Exercise 2.1

Question 1:

Find the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$

Answer

Let $\sin^{-1}\left(-\frac{1}{2}\right) = y.$ Then sin y = $-\frac{1}{2} = -\sin\left(\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{6}\right).$

We know that the range of the principal value branch of $\sin^{-1}$ is

$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}.$

Therefore, the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$ is $-\frac{\pi}{6}.$

Question 2:

Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Answer

Let $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = y.$ Then, $\cos y = \frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right).$

We know that the range of the principal value branch of $\cos^{-1}$ is

$[0, \pi]$ and $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.

Therefore, the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is $\frac{\pi}{6}$.

Question 3:

Find the principal value of $\operatorname{cosec}^{-1}(2)$

Answer

Let $\operatorname{cosec}^{-1}(2) = y$. Then, $\operatorname{cosec} y = 2 = \operatorname{cosec}\left(\frac{\pi}{6}\right)$.

We know that the range of the principal value branch of $\operatorname{cosec}^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

Therefore, the principal value of $\operatorname{cosec}^{-1}(2)$ is $\frac{\pi}{6}$.

Question 4:

Find the principal value of $\tan^{-1}\left(-\sqrt{3}\right)$

Answer

Let $\tan^{-1}\left(-\sqrt{3}\right) = y$. Then, $\tan y = -\sqrt{3} = -\tan\frac{\pi}{3} = \tan\left(-\frac{\pi}{3}\right)$.

We know that the range of the principal value branch of $\tan^{-1}$ is

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan\left(-\frac{\pi}{3}\right)$ is $-\sqrt{3}$.

Therefore, the principal value of $\tan^{-1}\left(\sqrt{3}\right)$ is $-\frac{\pi}{3}$.

Question 5:

Find the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$

Answer

Let $\cos^{-1}\left(-\frac{1}{2}\right) = y$. Then, $\cos y = -\frac{1}{2} = -\cos\left(\frac{\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right)$.

We know that the range of the principal value branch of $\cos^{-1}$ is

$[0, \pi]$ and $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$.

Therefore, the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is $\frac{2\pi}{3}$.

Question 6:

Find the principal value of $\tan^{-1}(-1)$

Answer

Let $\tan^{-1}(-1) = y$. Then, $\tan y = -1 = -\tan\left(\frac{\pi}{4}\right) = \tan\left(-\frac{\pi}{4}\right).$

We know that the range of the principal value branch of $\tan^{-1}$ is

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan\left(-\frac{\pi}{4}\right) = -1.$

Therefore, the principal value of $\tan^{-1}(-1)$ is $-\frac{\pi}{4}.$

Question 7:

Find the principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

Answer

Let $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = y$. Then, $\sec y = \frac{2}{\sqrt{3}} = \sec\left(\frac{\pi}{6}\right).$

We know that the range of the principal value branch of $\sec^{-1}$ is

$[0,\pi] - \left\{\frac{\pi}{2}\right\}$ and $\sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}.$

Therefore, the principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ is $\frac{\pi}{6}.$

Question 8:

Find the principal value of $\cot^{-1}\left(\sqrt{3}\right)$

Answer

Let $\cot^{-1}\left(\sqrt{3}\right) = y$. Then, $\cot y = \sqrt{3} = \cot\left(\frac{\pi}{6}\right).$

We know that the range of the principal value branch of $\cot^{-1}$ is $(0,\pi)$ and

$$\cot\left(\frac{\pi}{6}\right)=\sqrt{3}.$$

Therefore, the principal value of $\cot^{-1}\left(\sqrt{3}\right)$ is $\frac{\pi}{6}$.

Question 9:

Find the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Answer

$$\text{Let } \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)=y. \text{ Then, } \cos y=-\frac{1}{\sqrt{2}}=-\cos\left(\frac{\pi}{4}\right)=\cos\left(\pi-\frac{\pi}{4}\right)=\cos\left(\frac{3\pi}{4}\right).$$

We know that the range of the principal value branch of $\cos^{-1}$ is $[0,\pi]$ and

$$\cos\left(\frac{3\pi}{4}\right)=-\frac{1}{\sqrt{2}}.$$

Therefore, the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ is $\frac{3\pi}{4}$.

Question 10:

Find the principal value of $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$

Answer

$$\text{Let } \operatorname{cosec}^{-1}\left(-\sqrt{2}\right)=y. \text{ Then, } \operatorname{cosec} y=-\sqrt{2}=-\operatorname{cosec}\left(\frac{\pi}{4}\right)=\operatorname{cosec}\left(-\frac{\pi}{4}\right).$$

We know that the range of the principal value branch of $\operatorname{cosec}^{-1}$ is

$$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\} \text{ and } \operatorname{cosec}\left(-\frac{\pi}{4}\right)=-\sqrt{2}.$$

Therefore, the principal value of $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$ is $-\frac{\pi}{4}$.

Question 11:

Find the value of $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$

Answer

Let $\tan^{-1}(1)=x$. Then, $\tan x=1=\tan\frac{\pi}{4}$.

$\therefore \tan^{-1}(1)=\frac{\pi}{4}$

Let $\cos^{-1}\left(-\frac{1}{2}\right)=y$. Then, $\cos y=-\frac{1}{2}=-\cos\left(\frac{\pi}{3}\right)=\cos\left(\pi-\frac{\pi}{3}\right)=\cos\left(\frac{2\pi}{3}\right)$.

$\therefore \cos^{-1}\left(-\frac{1}{2}\right)=\frac{2\pi}{3}$

Let $\sin^{-1}\left(-\frac{1}{2}\right)=z$. Then, $\sin z=-\frac{1}{2}=-\sin\left(\frac{\pi}{6}\right)=\sin\left(-\frac{\pi}{6}\right)$.

$\therefore \sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$

$\therefore \tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$

$=\frac{\pi}{4}+\frac{2\pi}{3}-\frac{\pi}{6}$

$=\frac{3\pi+8\pi-2\pi}{12}=\frac{9\pi}{12}=\frac{3\pi}{4}$

Question 12:

Find the value of $\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$

Answer

Let $\cos^{-1}\left(\frac{1}{2}\right) = x$. Then, $\cos x = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right)$.

$\therefore \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

Let $\sin^{-1}\left(\frac{1}{2}\right) = y$. Then, $\sin y = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right)$.

$\therefore \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$\therefore \cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} + \frac{2\pi}{6} = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$

Question 13:

Find the value of if $\sin^{-1} x = y$, then

(A) $0 \le y \le \pi$ **(B)** $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(C) $0 < y < \pi$ **(D)** $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Answer

It is given that $\sin^{-1} x = y$.

We know that the range of the principal value branch of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Therefore, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Question 14:

Find the value of $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$ is equal to

(A) π (**B)** $-\frac{\pi}{3}$ (**C)** $\frac{\pi}{3}$ (**D)** $\frac{2\pi}{3}$

Answer

Let $\tan^{-1}\sqrt{3}=x$. Then, $\tan x=\sqrt{3}=\tan\frac{\pi}{3}$.

We know that the range of the principal value branch of $\tan^{-1}$ is $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$.

$\therefore \tan^{-1}\sqrt{3}=\frac{\pi}{3}$

Let $\sec^{-1}(-2)=y$. Then, $\sec y=-2=-\sec\left(\frac{\pi}{3}\right)=\sec\left(\pi-\frac{\pi}{3}\right)=\sec\frac{2\pi}{3}$.

We know that the range of the principal value branch of $\sec^{-1}$ is $[0,\pi]-\left\{\frac{\pi}{2}\right\}$.

$\therefore \sec^{-1}(-2)=\frac{2\pi}{3}$

Hence, $\tan^{-1}\left(\sqrt{3}\right)-\sec^{-1}(-2)=\frac{\pi}{3}-\frac{2\pi}{3}=-\frac{\pi}{3}$

Exercise 2.2

Question 1:

Prove $3\sin^{-1} x = \sin^{-1}\left(3x - 4x^3\right),\ x \in \left[-\frac{1}{2},\ \frac{1}{2}\right]$

Answer

To prove: $3\sin^{-1} x = \sin^{-1}\left(3x - 4x^3\right),\ x \in \left[-\frac{1}{2},\ \frac{1}{2}\right]$

Let $x = \sin\theta$. Then, $\sin^{-1} x = \theta.$

We have,

R.H.S. = $\sin^{-1}\left(3x - 4x^3\right) = \sin^{-1}\left(3\sin\theta - 4\sin^3\theta\right)$

$= \sin^{-1}\left(\sin 3\theta\right)$

$= 3\theta$

$= 3\sin^{-1} x$

= L.H.S.

Question 2:

Prove $3\cos^{-1} x = \cos^{-1}\left(4x^3 - 3x\right),\ x \in \left[\frac{1}{2},\ 1\right]$

Answer

To prove: $3\cos^{-1} x = \cos^{-1}\left(4x^3 - 3x\right),\ x \in \left[\frac{1}{2},\ 1\right]$

Let $x = \cos\theta$. Then, $\cos^{-1} x = \theta$.

We have,

$$\text{R.H.S.} = \cos^{-1}\left(4x^3 - 3x\right)$$
$$= \cos^{-1}\left(4\cos^3\theta - 3\cos\theta\right)$$
$$= \cos^{-1}\left(\cos 3\theta\right)$$
$$= 3\theta$$
$$= 3\cos^{-1}x$$
$$= \text{L.H.S.}$$

Question 3:

Prove $\tan^{-1}\dfrac{2}{11} + \tan^{-1}\dfrac{7}{24} = \tan^{-1}\dfrac{1}{2}$

Answer

To prove: $\tan^{-1}\dfrac{2}{11} + \tan^{-1}\dfrac{7}{24} = \tan^{-1}\dfrac{1}{2}$

$$\text{L.H.S.} = \tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24}$$
$$= \tan^{-1}\frac{\frac{2}{11} + \frac{7}{24}}{1 - \frac{2}{11}\cdot\frac{7}{24}} \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$
$$= \tan^{-1}\frac{\frac{48+77}{11\times 24}}{\frac{11\times 24 - 14}{11\times 24}}$$
$$= \tan^{-1}\frac{48+77}{264-14} = \tan^{-1}\frac{125}{250} = \tan^{-1}\frac{1}{2} = \text{R.H.S.}$$

Question 4:

Prove $2\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{7} = \tan^{-1}\dfrac{31}{17}$

Answer

To prove: $2\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{7} = \tan^{-1}\dfrac{31}{17}$

$$\text{L.H.S.} = 2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{2\cdot\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2} + \tan^{-1}\frac{1}{7} \qquad \left[2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \tan^{-1}\frac{1}{\left(\frac{3}{4}\right)} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\cdot\frac{1}{7}} \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\frac{\left(\frac{28+3}{21}\right)}{\left(\frac{21-4}{21}\right)}$$

$$= \tan^{-1}\frac{31}{17} = \text{R.H.S.}$$

Question 5:

Write the function in the simplest form:

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x},\ x \neq 0$$

Answer

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$$

Put $x=\tan\theta \Rightarrow \theta=\tan^{-1}x$

$$\therefore \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}=\tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$=\tan^{-1}\left(\tan\frac{\theta}{2}\right)=\frac{\theta}{2}=\frac{1}{2}\tan^{-1}x$$

Question 6:

Write the function in the simplest form:

$$\tan^{-1}\frac{1}{\sqrt{x^2-1}},\ |x|>1$$

Answer

$$\tan^{-1}\frac{1}{\sqrt{x^2-1}},\ |x|>1$$

Put x = cosec $\theta \Rightarrow \theta$ = $\text{cosec}^{-1}\, x$

$$\therefore \tan^{-1}\frac{1}{\sqrt{x^2-1}}=\tan^{-1}\frac{1}{\sqrt{\cos ec^2\theta-1}}$$

$$=\tan^{-1}\left(\frac{1}{\cot\theta}\right)=\tan^{-1}(\tan\theta)$$

$$=\theta=\text{cosec}^{-1}x=\frac{\pi}{2}-\sec^{-1}x \qquad \left[\text{cosec}^{-1}x+\sec^{-1}x=\frac{\pi}{2}\right]$$

Question 7:

Write the function in the simplest form:

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right), x<\pi$$

Answer

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right), x<\pi$$

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)=\tan^{-1}\left(\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right)$$

$$=\tan^{-1}\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)=\tan^{-1}\left(\tan\frac{x}{2}\right)$$

$$=\frac{x}{2}$$

Question 8:

Write the function in the simplest form:

$$\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right),\ 0<x<\pi$$

Answer

$$\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$$

$$=\tan^{-1}\left(\frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}\right)$$

$$=\tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right)$$

$$=\tan^{-1}(1)-\tan^{-1}(\tan x) \qquad \left[\tan^{-1}\frac{x-y}{1-xy}=\tan^{-1}x-\tan^{-1}y\right]$$

$$=\frac{\pi}{4}-x$$

Question 9:

Write the function in the simplest form:

$$\tan^{-1}\frac{x}{\sqrt{a^2-x^2}},\ |x|<a$$

Answer

$$\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$$

$$\text{Put } x=a\sin\theta \Rightarrow \frac{x}{a}=\sin\theta \Rightarrow \theta=\sin^{-1}\left(\frac{x}{a}\right)$$

$$\therefore \tan^{-1}\frac{x}{\sqrt{a^2-x^2}}=\tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{a\sin\theta}{a\sqrt{1-\sin^2\theta}}\right)=\tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right)$$

$$=\tan^{-1}(\tan\theta)=\theta=\sin^{-1}\frac{x}{a}$$

Question 10:

Write the function in the simplest form:

$$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right),\ a>0;\ \frac{-a}{\sqrt{3}}\le x\le\frac{a}{\sqrt{3}}$$

Answer

$$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)$$

Put $x=a\tan\theta \Rightarrow \frac{x}{a}=\tan\theta \Rightarrow \theta=\tan^{-1}\frac{x}{a}$

$$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)=\tan^{-1}\left(\frac{3a^2\cdot a\tan\theta-a^3\tan^3\theta}{a^3-3a\cdot a^2\tan^2\theta}\right)$$

$$=\tan^{-1}\left(\frac{3a^3\tan\theta-a^3\tan^3\theta}{a^3-3a^3\tan^2\theta}\right)$$

$$=\tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$$

$$=\tan^{-1}(\tan 3\theta)$$

$$=3\theta$$

$$=3\tan^{-1}\frac{x}{a}$$

Question 11:

Find the value of $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$

Answer

Let $\sin^{-1}\frac{1}{2}=x$. Then, $\sin x=\frac{1}{2}=\sin\left(\frac{\pi}{6}\right)$.

$$\therefore \sin^{-1}\frac{1}{2}=\frac{\pi}{6}$$

$$\therefore \tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]=\tan^{-1}\left[2\cos\left(2\times\frac{\pi}{6}\right)\right]$$

$$=\tan^{-1}\left[2\cos\frac{\pi}{3}\right]=\tan^{-1}\left[2\times\frac{1}{2}\right]$$

$$=\tan^{-1}1=\frac{\pi}{4}$$

Question 12:

Find the value of $\cot\left(\tan^{-1} a + \cot^{-1} a\right)$

Answer

$$\cot\left(\tan^{-1} a + \cot^{-1} a\right)$$
$$= \cot\left(\frac{\pi}{2}\right) \qquad \left[\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\right]$$
$$= 0$$

Question 13:

Find the value of $\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right],\ |x|<1,\ y>0 \text{ and } xy<1$

Answer

Let $x = \tan\theta$. Then, $\theta = \tan^{-1} x$.

$$\therefore \sin^{-1}\frac{2x}{1+x^2} = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1} x$$

Let $y = \tan\Phi$. Then, $\Phi = \tan^{-1} y$.

$$\therefore \cos^{-1}\frac{1-y^2}{1+y^2} = \cos^{-1}\left(\frac{1-\tan^2\phi}{1+\tan^2\phi}\right) = \cos^{-1}(\cos 2\phi) = 2\phi = 2\tan^{-1} y$$

$$\therefore \tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right]$$
$$= \tan\frac{1}{2}\left[2\tan^{-1} x + 2\tan^{-1} y\right]$$
$$= \tan\left[\tan^{-1} x + \tan^{-1} y\right]$$
$$= \tan\left[\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$
$$= \frac{x+y}{1-xy}$$

Question 14:

If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, then find the value of *x*.

Answer

$$\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$$

$$\Rightarrow \sin\left(\sin^{-1}\frac{1}{5}\right)\cos\left(\cos^{-1}x\right)+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1$$

$$\left[\sin(A+B)=\sin A\cos B+\cos A\sin B\right]$$

$$\Rightarrow \frac{1}{5}\times x+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1$$

$$\Rightarrow \frac{x}{5}+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1 \qquad \ldots(1)$$

Now, let $\sin^{-1}\frac{1}{5}=y$.

Then, $\sin y=\frac{1}{5}\Rightarrow \cos y=\sqrt{1-\left(\frac{1}{5}\right)^2}=\frac{2\sqrt{6}}{5}\Rightarrow y=\cos^{-1}\left(\frac{2\sqrt{6}}{5}\right)$.

$$\therefore \sin^{-1}\frac{1}{5}=\cos^{-1}\left(\frac{2\sqrt{6}}{5}\right) \qquad \ldots(2)$$

Let $\cos^{-1}x=z$.

Then, $\cos z=x\Rightarrow \sin z=\sqrt{1-x^2}\Rightarrow z=\sin^{-1}\left(\sqrt{1-x^2}\right)$.

$$\therefore \cos^{-1}x=\sin^{-1}\left(\sqrt{1-x^2}\right) \qquad \ldots(3)$$

From (1), (2), and (3) we have:

$$\frac{x}{5}+\cos\left(\cos^{-1}\frac{2\sqrt{6}}{5}\right)\cdot\sin\left(\sin^{-1}\sqrt{1-x^2}\right)=1$$

$$\Rightarrow \frac{x}{5}+\frac{2\sqrt{6}}{5}\cdot\sqrt{1-x^2}=1$$

$$\Rightarrow x+2\sqrt{6}\sqrt{1-x^2}=5$$

$$\Rightarrow 2\sqrt{6}\sqrt{1-x^2}=5-x$$

On squaring both sides, we get:

$$(4)(6)\left(1-x^2\right)=25+x^2-10x$$

$$\Rightarrow 24-24x^2=25+x^2-10x$$

$$\Rightarrow 25x^2-10x+1=0$$

$$\Rightarrow (5x-1)^2=0$$

$$\Rightarrow (5x-1)=0$$

$$\Rightarrow x=\frac{1}{5}$$

Hence, the value of *x* is $\frac{1}{5}$.

Question 15:

If $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1\pi}{x+2}=\frac{}{4}$, then find the value of *x*.

Answer

$$\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right]=\frac{\pi}{4} \qquad \left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]$$

$$\Rightarrow \tan^{-1}\left[\frac{(x-1)(x+2)+(x+1)(x-2)}{(x+2)(x-2)-(x-1)(x+1)}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{2x^2-4}{-3}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan\left[\tan^{-1}\frac{4-2x^2}{3}\right]=\tan\frac{\pi}{4}$$

$$\Rightarrow \frac{4-2x^2}{3}=1$$

$$\Rightarrow 4-2x^2=3$$

$$\Rightarrow 2x^2=4-3=1$$

$$\Rightarrow x=\pm\frac{1}{\sqrt{2}}$$

Hence, the value of *x* is $\pm\frac{1}{\sqrt{2}}$.

Question 16:

Find the values of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

Answer

$$\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$$

We know that $\sin^{-1}(\sin x) = x$ if $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which is the principal value branch of $\sin^{-1}x$.

Here, $\frac{2\pi}{3} \notin \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Now, $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ can be written as:

$$\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left[\sin\left(\pi - \frac{2\pi}{3}\right)\right] = \sin^{-1}\left(\sin\frac{\pi}{3}\right) \text{ where } \frac{\pi}{3} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

$$\therefore \sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$$

Question 17:

Find the values of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

Answer

$$\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$$

We know that $\tan^{-1}(\tan x) = x$ if $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which is the principal value branch of $\tan^{-1}x$.

Here, $\frac{3\pi}{4} \notin \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

Now, $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ can be written as:

$$\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}\left[-\tan\left(\frac{-3\pi}{4}\right)\right] = \tan^{-1}\left[-\tan\left(\pi - \frac{\pi}{4}\right)\right]$$

$$= \tan^{-1}\left[-\tan\frac{\pi}{4}\right] = \tan^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right] \textit{ where } -\frac{\pi}{4} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

$$\therefore \tan^{-1}\left(\tan\frac{3\pi}{4}\right)=\tan^{-1}\left[\tan\left(\frac{-\pi}{4}\right)\right]=\frac{-\pi}{4}$$

Question 18:

Find the values of $\tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$

Answer

Let $\sin^{-1}\frac{3}{5}=x$. Then, $\sin x=\frac{3}{5}\Rightarrow \cos x=\sqrt{1-\sin^2 x}=\frac{4}{5}\Rightarrow \sec x=\frac{5}{4}.$

$$\therefore \tan x=\sqrt{\sec^2 x-1}=\sqrt{\frac{25}{16}-1}=\frac{3}{4}$$

$$\therefore x=\tan^{-1}\frac{3}{4}$$

$$\therefore \sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4} \qquad ...(i)$$

$$\text{Now, } \cot^{-1}\frac{3}{2}=\tan^{-1}\frac{2}{3} \qquad ...(ii) \qquad \left[\tan^{-1}\frac{1}{x}=\cot^{-1}x\right]$$

$$\text{Hence, } \tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$$

$$=\tan\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}\right) \qquad [\text{Using (i) and (ii)}]$$

$$=\tan\left(\tan^{-1}\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right) \qquad \left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]$$

$$=\tan\left(\tan^{-1}\frac{9+8}{12-6}\right)$$

$$=\tan\left(\tan^{-1}\frac{17}{6}\right)=\frac{17}{6}$$

Question 19:

Find the values of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ is equal to

(A) $\frac{7\pi}{6}$ **(B)** $\frac{5\pi}{6}$ **(C)** $\frac{\pi}{3}$ **(D)** $\frac{\pi}{6}$

Answer

We know that $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi]$, which is the principal value branch of $\cos^{-1}x$.

Here, $\frac{7\pi}{6} \notin x \in [0, \pi]$.

Now, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ can be written as:

$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\frac{-7\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi - \frac{7\pi}{6}\right)\right] \quad [\cos(2\pi + x) = \cos x]$$

$$= \cos^{-1}\left[\cos\frac{5\pi}{6}\right] \text{ where } \frac{5\pi}{6} \in [0, \pi]$$

$$\therefore \cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\frac{5\pi}{6}\right) = \frac{5\pi}{6}$$

The correct answer is B.

Question 20:

Find the values of $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$ is equal to

(A) $\frac{1}{2}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{4}$ **(D)** 1

Answer

Let $\sin^{-1}\left(\frac{-1}{2}\right) = x$. Then, $\sin x = \frac{-1}{2} = -\sin\frac{\pi}{6} = \sin\left(\frac{-\pi}{6}\right)$.

We know that the range of the principal value branch of $\sin^{-1}$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

$$\therefore \sin^{-1}\left(\frac{-1}{2}\right)=\frac{-\pi}{6}$$

$$\therefore \sin\left(\frac{\pi}{3}-\sin^{-1}\left(\frac{-1}{2}\right)\right)=\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right)=\sin\left(\frac{3\pi}{6}\right)=\sin\left(\frac{\pi}{2}\right)=1$$

The correct answer is D.

Miscellaneous Solutions

Question 1:

Find the value of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$

Answer

We know that $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi]$, which is the principal value branch of $\cos^{-1}x$.

Here, $\frac{13\pi}{6} \notin [0, \pi]$.

Now, $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ can be written as:

$$\cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi + \frac{\pi}{6}\right)\right] = \cos^{-1}\left[\cos\left(\frac{\pi}{6}\right)\right], \text{ where } \frac{\pi}{6} \in [0, \pi].$$

$$\therefore \cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\left[\cos\left(\frac{\pi}{6}\right)\right] = \frac{\pi}{6}$$

Question 2:

Find the value of $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

Answer

We know that $\tan^{-1}(\tan x) = x$ if $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which is the principal value branch of $\tan^{-1}x$.

Here, $\frac{7\pi}{6} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Now, $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$ can be written as:

$$\tan^{-1}\left(\tan\frac{7\pi}{6}\right)=\tan^{-1}\left[\tan\left(2\pi-\frac{5\pi}{6}\right)\right] \qquad \left[\tan(2\pi-x)=-\tan x\right]$$

$$=\tan^{-1}\left[-\tan\left(\frac{5\pi}{6}\right)\right]=\tan^{-1}\left[\tan\left(-\frac{5\pi}{6}\right)\right]=\tan^{-1}\left[\tan\left(\pi-\frac{5\pi}{6}\right)\right]$$

$$=\tan^{-1}\left[\tan\left(\frac{\pi}{6}\right)\right],\text{ where }\frac{\pi}{6}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

$$\therefore \tan^{-1}\left(\tan\frac{7\pi}{6}\right)=\tan^{-1}\left(\tan\frac{\pi}{6}\right)=\frac{\pi}{6}$$

Question 3:

Prove $2\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{24}{7}$

Answer

Let $\sin^{-1}\frac{3}{5}=x$. Then, $\sin x=\frac{3}{5}$.

$$\Rightarrow \cos x=\sqrt{1-\left(\frac{3}{5}\right)^2}=\frac{4}{5}$$

$$\therefore \tan x=\frac{3}{4}$$

$$\therefore x=\tan^{-1}\frac{3}{4}\Rightarrow \sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4}$$

Now, we have:

$$\text{L.H.S.} = 2\sin^{-1}\frac{3}{5} = 2\tan^{-1}\frac{3}{4}$$

$$= \tan^{-1}\left(\frac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2}\right) \qquad \left[2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \tan^{-1}\left(\frac{\frac{3}{2}}{\frac{16-9}{16}}\right) = \tan^{-1}\left(\frac{3}{2}\times\frac{16}{7}\right)$$

$$= \tan^{-1}\frac{24}{7} = \text{R.H.S.}$$

Question 4:

Prove $\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$

Answer

Let $\sin^{-1}\frac{8}{17} = x$. Then, $\sin x = \frac{8}{17} \Rightarrow \cos x = \sqrt{1-\left(\frac{8}{17}\right)^2} = \sqrt{\frac{225}{289}} = \frac{15}{17}$.

$$\therefore \tan x = \frac{8}{15} \Rightarrow x = \tan^{-1}\frac{8}{15}$$

$$\therefore \sin^{-1}\frac{8}{17} = \tan^{-1}\frac{8}{15} \qquad ...(1)$$

Now, let $\sin^{-1}\frac{3}{5} = y$. Then, $\sin y = \frac{3}{5} \Rightarrow \cos y = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}$.

$$\therefore \tan y = \frac{3}{4} \Rightarrow y = \tan^{-1}\frac{3}{4}$$

$$\therefore \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4} \qquad ...(2)$$

Now, we have:

$$\text{L.H.S.} = \sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5}$$

$$= \tan^{-1}\frac{8}{15} + \tan^{-1}\frac{3}{4} \qquad \left[\text{Using (1) and (2)}\right]$$

$$= \tan^{-1}\frac{\frac{8}{15}+\frac{3}{4}}{1-\frac{8}{15}\times\frac{3}{4}}$$

$$= \tan^{-1}\left(\frac{32+45}{60-24}\right) \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\frac{77}{36} = \text{R.H.S.}$$

Question 5:

Prove $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$

Answer

Let $\cos^{-1}\frac{4}{5}=x$. Then, $\cos x=\frac{4}{5}\Rightarrow \sin x=\sqrt{1-\left(\frac{4}{5}\right)^2}=\frac{3}{5}$.

$\therefore \tan x=\frac{3}{4}\Rightarrow x=\tan^{-1}\frac{3}{4}$

$\therefore \cos^{-1}\frac{4}{5}=\tan^{-1}\frac{3}{4} \qquad ...(1)$

Now, let $\cos^{-1}\frac{12}{13}=y$. Then, $\cos y=\frac{12}{13}\Rightarrow \sin y=\frac{5}{13}$.

$\therefore \tan y=\frac{5}{12}\Rightarrow y=\tan^{-1}\frac{5}{12}$

$\therefore \cos^{-1}\frac{12}{13}=\tan^{-1}\frac{5}{12} \qquad ...(2)$

Let $\cos^{-1}\frac{33}{65}=z$. Then, $\cos z=\frac{33}{65}\Rightarrow \sin z=\frac{56}{65}$.

$\therefore \tan z=\frac{56}{33}\Rightarrow z=\tan^{-1}\frac{56}{33}$

$\therefore \cos^{-1}\frac{33}{65}=\tan^{-1}\frac{56}{33} \qquad ...(3)$

Now, we will prove that:

$$\begin{aligned}
\text{L.H.S.} &= \cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13} \\
&= \tan^{-1}\frac{3}{4}+\tan^{-1}\frac{5}{12} && \left[\text{Using } (1) \text{ and } (2)\right] \\
&= \tan^{-1}\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}.\frac{5}{12}} && \left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right] \\
&= \tan^{-1}\frac{36+20}{48-15} \\
&= \tan^{-1}\frac{56}{33} \\
&= \tan^{-1}\frac{56}{33} && \left[\text{by } (3)\right] \\
&= \text{R.H.S.}
\end{aligned}$$

Question 6:

Prove $\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{56}{65}$

Answer

Let $\sin^{-1}\frac{3}{5}=x$. Then, $\sin x=\frac{3}{5}\Rightarrow \cos x=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{\frac{16}{25}}=\frac{4}{5}$.

$\therefore \tan x=\frac{3}{4}\Rightarrow x=\tan^{-1}\frac{3}{4}$

$\therefore \sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4}$ ...(1)

Now, let $\cos^{-1}\frac{12}{13}=y$. Then, $\cos y=\frac{12}{13}\Rightarrow \sin y=\frac{5}{13}$.

$\therefore \tan y=\frac{5}{12}\Rightarrow y=\tan^{-1}\frac{5}{12}$

$\therefore \cos^{-1}\frac{12}{13}=\tan^{-1}\frac{5}{12}$ ...(2)

Let $\sin^{-1}\frac{56}{65}=z$. Then, $\sin z=\frac{56}{65}\Rightarrow \cos z=\frac{33}{65}$.

$\therefore \tan z=\frac{56}{33}\Rightarrow z=\tan^{-1}\frac{56}{33}$

$\therefore \sin^{-1}\frac{56}{65}=\tan^{-1}\frac{56}{33}$ ...(3)

Now, we have:

$$\text{L.H.S.} = \cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5}$$

$$= \tan^{-1}\frac{5}{12} + \tan^{-1}\frac{3}{4} \qquad \left[\text{Using } (1) \text{ and } (2)\right]$$

$$= \tan^{-1}\frac{\frac{5}{12}+\frac{3}{4}}{1-\frac{5}{12}.\frac{3}{4}} \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\frac{20+36}{48-15}$$

$$= \tan^{-1}\frac{56}{33}$$

$$= \sin^{-1}\frac{56}{65} = \text{R.H.S.} \qquad \left[\text{Using } (3)\right]$$

Question 7:

Prove $\tan^{-1}\frac{63}{16} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$

Answer

$$\text{Let } \sin^{-1}\frac{5}{13} = x. \text{ Then, } \sin x = \frac{5}{13} \Rightarrow \cos x = \frac{12}{13}.$$

$$\therefore \tan x = \frac{5}{12} \Rightarrow x = \tan^{-1}\frac{5}{12}$$

$$\therefore \sin^{-1}\frac{5}{13} = \tan^{-1}\frac{5}{12} \qquad ...(1)$$

$$\text{Let } \cos^{-1}\frac{3}{5} = y. \text{ Then, } \cos y = \frac{3}{5} \Rightarrow \sin y = \frac{4}{5}.$$

$$\therefore \tan y = \frac{4}{3} \Rightarrow y = \tan^{-1}\frac{4}{3}$$

$$\therefore \cos^{-1}\frac{3}{5} = \tan^{-1}\frac{4}{3} \qquad ...(2)$$

Using (1) and (2), we have

$$\text{R.H.S.} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$$

$$= \tan^{-1}\frac{5}{12} + \tan^{-1}\frac{4}{3}$$

$$= \tan^{-1}\left(\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\times\frac{4}{3}}\right) \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\left(\frac{15+48}{36-20}\right)$$

$$= \tan^{-1}\frac{63}{16}$$

$$= \text{L.H.S.}$$

Question 8:

Prove $\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

Answer

$$\text{L.H.S.} = \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8}$$

$$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right) + \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right) \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\left(\frac{7+5}{35-1}\right) + \tan^{-1}\left(\frac{8+3}{24-1}\right)$$

$$= \tan^{-1}\frac{12}{34} + \tan^{-1}\frac{11}{23}$$

$$= \tan^{-1}\frac{6}{17} + \tan^{-1}\frac{11}{23}$$

$$= \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right)$$

$$= \tan^{-1}\left(\frac{138+187}{391-66}\right)$$

$$= \tan^{-1}\left(\frac{325}{325}\right) = \tan^{-1}1$$

$$= \frac{\pi}{4} = \text{R.H.S.}$$

Question 9:

Prove $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right),\ x \in [0,\ 1]$

Answer

Let $x = \tan^2\theta$. Then, $\sqrt{x} = \tan\theta \Rightarrow \theta = \tan^{-1}\sqrt{x}$.

$$\therefore \frac{1-x}{1+x} = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$$

Now, we have:

$$\text{R.H.S.} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\cos^{-1}(\cos 2\theta) = \frac{1}{2}\times 2\theta = \theta = \tan^{-1}\sqrt{x} = \text{L.H.S.}$$

Question 10:

Prove $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)=\frac{x}{2},\ x\in\left(0,\ \frac{\pi}{4}\right)$

Answer

Consider $\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$

$$=\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}\right)^2-\left(\sqrt{1-\sin x}\right)^2} \qquad \text{(by rationalizing)}$$

$$=\frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1+\sin x)(1-\sin x)}}{1+\sin x-1+\sin x}$$

$$=\frac{2\left(1+\sqrt{1-\sin^2 x}\right)}{2\sin x}=\frac{1+\cos x}{\sin x}=\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\cot\frac{x}{2}$$

$$\therefore \text{L.H.S.}=\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)=\cot^{-1}\left(\cot\frac{x}{2}\right)=\frac{x}{2}=\text{R.H.S.}$$

Question 11:

Prove $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x,\ -\frac{1}{\sqrt{2}}\le x\le 1$ [**Hint:** put $x = \cos 2\theta$]

Answer

Put $x=\cos 2\theta$ so that $\theta=\frac{1}{2}\cos^{-1}x$. Then, we have:

$$\text{L.H.S.}=\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$$

$$=\tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right)=\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$=\tan^{-1}1-\tan^{-1}(\tan\theta) \qquad \left[\tan^{-1}\left(\frac{x-y}{1+xy}\right)=\tan^{-1}x-\tan^{-1}y\right]$$

$$=\frac{\pi}{4}-\theta=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x=\text{R.H.S.}$$

Question 12:

Prove $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}=\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

Answer

$$\text{L.H.S.} = \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3}$$
$$= \frac{9}{4}\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{3}\right)$$
$$= \frac{9}{4}\left(\cos^{-1}\frac{1}{3}\right) \quad . \quad ....(1) \quad \left[\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$$

Now, let $\cos^{-1}\frac{1}{3} = x$. Then, $\cos x = \frac{1}{3} \Rightarrow \sin x = \sqrt{1-\left(\frac{1}{3}\right)^2} = \frac{2\sqrt{2}}{3}$.

$$\therefore x = \sin^{-1}\frac{2\sqrt{2}}{3} \Rightarrow \cos^{-1}\frac{1}{3} = \sin^{-1}\frac{2\sqrt{2}}{3}$$

$$\therefore \text{L.H.S.} = \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3} = \text{R.H.S.}$$

Question 13:

Solve $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

Answer

$$2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$$
$$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\operatorname{cosec} x) \qquad \left[2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$
$$\Rightarrow \frac{2\cos x}{1-\cos^2 x} = 2\operatorname{cosec} x$$
$$\Rightarrow \frac{2\cos x}{\sin^2 x} = \frac{2}{\sin x}$$
$$\Rightarrow \cos x = \sin x$$
$$\Rightarrow \tan x = 1$$
$$\therefore x = \frac{\pi}{4}$$

Question 14:

Solve $\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1}x, (x>0)$

Answer

$$\tan^{-1}\frac{1-x}{1+x}=\frac{1}{2}\tan^{-1}x$$

$$\Rightarrow \tan^{-1}1-\tan^{-1}x=\frac{1}{2}\tan^{-1}x \qquad \left[\tan^{-1}x-\tan^{-1}y=\tan^{-1}\frac{x-y}{1+xy}\right]$$

$$\Rightarrow \frac{\pi}{4}=\frac{3}{2}\tan^{-1}x$$

$$\Rightarrow \tan^{-1}x=\frac{\pi}{6}$$

$$\Rightarrow x=\tan\frac{\pi}{6}$$

$$\therefore x=\frac{1}{\sqrt{3}}$$

Question 15:

Solve $\sin\left(\tan^{-1}x\right),\ |x|<1$ is equal to

(A) $\frac{x}{\sqrt{1-x^2}}$ (**B**) $\frac{1}{\sqrt{1-x^2}}$ (**C**) $\frac{1}{\sqrt{1+x^2}}$ (**D**) $\frac{x}{\sqrt{1+x^2}}$

Answer

Let $\tan^{-1}x = y$. Then, $\tan y = x \Rightarrow \sin y=\frac{x}{\sqrt{1+x^2}}$.

$$\therefore y=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\Rightarrow \tan^{-1}x=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$$

$$\therefore \sin\left(\tan^{-1}x\right)=\sin\left(\sin^{-1}\frac{x}{\sqrt{1+x^2}}\right)=\frac{x}{\sqrt{1+x^2}}$$

The correct answer is D.

Question 16:

Solve $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$, then x is equal to

(A) $0, \frac{1}{2}$ (**B**) $1, \frac{1}{2}$ (**C**) 0 (**D**) $\frac{1}{2}$

Answer

$$\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$$

$$\Rightarrow -2\sin^{-1}x=\frac{\pi}{2}-\sin^{-1}(1-x)$$

$$\Rightarrow -2\sin^{-1}x=\cos^{-1}(1-x) \qquad ...(1)$$

$$\text{Let } \sin^{-1}x=\theta \Rightarrow \sin\theta = x \Rightarrow \cos\theta=\sqrt{1-x^2}.$$

$$\therefore \theta=\cos^{-1}\left(\sqrt{1-x^2}\right)$$

$$\therefore \sin^{-1}x=\cos^{-1}\left(\sqrt{1-x^2}\right)$$

Therefore, from equation (1), we have

$$-2\cos^{-1}\left(\sqrt{1-x^2}\right)=\cos^{-1}(1-x)$$

Put x = sin y. Then, we have:

$$-2\cos^{-1}\left(\sqrt{1-\sin^2 y}\right)=\cos^{-1}(1-\sin y)$$

$$\Rightarrow -2\cos^{-1}(\cos y)=\cos^{-1}(1-\sin y)$$

$$\Rightarrow -2y=\cos^{-1}(1-\sin y)$$

$$\Rightarrow 1-\sin y=\cos(-2y)=\cos 2y$$

$$\Rightarrow 1-\sin y=1-2\sin^2 y$$

$$\Rightarrow 2\sin^2 y-\sin y=0$$

$$\Rightarrow \sin y(2\sin y-1)=0$$

$$\Rightarrow \sin y=0 \text{ or } \frac{1}{2}$$

$$\therefore x=0 \text{ or } x=\frac{1}{2}$$

But, when $x=\frac{1}{2}$, it can be observed that:

$$\text{L.H.S.} = \sin^{-1}\left(1-\frac{1}{2}\right) - 2\sin^{-1}\frac{1}{2}$$

$$= \sin^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\frac{1}{2}$$

$$= -\sin^{-1}\frac{1}{2}$$

$$= -\frac{\pi}{6} \neq \frac{\pi}{2} \neq \text{R.H.S.}$$

$\therefore x = \frac{1}{2}$ is not the solution of the given equation.

Thus, $x = 0$.

Hence, the correct answer is **C**.

Question 17:

Solve $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\frac{x-y}{x+y}$ is equal to

(A) $\frac{\pi}{2}$ **(B).** $\frac{\pi}{3}$ **(C)** $\frac{\pi}{4}$ **(D)** $\frac{-3\pi}{4}$

Answer

$$\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\frac{x-y}{x+y}$$

$$= \tan^{-1}\left[\frac{\frac{x}{y} - \frac{x-y}{x+y}}{1+\left(\frac{x}{y}\right)\left(\frac{x-y}{x+y}\right)}\right] \qquad \left[\tan^{-1} y - \tan^{-1} y = \tan^{-1}\frac{x-y}{1+xy}\right]$$

$$= \tan^{-1}\left[\frac{\frac{x(x+y)-y(x-y)}{y(x+y)}}{\frac{y(x+y)+x(x-y)}{y(x+y)}}\right]$$

$$= \tan^{-1}\left(\frac{x^2+xy-xy+y^2}{xy+y^2+x^2-xy}\right)$$

$$= \tan^{-1}\left(\frac{x^2+y^2}{x^2+y^2}\right) = \tan^{-1} 1 = \frac{\pi}{4}$$

Hence, the correct answer is **C.**