7 INTEGRALS

KEY CONCEPTS INVOLVED

1. **Integration** – The process of finding the function f(x) whose differential coeffiicient w.r.t. 'x', denoted by F (x) is given, is called the integration of f(x) w.r.t. x and is written as $\int F(x)\,dx = f(x)$
Thus, integration is an inverse process of differentiation or integration is anti of differentiation.
The differential coefficient of a constant is zero. Thus if c is an arbitrary constant independent of x. then $\frac{d}{dx}[f(x)+c] = F(x)$ Thus $\int F(x)\,dx = f(x)+c$
The arbitrary constant c is called the constant of integration.

2. **Integration by Substitution**

(a) **To evaluate the integral $\int f(ax+b)\,dx$**

Put ax + b = t, so that adx = dt *i.e.,* $dx = \frac{1}{a}dt$

$$\int f(ax+b)\,dx = \int f(t)\cdot\frac{1}{a}dt = \frac{1}{a}F(t),\ \text{where} \int f(t)\,dt = F(t) = F(ax+b)$$

If a function is not in some suitable form to find the integration, then we transform it into some suitable form by changing the independent variable x to t by substituting x = g (t).

Consider $I = \int f(x)\,dx$

Put $x = g(t)$, so that $\frac{dx}{dt} = g'(t)$

We write $dx = g'(t)\,dt$

Thus $I = \int f(x)\cdot dx = \int f(g(t)\,g'(t)\,dt$

But it is very important to guess, what will be the useful substitution.

(b) $\int \frac{f'(x)}{f(x)}dx = \log f(x) + c$

(c) $\int [f(x)]^n f'(x)\,dx = f(x)^{n+1}/(n+1) + c$

(d) Some important substitutions

function	Substitutions
$\sqrt{a^2-x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$

3. **Trigonometrical transformations** – For the integration of the trigonometrical products such as $\sin^2 x$, $\cos^2 x$, $\sin^3 x$, $\cos^3 x$, sin ax cos bx etc.they are expressed as the sum or difference of the sines and cosines of multiples of angles.

4. Integration of Some Special Integrals –

(a) For $\int \frac{dx}{ax^2+bx+c}$, $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ and $\int \sqrt{ax^2+bx+c}\, dx$

$$ax^2+bx+c = a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right] = a\left[\left(x+\frac{b}{2a}\right)^2+\frac{c}{a}-\frac{b^2}{4a^2}\right] = a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right]$$

Put $x+\frac{b}{2a}=t$, $\therefore$ $dx = dt$, $\frac{4ac-b^2}{4a^2} = \pm k^2$, ax^2+bx+c changes to t^2+k^2, t^2-k^2 or k^2-t^2

(b) For $\int \frac{(px+q)\,dx}{ax^2+bx+c}$, $\int \frac{(px+q)\,dx}{\sqrt{ax^2+bx+c}}$, $\int (px+q)\sqrt{(ax^2+bx+c)}\,dx$

Put $px+q = A\frac{d}{dx}(ax^2+bx+c)+B$

Compare the two sides and find the value of A and B.

$$\text{Thus } \int \frac{px+q}{ax^2+bx+c}\,dx = \int \frac{A\frac{d}{dx}(ax^2+bx+c)+B}{(ax^2+bx+c)}$$

$$= A\int \frac{\frac{d}{dx}(ax^2+bx+c)}{(ax^2+bx+c)}\,dx + B\int \frac{dx}{(ax^2+bx+c)}$$

$$\text{Similarly} \int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx = A\int \frac{\frac{d}{dx}(ax^2+bx+c)}{\sqrt{ax^2+bx+c}}\,dx + B\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

same as do $\int (px+q)\sqrt{ax^2+bx+c}\,dx$.

(c) For $\int \frac{dx}{(x+k)\sqrt{ax^2+bx+c}}$ put $x+k=\frac{1}{t}$

(d) For $\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$, $\int \sqrt{\frac{x-\alpha}{\beta-x}}\,dx$

$\int \sqrt{(x-\alpha)(x-\beta)}\,dx$, Put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

(e) For $\int \frac{dx}{a+b\cos x}$, $\int \frac{dx}{a+b\sin x}$, $\int \frac{dx}{a+b\cos x+c\sin x}$

$\sin x = \left(2\tan\frac{x}{2}\right)\Big/\left(1+\tan^2 x/2\right)$, $\cos x = \left(1-\tan\frac{x}{2}\right)\Big/\left(1+\tan^2 x/2\right)$ then put $\tan x/2 = t$

(f) For $\int \frac{p\cos x+q\sin x}{a+b\cos x+b\sin x}\,dx$

Put p cos x + q sin x = A (a + b cos x + b sin x) + B differential of (a + b cos x + b sin x) + C

A, B and C can be calculated by equating the coefficients of cos x. sin x and the constant terms.

5. Integration by parts $\int u\cdot v\,dx = u\cdot\int v\,dx - \int\left[\frac{du}{dx}\cdot\int v\,dx\right]dx$

i.e., the integral of the product of two functions = (first function) × (Integral of the second function – Integral of {(dfferential of first function) x (Integral of second function)}

This formula is called integration by parts.

6. **Partial Integration** – To Evaluate $\int \frac{P(x)}{Q(x)}\, dx$

The rational functions which we shall consider here for integration purposes will be those whose denominators can be factorised into linear and quadratic factors.

If $\frac{P(x)}{Q(x)}$ is improper fraction, i.e., degree of numerator is equal or greater than the degree of denominator.

Then first we reduce in proper rational function as $\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$ where T (x) is a polynomial in x

and $\frac{P_1(x)}{Q(x)}$ is a proper rational function.

After this, the integration can be carried out easily using the already known methods. The following Table 7.1 indicates the types of simpler partial fractions that are to be associated with various kind of rational functions.

Table 7.1

S. No.	**Form of the rational function**	**Form of the partial fraction**
1.	$\frac{px - q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
2.	$\frac{px + q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-b)^2}$
3.	$\frac{px^2 + qx + r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
4.	$\frac{px^2 + qx + r}{(x-a)^2(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
5.	$\frac{px^2 + qx + r}{(x-a)(x^2 + bx + c)}$ Where $x^2 + bx + c$ can not be factorised further	$\frac{A}{x-a} + \frac{Bx + c}{x^2 + bx + c}$

In the above table, A, B and C are real numbers to be determined suitably.

7. **Definite Integral** – The definite integral of f(x) between the limits a to b i.e. in the interval [a,b] is denoted by $\int_a^b f(x)\, dx$ and is defined as follows. $\int_a^b f(x)\, dx = [F(x)]_a^b = F(b) - F(a)$ where $\int f(x)\, dx = F(x)$

8. **General Properties of Definite Integrals** –

Prop. I $\int_a^b f(x)\, dx = \int_a^b f(t)\, dt$

Prop. II $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

Prop. III $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$ where $a < c < b$

Prop. IV $\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$

In particualr $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Prop. V $\int_0^{2a} f(x)\,dx$

Prop. V $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f(x)$ is even function

$\int_{-a}^{a} f(x)\,dx = 0$, if $f(x)$ is odd function

Prop. VI $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

Prop. VII $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f(2a-x) = f(x)$

$\int_0^{2a} f(x)\,dx = 0$, if $f(2a-x) = -f(x)$

9. Definite Integral as the limit of a sum

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\,[f(a)+f(a+h)+f(a+2h)+\cdots+f<a+(n-1)h)]$$

or $$\int_a^b f(x)\,dx = \lim_{h\to 0} h\,[f(a+h)+f(a+2h)+f(a+3h)+\cdots+f(a+nh)$$

where, $h = \dfrac{b-a}{n}$

$\dfrac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f\{v(x)\}\dfrac{d}{dx}v(x) - f\{u(x)\}\dfrac{d}{dx}u(x)$ this rule is called leibnitz's is Rule.

CONNECTING CONCEPTS

1. Integration is an operation on function

2. $\int [k_1 f_1(x) + k_2 f_2(x) + \ldots\ldots\ldots\ldots + k_n f_n(x)]dx$

$= k_1\int f_1(x)\,dx + k_2\int f_2(x)\,dx + \ldots\ldots\ldots\ldots + k_n\int f_n(x)\,dx$

3. All functions are not integrable and the integral of a function is not unique.

4. If a polynomial function of a degree n is integrated we get a polynomial of degree n + 1

4. **Integration by using standard formulae** –

1. $\int k\,dx = kx + c$, k is constant
2. $\int kf(x)\,dx = k\int f(x)\,dx + c$
3. $\int (f_1(x) \pm f_2(x)]\,dx = \int f_1(x)dx \pm \int f_2(x)\,dx + c$
4. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c\ (n \neq -1)$
5. $\int \dfrac{1}{x}\,dx = \log_e|x| + c$
6. $\int a^x\,dx = \dfrac{a^x}{\log_e a} + c,\ a > 0$
7. $\int e^x\,dx = e^x + c$

5. $\int \sin x \, dx = -\cos x + c$

6. $\int \cos x \, dx = \sin x + c$

7. $\int \sec^2 x \, dx = \tan x + c$

8. $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$

9. $\int \sec x \tan x \, dx = \sec x + c$

10. $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$

11. $\int \tan x \, dx = \log |\sec x| + c = -\log |\cos x| + c$

12. $\int \cot x \, dx = \log |\sin x| + c$

13. $\int \sec x \, dx = \log |\sec x + \tan x| + c$

14. $\int \operatorname{cosec} x \, dx = \log |\operatorname{cosec} x - \cot x| + c$

15. $\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + c \quad \text{or} \quad -\cos^{-1} x + c$

16. $\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + c \quad \text{or} \quad -\cot^{-1} x + c$

17. $\int \frac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1} x + c \quad \text{or} \quad -\operatorname{cosec}^{-1} x + c$

18. $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

19. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log \left|\frac{x-a}{x+a}\right| + c, \; x > a$

20. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log \left|\frac{a+x}{a-x}\right| + c, \; x < a$

21. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$

22. $\int \frac{dx}{\sqrt{x^2+a^2}} = \log x + \sqrt{a^2+x^2} + c$

23. $\int \frac{dx}{\sqrt{x^2-a^2}} = \log x + \sqrt{x^2-a^2} + c$

24. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + c$

25. $\int \sqrt{a^2-x^2} \, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{1}{2} a^2 \sin^{-1}\left(\frac{x}{a}\right) + c$

26. $\int \sqrt{x^2 + a^2}\ dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{1}{2}a^2 \log\left|x + \sqrt{x^2 + a^2}\right| + c$

27. $\int \sqrt{x^2 - a^2}\ dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{1}{2}a^2 \log x + \sqrt{x^2 - a^2} + c$

28. $\int e^x\ [f(x) + f'(x)]\ dx = e^x f(x) + c$

29. **Use of Trigonometric Identities in Integration.**

(i) $\sin^2 x = \frac{1 - \cos 2x}{2}$, $\cos^2 x = \frac{1 + \cos 2x}{2}$

(ii) $\sin^3 x = \frac{3\sin x - \sin 3x}{4}$, $\cos^3 x = \frac{3\cos x + \cos 3x}{4}$

(iii) $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$
$2 \cos A \sin B = \sin (A + B) - \sin (A - B)$
$2 \cos A \cos B = \cos (A + B) + \cos (A - B)$
$2 \sin A \sin B = \cos (A - B) + \cos (A + B)$

(iv) $\sin x = 2 \sin\left(\frac{x}{2}\right) \cdot \cos\left(\frac{x}{2}\right)$

30.(i) $1 + 2 + 3 + \cdots\cdots + n = \frac{n\ (n+1)}{2}$

(ii) $1^2 + 2^2 + 3^2 + \cdots\cdots + n^2 = \frac{n\ (n+1)\ (2n-1)}{6}$

(iii) $1^3 + 2^3 + 3^2 + \cdots\cdots + n^3 = \left[\frac{n\ (n+1)}{2}\right]^2$

(iv) $a + (a + d) + (a + 2d) + \cdots\cdots + [a + (n - 1)\ d] = \frac{n}{2}\ [2a + (n - 1)\ d]$

(v) $a + ar + ar^2 + \cdots\cdots + ar^{n+1} = \frac{a(r^n - 1)}{r - 1}$

Class 12 Maths NCERT Solutions

NCERT Solutions	Important Questions	NCERT Exemplar
Chapter 1 Relations and Functions	**Relations and Functions**	Chapter 1 Relations and Functions
Chapter 2 Inverse Trigonometric Functions	Concept of Relations and Functions	Chapter 2 Inverse Trigonometric Functions
Chapter 3 Matrices	Binary Operations	Chapter 3 Matrices
Chapter 4 Determinants	Inverse Trigonometric Functions	Chapter 4 Determinants
Chapter 5 Continuity and Differentiability	**Matrices**	Chapter 5 Continuity and Differentiability
Chapter 6 Application of Derivatives	Matrix and Operations of Matrices	Chapter 6 Application of Derivatives
Chapter 7 Integrals Ex 7.1	Transpose of a Matrix and Symmetric Matrix	Chapter 7 Integrals
Integrals Class 12 Ex 7.2	Inverse of a Matrix by Elementary Operations	Chapter 8 Applications of Integrals
Integrals Class 12 Ex 7.3	**Determinants**	Chapter 9 Differential Equations
Integrals Class 12 Ex 7.4	Expansion of Determinants	Chapter 10 Vector Algebra
Integrals Class 12 Ex 7.5	Properties of Determinants	Chapter 11 Three Dimensional Geometry
Integrals Class 12 Ex 7.6	Inverse of a Matrix and Application of Determinants and Matrix	Chapter 12 Linear Programming
Integrals Class 12 Ex 7.7	**Continuity and Differentiability**	Chapter 13 Probability
Integrals Class 12 Ex 7.8	Continuity	
Integrals Class 12 Ex 7.9	Differentiability	
Integrals Class 12 Ex 7.10	**Application of Derivatives**	
Integrals Class 12 Ex 7.11	Rate Measure Approximations and Increasing-Decreasing Functions	
Integrals Class 12 Miscellaneous Exercise	Tangents and Normals	
Chapter 8 Application of Integrals	Maxima and Minima	
Chapter 9 Differential Equations	**Integrals**	
Chapter 10 Vector Algebra	Types of Integrals	
Chapter 11 Three Dimensional Geometry	**Differential Equation**	
Chapter 12 Linear Programming	Formation of Differential Equations	
Chapter 13 Probability Ex	Solution of Different Types of Differential	

13.1	Equations	
Probability Solutions Ex 13.2	**Vector Algebra**	
Probability Solutions Ex 13.3	Algebra of Vectors	
Probability Solutions Ex 13.4	Dot and Cross Products of Two Vectors	
Probability Solutions Ex 13.5	**Three Dimensional Geometry**	
	Direction Cosines and Lines	
	Plane	
	Linear Programming	
	Probability	
	Conditional Probability and Independent Events	
	Baye's Theorem and Probability Distribution	

RD Sharma Class 12 Solutions

Chapter 1: Relations	Chapter 12: Higher Order Derivatives	Chapter 23 Algebra of Vectors
Chapter 2: Functions	Chapter 13: Derivative as a Rate Measurer	Chapter 24: Scalar Or Dot Product
Chapter 3: Binary Operations	Chapter 14: Differentials, Errors and Approximations	Chapter 25: Vector or Cross Product
Chapter 4: Inverse Trigonometric Functions	Chapter 15: Mean Value Theorems	Chapter 26: Scalar Triple Product
Chapter 5: Algebra of Matrices	Chapter 16: Tangents and Normals	Chapter 27: Direction Cosines and Direction Ratios
Chapter 6: Determinants	Chapter 17: Increasing and Decreasing Functions	Chapter 28 Straight line in space
Chapter 7: Adjoint and Inverse of a Matrix	Chapter 18: Maxima and Minima	Chapter 29: The plane
Chapter 8: Solution of Simultaneous Linear Equations	Chapter 19: Indefinite Integrals	Chapter 30: Linear programming
Chapter 9: Continuity	Chapter 20: Definite Integrals	Chapter 31: Probability
Chapter 10: Differentiability	Chapter 21: Areas of Bounded Regions	Chapter 32: Mean and variance of a random variable
Chapter 11: Differentiation	Chapter 22: Differential Equations	Chapter 33: Binomial Distribution

JEE Main Maths Chapter wise Previous Year Questions

1. Relations, Functions and Reasoning
2. Complex Numbers
3. Quadratic Equations And Expressions
4. Matrices, Determinatnts and Solutions of Linear Equations
5. Permutations and Combinations
6. Binomial Theorem and Mathematical Induction
7. Sequences and Series
8. Limits,Continuity,Differentiability and Differentiation
9. Applications of Derivatives
10. Indefinite and Definite Integrals
11. Differential Equations and Areas
12. Cartesian System and Straight Lines
13. Circles and System of Circles
14. Conic Sections
15. Three Dimensional Geometry
16. Vectors
17. Statistics and Probability
18. Trignometry
19. Miscellaneous

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- NCERT Solutions for Class 12 Maths
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- NCERT Solutions for Class 12 English Vistas
- NCERT Solutions for Class 12 English Flamingo
- NCERT Solutions for Class 12 Hindi
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