}_2\ (\vec{a}_2 - \vec{a}_1)] \neq 0$.

**(iii)** Shortest distance between the two parallel lines

$\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}$ is $d = \left|\frac{(\vec{a}_2 - \vec{a}_1)\times\vec{b}}{|\vec{b}|}\right|$.

## 28. COPLANARITY OF FOUR POINTS

The points $A(x_1\ y_1\ z_1)$, $B(x_2\ y_2\ z_2)$ $C(x_3\ y_3\ z_3)$ and $D(x_4\ y_4\ z_4)$ are coplaner then

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \\ x_4-x_1 & y_4-y_1 & z_4-z_1 \end{vmatrix} = 0$$

very similar in vector method the points A $(\vec{r}_1)$, $B(\vec{r}_2)$, $C(\vec{r}_3)$ and $D(\vec{r}_4)$ are coplanar if $= \quad 4\ \hat{i} \ + \ 2\ \hat{k} \quad = 0$

## 29. SIDES OF A PLANE

A plane divides the three dimensional space in two equal parts. Two points A $(x_1\ y_1\ z_1)$ and B $(x_2\ y_2\ z_2)$ are on the same side of the plane $ax + by + cz + d = 0$ if $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ and both positive or both negative and are opposite side of plane if both of these values are in opposite sign.

## 30. LINE PASSING THROUGH THE GIVEN POINT $(x_1\ y_1\ z_1)$ AND INTERSECTING BOTH THE LINES $(P_1 = 0, P_2 = 0)$ AND $(P_3 = 0, P_4 = 0)$

Get a plane through $(x_1, y_1, z_1)$ and containing the line $(P_1 = 0, P_2 = 0)$ as $P_5 = 0$

Also get a plane through $(x_1, y_1, z_1)$ and containing the line $P_3 = 0, P_4 = 0$ as $P_6 = 0$

equation of the required line is $(P_5 = 0, P_6 = 0)$

## 31. TO FIND IMAGE OF A POINT W.R.T. A LINE

Let $L \equiv \dfrac{x-x_2}{a} = \dfrac{y-y_2}{b} = \dfrac{z-z_2}{c}$ is a given line

Let $(x', y', z')$ is the image of the point P $(x_1, y_1, z_1)$ with respect to the line L.

Then

**(i)** $a(x_1 - x') + b(y_1 - y') + c(z_1 - z') = 0$

**(ii)** $$\frac{\frac{x_1+x'}{2}-x_2}{a} = \frac{\frac{y_1+y'}{2}-y_2}{b} = \frac{\frac{z_1+z'}{2}-z_2}{c} = \lambda$$

from (ii) get the value of $x', y', z'$ in terms of $\lambda$ as

$x' = 2a\lambda + 2x_2 - x_1$, $y' = 2b\alpha - 2y_2 - y_1$,

$z' = 2c\lambda + 2z_2 - z_1$

now put the values of $x', y', z'$ in (i) get $\lambda$ and resubstitute the value of $\lambda$ to get $(x'\ y'\ z')$.

## 32. TO FIND IMAGE OF A POINT W.R.T. A PLANE

Let P $(x_1, y_1, z_1)$ is a given point and $ax + by + cz + d = 0$ is given plane Let $(x', y', z')$ is the image point.

then

**(i)** $x' - x_1 = \lambda a,\ y' - y_1 = \lambda b,\ z' - z_1 = \lambda c$

$\Rightarrow \quad x' = \lambda a + x,\ y' = \lambda b + y_1,\ z' = \lambda c + z_1$

**(ii)** $$a\left(\frac{x'+x_1}{2}\right) + b\left(\frac{y'+y_1}{2}\right) + c\left(\frac{z'+z_1}{2}\right) + d = 0$$

from (i) put the values of $x', y', z'$ in (ii) and get the values of $\lambda$ and resubstitute in (i) to get $(x'\ y'\ z')$.