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MATHEMATICS

Examination Papers
2008–2014

CONTENT

EXAMINATION PAPERS – 2008

MATHEMATICS CBSE (Delhi)
CLASS – XII

Time allowed: 3 hours **Maximum marks: 100**

General Instructions:

1. *All questions are compulsory.*
2. *The question paper consists of 29 questions divided into three sections-A, B and C. Section A comprises of 10 questions of one mark each, Section B comprises of 12 questions of four marks each and Section C comprises of 7 questions of six marks each.*
3. *All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.*
4. *There is no overall choice. However, internal choice has been provided in 4 questions of four marks each and 2 questions of six marks each. You have to attempt only one of the alternatives in all such questions.*
5. *Use of calculators is not permitted.*

Set–I

SECTION–A

1. If $f(x) = x + 7$ and $g(x) = x - 7$, $x \in R$, find $(fog)(7)$
2. Evaluate : $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$
3. Find the value of x and y if : $2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$
4. Evaluate: $\begin{vmatrix} a + ib & c + id \\ -c + id & a - ib \end{vmatrix}$
5. Find the cofactor of a_{12} in the following: $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$
6. Evaluate: $\int \frac{x^2}{1 + x^3}\, dx$
7. Evaluate: $\int_0^1 \frac{dx}{1 + x^2}$
8. Find a unit vector in the direction of $\vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$

9. Find the angle between the vectors $\vec{a} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - \hat{k}$

10. For what value of λ are the vectors $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$ perpendicular to each other?

SECTION–B

11. (*i*) Is the binary operation defined on set N, given by $a * b = \dfrac{a+b}{2}$ for all $a, b \in N$, commutative?

(*ii*) Is the above binary operation associative?

12. Prove the following:

$$\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$$

13. Let $A = \begin{bmatrix} 3 & 2 & 5 \\ 4 & 1 & 3 \\ 0 & 6 & 7 \end{bmatrix}$.

Express A as sum of two matrices such that one is symmetric and the other is skew symmetric.

OR

If $A = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix}$, verify that $A^2 - 4A - 5I = 0$

14. For what value of k is the following function continuous at $x = 2$?

$$f(x) = \begin{cases} 2x + 1 & ; x < 2 \\ k & ; x = 2 \\ 3x - 1 & ; x > 2 \end{cases}$$

15. Differentiate the following with respect to x : $\tan^{-1}\left(\dfrac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right)$

16. Find the equation of tangent to the curve $x = \sin 3t$, $y = \cos 2t$ at $t = \dfrac{\pi}{4}$

17. Evaluate: $\displaystyle\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx$

18. Solve the following differential equation:

$$(x^2 - y^2)\, dx + 2xy\, dy = 0$$

given that $y = 1$ when $x = 1$

OR

Solve the following differential equation:

$$\frac{dy}{dx} = \frac{x\,(2y - x)}{x(2y + x)}, \quad \text{if } y = 1 \text{ when } x = 1$$

19. Solve the following differential equation : $\cos^2 x \dfrac{dy}{dx} + y = \tan x$

20. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$, find a vector $\vec{c}$ such that $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a} \cdot \vec{c} = 3$.

OR

If $\vec{a} + \vec{b} + \vec{c} = 0$ and $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 7$, show that the angle between $\vec{a}$ and $\vec{b}$ is $60°$.

21. Find the shortest distance between the following lines :

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \text{ and } \frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$$

OR

Find the point on the line $\dfrac{x+2}{3} = \dfrac{y+1}{2} = \dfrac{z-3}{2}$ at a distance $3\sqrt{2}$ from the point (1, 2, 3).

22. A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability distribution of number of successes.

SECTION–C

23. Using properties of determinants, prove the following :

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma)$$

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix} + \beta + \gamma) \ \alpha^2 \begin{vmatrix} \alpha & \beta & \\ & \gamma = (\alpha & \\ \beta^2 & \gamma^2 & \\ 1 & 1 & 1 \end{vmatrix}$$

24. Show that the rectangle of maximum area that can be inscribed in a circle is a square.

OR

Show that the height of the cylinder of maximum volume that can be inscribed in a cone of height h is $\dfrac{1}{3}h$.

25. Using integration find the area of the region bounded by the parabola $y^2 = 4x$ and the circle $4x^2 + 4y^2 = 9$.

26. Evaluate: $\displaystyle\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}\,dx$

27. Find the equation of the plane passing through the point $(-1, -1, 2)$ and perpendicular to each of the following planes:

$2x + 3y - 3z = 2$ and $5x - 4y + z = 6$

OR

Find the equation of the plane passing through the points (3, 4, 1) and (0, 1, 0) and parallel to the line $\dfrac{x+3}{2} = \dfrac{y-3}{7} = \dfrac{z-2}{5}$

28. A factory owner purchases two types of machines, A and B for his factory. The requirements and the limitations for the machines are as follows :

Machine	Area occupied	Labour force	Daily output (in units)
A	$1000\ m^2$	12 men	60
B	$1200\ m^2$	8 men	40

He has maximum area of $9000\ m^2$ available, and 72 skilled labourers who can operate both the machines. How many machines of each type should he buy to maximise the daily output?

29. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accident involving a scooter, a car and a truck are 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver.

Set–II

Only those questions, not included in Set I, are given

20. Solve for x : $\tan^{-1}(2x) + \tan^{-1}(3x) = \dfrac{\pi}{4}$.

21. Evaluate: $\displaystyle\int_0^{\pi} \frac{x \tan x}{\sec x \operatorname{cosec} x}\, dx.$

22. If $y = \sqrt{x^2+1} - \log\left(\dfrac{1}{x} + \sqrt{1+\dfrac{1}{x^2}}\right)$, find $\dfrac{dy}{dx}$.

23. Using properties of determinants, prove the following :

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3.$$

24. Evaluate: $\displaystyle\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx.$

25. Using integration, find the area of the region enclosed between the circles $x^2+y^2=4$ and $(x-2)^2+y^2=4$.

Set–III

Only those questions, not included in Set I and Set II, are given.

20. Solve for x : $\tan^{-1}\left(\dfrac{x-1}{x-2}\right) + \tan^{-1}\left(\dfrac{x+1}{x+2}\right) = \dfrac{\pi}{4}$

21. If $y = \cot^{-1}\left[\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$, find $\dfrac{dy}{dx}$

22. Evaluate: $\displaystyle\int_0^1 \cot^{-1}[1-x+x^2]\, dx$

23. Using properties of determinants, prove the following :

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$

24. Using integration, find the area lying above x-axis and included between the circle $x^2 + y^2 = 8x$ and the parabola $y^2 = 4x$.

25. Using properties of definite integrals, evaluate the following: $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\, dx$

SOLUTIONS

Set–I

SECTION–A

1. Given $f(x) = x + 7$ and $g(x) = x - 7,\ x \in R$

$$\text{fog}(x) = f(g(x)) = g(x) + 7 = (x - 7) + 7 = x$$

$\Rightarrow$ $(\text{fog})(7) = 7.$

2. $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right] = \sin\left[\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right] = \sin\frac{\pi}{2} = 1$

3. $2\begin{bmatrix}1 & 3\\ 0 & x\end{bmatrix} + \begin{bmatrix}y & 0\\ 1 & 2\end{bmatrix} = \begin{bmatrix}5 & 6\\ 1 & 8\end{bmatrix}$

$\Rightarrow$ $\begin{bmatrix}2 & 6\\ 0 & 2x\end{bmatrix} + \begin{bmatrix}y & 0\\ 1 & 2\end{bmatrix} = \begin{bmatrix}5 & 6\\ 1 & 8\end{bmatrix}$

$\Rightarrow$ $\begin{bmatrix}2+y & 6\\ 1 & 2x+2\end{bmatrix} = \begin{bmatrix}5 & 6\\ 1 & 8\end{bmatrix}$

Comparing both matrices

$2 + y = 5$ and $2x + 2 = 8$

$\Rightarrow$ $y = 3$ and $2x = 6$

$\Rightarrow$ $x = 3,\ y = 3.$

4. $\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}$

$= (a+ib)(a-ib) - (c+id)(-c+id)$

$= [a^2 - i^2b^2] - [i^2d^2 - c^2]$

$= (a^2 + b^2) - (-d^2 - c^2)$

$= a^2 + b^2 + c^2 + d^2$

5. Minor of a_{12} is $M_{12} = \begin{matrix} & 4 \\ 1 & \\ & -7 \end{matrix} = -42 - 4 = -46$

Cofactor $C_{12} = (-1)^{1+2} M_{12} = (-1)^3 (-46) = 46$

6. Let $I = \int \dfrac{x^2}{1+x^3}\, dx$

Putting $1 + x^3 = t$

$\Rightarrow \quad 3x^2 dx = dt$

or $\quad x^2 dx = \dfrac{dt}{3}$

$\therefore \qquad I = \dfrac{1}{3}\int \dfrac{dt}{t} = \dfrac{1}{3}\log|t| + C$

$\qquad\quad = \dfrac{1}{3}\log|1+x^3| + C$

7. $\displaystyle\int_0^1 \frac{dx}{1+x^2}$

$= \tan^{-1} x \Big|_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$

$= \dfrac{\pi}{4} - 0 = \dfrac{\pi}{4}.$

8. $\vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$

Unit vector in direction of $\vec{a} = \dfrac{\vec{a}}{|\vec{a}|}$

$= \dfrac{3\hat{i} - 2\hat{j} + 6\hat{k}}{\sqrt{3^2 + (-2)^2 + 6^2}} = \dfrac{1}{7}(3\hat{i} - 2\hat{j} + 6\hat{k})$

9. $\vec{a} = \hat{i} - \hat{j} + \hat{k} \qquad \Rightarrow \qquad |\vec{a}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$

$\vec{b} = \hat{i} + \hat{j} - \hat{k} \qquad \Rightarrow \qquad |\vec{b}| = \sqrt{(1)^2 + (1)^2 + (-1)^2} = \sqrt{3}$

$\vec{a}\,.\,\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

$\Rightarrow \sqrt{}\sqrt{} -1 - 1 = \quad 3. \;\; 3 \qquad -1 = 3\cos\theta$

$\cos\theta \qquad \Rightarrow \Rightarrow 1 \qquad \cos\theta = -\dfrac{}{3} \qquad \theta = \cos^{-1}\left(-\dfrac{1}{3}\right)$

$\rightarrow \qquad \rightarrow$

$\Rightarrow$

10. a and $\vec{b}$ are perpendicular if

$\vec{a}\,.\,b = 0$

$\Rightarrow \quad (2\hat{i} + \lambda\hat{j} + \hat{k})\,.\,(\hat{i} - 2\hat{j} + 3\hat{k}) = 0$

$\Rightarrow \quad 2 - 2\lambda + 3 = 0 \qquad \Rightarrow \qquad \lambda = \dfrac{5}{2}.$

SECTION–B

11. (*i*) Given N be the set

$$a * b = \frac{a+b}{2} \ \forall \ a, b \in N$$

To find * is commutative or not.

Now, $a * b = \dfrac{a+b}{2} = \dfrac{b+a}{2}$ $\quad \therefore$ (addition is commulative on N)

$$= b * a$$

So $\quad a * b = b * a$

$\therefore \quad$ * is commutative.

(*ii*) To find $a * (b * c) = (a * b) * c$ or not

Now $a * (b * c) = a * \left(\dfrac{b+c}{2}\right) = \dfrac{a + \left(\dfrac{b+c}{2}\right)}{2} = \dfrac{2a+b+c}{4}$ ...(*i*)

$$(a * b) * c = \left(\frac{a+b}{2}\right) * c = \frac{\frac{a+b}{2} + c}{2}$$

$$= \frac{a+b+2c}{4}$$ ...(*ii*)

From (*i*) and (*ii*)

$$(a * b) * c \neq a * (b * c)$$

Hence the operation is not associative.

12. L.H.S. $= \tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{1}{5} + \tan^{-1}\dfrac{1}{7} + \tan^{-1}\dfrac{1}{8}$

$$= \tan^{-1}\frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{3}\times\frac{1}{5}} + \tan^{-1}\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\times\frac{1}{8}}$$

$$= \tan^{-1}\frac{8}{14} + \tan^{-1}\frac{15}{55}$$

$$= \tan^{-1}\frac{4}{7} + \tan^{-1}\frac{3}{11} = \tan^{-1}\frac{\frac{4}{7}+\frac{3}{11}}{1-\frac{4}{7}\times\frac{3}{11}}$$

$$= \tan^{-1}\frac{65}{77-12} = \tan^{-1}\frac{65}{65} = \tan^{-1}1 = \frac{\pi}{4} \qquad = \text{R.H.S}$$

13. We know that any matrix can be expressed as the sum of symmetric and skew symmetric.

So, $A = \dfrac{1}{2}(A^T + A) + \dfrac{1}{2}(A - A^T)$

or $A = P + Q$ where P is symmetric matrix and Q skew symmetric matrix.

$$\therefore P = \frac{1}{2}(A + A^T) = \frac{1}{2}\left\{\begin{bmatrix} 3 & 2 & 5 \\ 4 & 1 & 3 \\ 0 & 6 & 7 \end{bmatrix} + \begin{bmatrix} 3 & 4 & 0 \\ 2 & 1 & 6 \\ 5 & 3 & 7 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 6 & 6 & 5 \\ 6 & 2 & 9 \\ 5 & 9 & 14 \end{bmatrix} = \begin{bmatrix} 3 & 3 & \frac{5}{2} \\ 3 & 1 & \frac{9}{2} \\ \frac{5}{2} & \frac{9}{2} & 7 \end{bmatrix}$$

$$Q = \frac{1}{2}(A - A^T)$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 3 & 2 & 5 \\ 4 & 1 & 3 \\ 0 & 6 & 7 \end{bmatrix} + \begin{bmatrix} -3 & -4 & 0 \\ -2 & -1 & -6 \\ -5 & -3 & -7 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & -2 & 5 \\ 2 & 0 & -3 \\ -5 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & \frac{5}{2} \\ 1 & 0 & -\frac{3}{2} \\ -\frac{5}{2} & \frac{3}{2} & 0 \end{bmatrix}$$

OR

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\therefore \quad A^2 = A \times A$$

$$\therefore \quad = \begin{bmatrix} 1\times1+2\times2+2\times2 & 1\times2+2\times1+2\times2 & 1\times2+2\times2+2\times1 \\ 2\times1+1\times2+2\times2 & 2\times2+1\times1+2\times2 & 2\times2+1\times2+2\times1 \\ 2\times1+2\times2+1\times2 & 2\times2+2\times1+1\times2 & 2\times2+2\times2+1\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$4A = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \end{bmatrix} \text{ and } 5I = \begin{bmatrix} 5\times1 & 0 & 0 \\ 0 & 5\times1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}$$

$$\lfloor 8 \quad 8 \quad 4 \rfloor \qquad\qquad \lfloor\ 0 \qquad 0 \qquad 5\times 1 \rfloor \ \lfloor 0 \quad 0 \quad 5 \rfloor$$

$$A^2 - 4A - 5I = \begin{bmatrix} 9-4-5 & 8-8 & 8-8 \\ 8-8 & 9-4-5 & 8-8 \\ 8-8 & 8-8 & 9-4-5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

14. For continuity of the function at $x = 2$

$$\lim_{h \to 0} f(2-h) = f(2) = \lim_{h \to 0} f(2+h)$$

Now, $\quad f(2-h) = 2(2-h) + 1 = 5 - 2h$

$\therefore \quad \lim_{h \to 0} f(2-h) = 5$

Also, $\quad f(2+h) = 3(2+h) - 1 = 5 + 3h$

$$\lim_{h \to 0} f(2+h) = 5$$

So, for continuity $f(2) = 5$.

$\therefore \quad k = 5.$

15. Let $\tan^{-1}\left(\dfrac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right) = y$

$$y = \tan^{-1}\left(\frac{1 - \sqrt{\dfrac{1-x}{1+x}}}{1 + \sqrt{\dfrac{1-x}{1+x}}}\right)$$

$$\Rightarrow \quad y = \tan^{-1} 1 - \tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)$$

$$\frac{dy}{dx} = 0 - \frac{1}{1 + \left(\sqrt{\dfrac{1-x}{1+x}}\right)^2} \cdot \frac{d}{dx}\left(\sqrt{\frac{1-x}{1+x}}\right)$$

$$= -\frac{1+x}{2}\left\{\frac{\dfrac{-1}{2\sqrt{1-x}}\sqrt{1+x} - \dfrac{1}{2\sqrt{1+x}}\sqrt{1-x}}{1+x}\right\}$$

$$= \frac{1+x}{4}\left\{\frac{\dfrac{\sqrt{1+x} \times \sqrt{1+x}}{\sqrt{1-x} \times \sqrt{1+x}} + \dfrac{\sqrt{1-x} \times \sqrt{1-x}}{\sqrt{1+x} \times \sqrt{1-x}}}{1+x}\right\}$$

$$= \frac{1}{4} \cdot \frac{2}{\sqrt{1 - x^2}} = \frac{1}{2\sqrt{1 - x^2}}$$

16. Slope of tangent $= \dfrac{dy}{dx}$

$$= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d(\cos 2t)}{dt}}{\frac{d(\sin 3t)}{dt}} = \frac{-2\sin 2t}{3\cos 3t}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{\text{at } t = \frac{\pi}{4}} = \frac{-2 \times \sin\frac{\pi}{2}}{3 \times \cos\frac{3\pi}{4}} = \frac{-2 \times 1}{3 \times \left(-\frac{1}{\sqrt{2}}\right)} = \frac{2\sqrt{2}}{3}$$

Now $\quad x = \sin\left(\dfrac{3\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$

$$y = \cos\left(\frac{2\pi}{4}\right) = 0$$

$\therefore$ Equation of tangent is

$$y - 0 = \frac{dy}{dx}\left(x - \left(\frac{1}{\sqrt{2}}\right)\right)$$

$$y = \frac{2\sqrt{2}}{3}\left(x - \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow y = \frac{2\sqrt{2}}{3}x - \frac{2}{3}$$

or $\quad 3y = 2\sqrt{2}\,x - 2.$

17. Let $I = \displaystyle\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx$

Apply the property $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx$

$$I = \int_0^{\pi} \frac{(\pi - x)\sin x\,dx}{1 + \cos^2 x}$$

$$I = \pi\int_0^{\pi} \frac{dx}{1 + \cos^2 x} - I \qquad \Rightarrow \qquad 2I = \pi\int_0^{\pi} \frac{dx}{1 + \cos^2 x}$$

$$I = \pi\int_0^{\pi/2} \frac{\sec^2 x}{1 + \sec^2 x}\,dx \qquad \left[\text{Using } \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx\right]$$

$$I = \pi\int_0^{\pi/2} \frac{\sec^2 x}{2 + \tan^2 x}\,dx$$

Putting $\quad \tan x = t \qquad$ *if* $\; x = 0, \; t = 0$

$\qquad \sec^2 x\,dx = dt \qquad$ *if* $\; x = \dfrac{\pi}{2}, \; t = \infty$

$$I = \pi \int_0^\infty \frac{dt}{(\sqrt{2})^2 + t^2}$$

$$I = \pi \left| \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) \right|_0^\infty$$

$$I = \frac{\pi}{\sqrt{2}}\left(\frac{\pi}{2}\right)$$

$$I = \frac{\pi^2}{2\sqrt{2}}$$

18. $(x^2 - y^2)\, dx + 2xy\, dy = 0$

$$\frac{dy}{dx} = -\frac{(x^2 - y^2)}{2xy} \qquad \text{...}(i)$$

It is homogeneous differential equation.

Putting $y = ux \Rightarrow u + x\frac{du}{dx} = \frac{dy}{dx}$

From (i) $u + x\dfrac{du}{dx} = -x^2\dfrac{(1-u^2)}{2x^2 u} = -\left(\dfrac{1-u^2}{2u}\right)$

$\Rightarrow$ $$x\frac{du}{dx} = -\left[\frac{1-u^2}{2u} + u\right]$$

$\Rightarrow$ $$x\frac{du}{dx} = -\left[\frac{1+u^2}{2u}\right]$$

$\Rightarrow$ $$\frac{2u}{1+u^2}\,du = -\frac{dx}{x}$$

Integrating both sides, we get

$\Rightarrow$ $$\int \frac{2u\,du}{1+u^2} = -\int \frac{dx}{x}$$

$\Rightarrow$ $$\log|1+u^2| = -\log|x| + \log C$$

$\Rightarrow$ $$\log\left|\frac{x^2+y^2}{x^2}\right| |x| = \log C$$

$\Rightarrow$ $$\frac{x^2+y^2}{x} = C$$

$\Rightarrow$ $$x^2 + y^2 = Cx$$

Given that $y = 1$ when $x = 1$

$\Rightarrow$ $1 + 1 = C \Rightarrow C = 2.$

$\therefore$ Solution is $x^2 + y^2 = 2x$.

OR

$$\frac{dy}{dx} = \frac{x(2y - x)}{x(2y + x)} \quad \text{...}(i)$$

Let $y = ux$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

$$\Rightarrow u + x.\frac{du}{dx} = \left(\frac{2u - 1}{2u + 1}\right) \quad \text{[from}(i)\text{]}$$

$$x\frac{du}{dx} = \frac{2u - 1}{2u + 1} - u$$

$$x\frac{du}{dx} = \frac{2u - 1 - 2u^2 - u}{2u + 1}$$

$$\Rightarrow \int \frac{2u + 1}{u - 1 - 2u^2}\, du = \int \frac{dx}{x}$$

$$\Rightarrow \int \frac{2u + 1}{2u^2 - u + 1}\, du = -\int \frac{dx}{x}$$

Let $2u + 1 = A(4u - 1) + B;\ A = \frac{1}{2},\ B = \frac{3}{2}$

$$\Rightarrow \frac{1}{2}\int \frac{4u - 1}{2u^2 - u + 1}\, du + \int \frac{\frac{3}{2}}{2u^2 - u + 1}\, du = -\log x + k$$

$$\Rightarrow \frac{1}{2}\log(2u^2 - u + 1) + \frac{3}{4}\int \frac{du}{\left(u - \frac{1}{4}\right)^2 + \frac{7}{16}} = -\log x + k$$

$$\log(2u^2 - u + 1) + \frac{3}{2}\frac{1}{\sqrt{7}/4}\tan^{-1}\left[\frac{\left(u - \frac{1}{4}\right)}{\frac{\sqrt{7}}{4}}\right] = -2\log x + k'$$

Putting $u = \frac{y}{x}$ and then $y = 1$ and $x = 1$, we get

$$k' = \log 2 + \frac{6}{\sqrt{7}}\tan^{-1}\frac{3}{\sqrt{7}}$$

$\therefore$ Solution is $\log\left(\frac{2y^2 - xy + x^2}{x^2}\right) + \frac{6}{\sqrt{7}}\tan^{-1}\left(\frac{4y - x}{\sqrt{7}x}\right) + 2\log x = \log 2 + \frac{6}{\sqrt{7}}\tan^{-1}\frac{3}{\sqrt{7}}$

19. $\cos^2 x\frac{dy}{dx} + y = \tan x$

$$\frac{dy}{dx} + \sec^2 x \times y = \sec^2 x \tan x$$

It is a linear differential equation.

Integrating factor $= e^{\int \sec^2 x\, dx}$

$= e^{\tan x}$

General solution : $y.\, IF = \int Q.\, IF\, dx$

$$y.\, e^{\tan x} = \int e^{\tan x} . \tan x . \sec^2 x\, dx$$

Putting $\tan x = t \quad \Rightarrow \sec^2 x\, dx = dt$

$$\therefore \quad y e^{\tan x} = \int e^t . t . dt$$

$$= e^t . t - \int e^t dt = e^t . t - e^t + k$$

$$= e^{\tan x} (\tan x - 1) + k$$

$$\therefore \quad y. e^{\tan x} = e^{\tan x} (\tan x - 1) + k$$

where k is some constant.

20. Given $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$

Let $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{a} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = \hat{i}(z - y) + \hat{j}(x - z) + \hat{k}(y - x)$$

Given $\vec{a} \times \vec{c} = \vec{b}$

$$(z - y)\hat{i} + (x - z)\hat{j} + (y - x)\hat{k} = \hat{j} - \hat{k}.$$

Comparing both sides

$z - y = 0 \quad \therefore \quad z = y$

$x - z = 1 \quad \therefore \quad x = 1 + z$

$y - x = -1 \quad \therefore \quad y = x - 1$

Also, $\vec{a} . \vec{c} = 3$

$(\hat{i} + \hat{j} + \hat{k}) . (x\hat{i} + y\hat{j} + z\hat{k}) = 3$

$x + y + z = 3$

$(1 + z) + z + z = 3$

$3z = 2 \quad \therefore \quad z = 2/3$

$y = 2/3$

$x = 1 + \frac{2}{3} = \frac{5}{3}$

$$\vec{c} = \frac{1}{3}(5\hat{i} + 2\hat{j} + 2\hat{k})$$

OR

$$\vec{a} + \vec{b} + \vec{c} = 0$$

$$\Rightarrow \quad (\vec{a} + \vec{b})^2 = (-\vec{c})^2$$

$$\Rightarrow \quad (\vec{a} + \vec{b}).(\vec{a} + \vec{b}) = \vec{c}.\vec{c}$$

$$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + 2\,\vec{a}.\vec{b} = |\vec{c}|^2$$

$$\Rightarrow \quad 9 + 25 + 2\,\vec{a}.\vec{b} = 49$$

$$\Rightarrow \quad 2\,\vec{a}.\vec{b} = 49 - 25 - 9$$

$$\Rightarrow \quad 2|\vec{a}||\vec{b}|\cos\theta = 15$$

$$\Rightarrow \quad 30\cos\theta = 15$$

$$\Rightarrow \quad \cos\theta = \frac{1}{2} = \cos 60^\circ$$

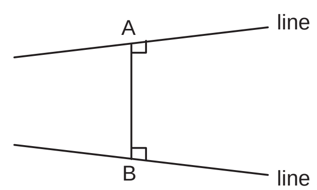
$$\Rightarrow \quad \theta = 60^\circ$$

21. Let $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} = \lambda$ and $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} = k$

Now, let's take a point on first line as

$A(\lambda + 3, -2\lambda + 5, \lambda + 7)$ and let

$B(7k - 1, -6k - 1, k - 1)$ be point on the second line



The direction ratio of the line AB

$7k - \lambda - 4, -6k + 2\lambda - 6, k - \lambda - 8$

Now as AB is the shortest distance between line 1 and line 2 so,

$$(7k - \lambda - 4) \times 1 + (-6k + 2\lambda - 6) \times (-2) + (k - \lambda - 8) \times 1 = 0 \quad ...(i)$$

and
$$(7k - \lambda - 4) \times 7 + (-6k + 2\lambda - 6) \times (-6) + (k - \lambda - 8) \times 1 = 0 \quad ...(ii)$$

Solving equation (*i*) and (*ii*) we get

$\lambda = 0$ and $k = 0$

$\therefore$ $A \equiv (3, 5, 7)$ and $B \equiv (-1, -1, -1)$

$\therefore$ $AB = \sqrt{(3+1)^2 + (5+1)^2 + (7+1)^2} = \sqrt{16 + 36 + 64} = \sqrt{116}$ units $= 2\sqrt{29}$ units

OR

Let $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2} = \lambda$

$\therefore$ $(3\lambda - 2, 2\lambda - 1, 2\lambda + 3)$ is any general point on the line

Now if the distance of the point from (1, 2, 3) is $3\sqrt{2}$, then

$$\sqrt{(3\lambda - 2 - 1)^2 + (2\lambda - 1 - 2)^2 + (2\lambda + 3 - 3)^2} = (3\sqrt{2})$$

$$\Rightarrow \quad (3\lambda - 3)^2 + (2\lambda - 3)^2 + 4\lambda^2 = 18$$

$$\Rightarrow \quad 9\lambda^2 - 18\lambda + 9 + 4\lambda^2 - 12\lambda + 9 + 4\lambda^2 = 18$$

$$\Rightarrow \qquad 17\lambda^2 - 30\lambda = 0$$

$$\Rightarrow \qquad \lambda(17\lambda - 30) = 0$$

$$\Rightarrow \qquad \lambda = 0 \quad \text{or} \quad \lambda = \frac{30}{17}$$

$\therefore$ Required point on the line is $(-2, -1, 3)$ *or* $\left(\frac{56}{17}, \frac{43}{17}, \frac{77}{17}\right)$

22. Let X be the numbers of doublets. Then, $X = 0, 1, 2, 3$ or 4

$P(X = 0) = P$ (non doublet in each case)

$$P(\overline{D}_1\overline{D}_2\overline{D}_3\overline{D}_4) = \left(\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}\right) = \frac{625}{1296}$$

$P(X = 1) = P$ (one doublet) $\left[\text{Alternatively use } {}^nC_r\, p^r q^r \text{ where } p = \frac{1}{6}, q = \frac{5}{6}\right]$

$$= P(D_1\overline{D}_2\overline{D}_3\overline{D}_4) \text{ or } P(\overline{D}_1 D_2\overline{D}_3\overline{D}_4) \text{ or } P(\overline{D}_1\overline{D}_2 D_3\overline{D}_4) \text{ or } P(\overline{D}_1\overline{D}_2\overline{D}_3 D_4)$$

$$= \left(\frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}\right) + \left(\frac{5}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6}\right) + \left(\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} \times \frac{5}{6}\right) + \left(\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}\right)$$

$$= \left(4 \times \frac{125}{1296}\right) = \frac{125}{324}$$

$P(X = 2) = P$ (two doublets)

$$= P(D_1D_2\overline{D}_3\overline{D}_4) \text{ or } P(D_1\overline{D}_2D_3\overline{D}_4) \text{ or } P(D_1\overline{D}_2\overline{D}_3D_4) \text{ or } P(\overline{D}_1D_2D_3\overline{D}_4)$$
$$\text{or } P(\overline{D}_1D_2\overline{D}_3D_4) \text{ or } P(\overline{D}_1\overline{D}_2D_3D_4)$$

$$= \left(\frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6}\right) + \left(\frac{1}{6} \times \frac{5}{6} \times \frac{1}{6} \times \frac{5}{6}\right) + \left(\frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}\right)$$
$$+ \left(\frac{5}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6}\right) + \left(\frac{5}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{1}{6}\right) + \left(\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} \times \frac{1}{6}\right)$$

$$= \left(6 \times \frac{25}{1296}\right) = \frac{25}{216}$$

$P(X = 3) = P$ (three doublets)

$$= P(D_1D_2D_3\overline{D}_4) \text{ or } P(D_1D_2\overline{D}_3D_4) \text{ or } P(D_1\overline{D}_2D_3D_4) \text{ or } P(\overline{D}_1D_2D_3D_4)$$

$$= \left(\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6}\right) + \left(\frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{1}{6}\right) + \left(\frac{1}{6} \times \frac{5}{6} \times \frac{1}{6} \times \frac{1}{6}\right) + \left(\frac{5}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}\right)$$

$$= \left(4 \times \frac{5}{1296}\right) = \frac{5}{324}$$

$P(X = 4) = P$ (four doublets) $= P(D_1D_2D_3D_4)$

$$= \left(\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}\right) = \frac{1}{1296}$$

Thus, we have

$X = x_i$	0	1	2		
P_i	$\frac{625}{1296}$	$\frac{125}{324}$	$\frac{25}{216}$	1296	324

3

4 5

1

324 1296

SECTION–C

23.

$$\text{L.H.S.} = \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix}$$

Applying $R_3 \to R_3 + R_1$ and taking common $(\alpha+\beta+\gamma)$ from R_3.

$$= (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \beta-\alpha & \gamma-\alpha \\ \alpha^2 & \beta^2-\alpha^2 & \gamma^2-\alpha^2 \\ 1 & 0 & 0 \end{vmatrix} \quad (\text{Applying } C_2 \to C_2 - C_1, C_3 \to C_3 - C_1)$$

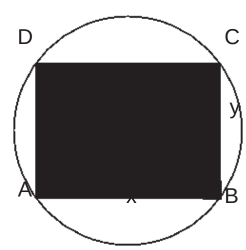
$= (\alpha+\beta+\gamma)[(\gamma^2-\alpha^2)(\beta-\alpha) - (\gamma-\alpha)(\beta^2-\alpha^2)]$ (Expanding along R_3)

$= (\alpha+\beta+\gamma)(\gamma-\alpha)(\beta-\alpha)[(\gamma+\alpha)-(\beta+\alpha)]$

$= (\alpha+\beta+\gamma)(\gamma-\alpha)(\beta-\alpha)(\gamma-\beta)$

$= (\alpha+\beta+\gamma)(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$

24. Let x and y be the length and breadth of rectangle and R be the radius of given circle, ($i.e. R$ is constant).



Now, in right ΔABC, we have

$$x^2 + y^2 = (2R)^2$$

$$x^2 + y^2 = 4R^2 \Rightarrow y = \sqrt{4R^2 - x^2} \quad ....(i)$$

Now, area, of rectangle $ABCD$.

$$A = xy$$

$$\Rightarrow \quad A = x\sqrt{4R^2 - x^2} \quad [\text{from } (i)]$$

For area to be maximum or minimum

$$\frac{dA}{dx} = 0$$

$$\Rightarrow \quad x \times \frac{1}{2\sqrt{4R^2 - x^2}} \times -2x + \sqrt{4R^2 - x^2} \times 1 = 0$$

$$\Rightarrow \quad \frac{-x^2}{\sqrt{4R^2 - x^2}} + \sqrt{4R^2 - x^2} = 0 \quad \Rightarrow \quad \frac{(\sqrt{4R^2 - x^2})^2 - x^2}{\sqrt{4R^2 - x^2}} = 0$$

$$\Rightarrow \quad 4R^2 - x^2 - x^2 = 0 \quad \Rightarrow \quad 4R^2 - 2x^2 = 0$$

$$x^2 - 2R^2 = 0 \quad \Rightarrow \quad x = \sqrt{2}R$$

Now, $\dfrac{d^2A}{dx^2} = \dfrac{2x(x^2 - 6R^2)}{(4R^2 - x^2)^{3/2}}$

$\therefore$ $\left.\dfrac{d^2A}{dx^2}\right|_{at\ x=\sqrt{2}R} = \dfrac{-8\sqrt{2}R^3}{(2R^2)^{3/2}} < 0$

So, area will be maximum at $x = \sqrt{2}R$

Now, from (i), we have

$$y = \sqrt{4R^2 - x^2} = \sqrt{4R^2 - 2R^2} = \sqrt{2R^2}$$

$$y = \sqrt{2}R$$

Here $x = y = \sqrt{2}R$

So the area will be maximum when $ABCD$ is a square.

OR

Let radius CD of inscribed cylinder be x and height OC be H and θ be the semi-vertical angle of cone.

Therefore,

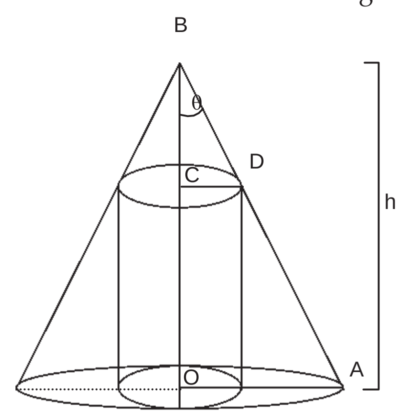
$OC = OB - BC$

$\Rightarrow$ $H = h - x\cot\theta$

Now, volume of cylinder

$V = \pi x^2 (h - x\cot\theta)$

$\Rightarrow$ $V = \pi(x^2 h - x^3 \cot\theta)$



For maximum or minimum value

$\dfrac{dV}{dx} = 0 \qquad \Rightarrow \quad \pi(2xh - 3x^2 \cot\theta) = 0$

$\Rightarrow$ $\pi x(2h - 3x\cot\theta) = 0$

$\therefore$ $2h - 3x\cot\theta = 0$ (as $x = 0$ is not possible)

$\Rightarrow$ $x = \dfrac{2h}{3}\tan\theta$

Now, $\dfrac{d^2V}{dx^2} = \pi(2h - 6x\cot\theta)$

$\Rightarrow$ $\dfrac{d^2V}{dx^2} = 2\pi h - 6\pi x\cot\theta$

$\Rightarrow$ $\left.\dfrac{d^2V}{dx^2}\right|_{at\ x=\frac{2h\tan\theta}{3}} = 2\pi h - 6\pi \times \dfrac{2h}{3}\tan\theta\cot\theta$

$= 2\pi h - 4\pi h = -2\pi h < 0$

Hence, volume will be maximum when $x = \frac{2h}{3}\tan\theta$.

Therefore, height of cylinder

$$H = h - x\cot\theta$$

$$= h - \frac{2h}{3}\tan\theta\cot\theta = h - \frac{2h}{3} = \frac{h}{3}.$$

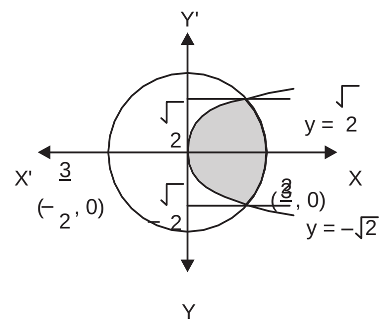
Thus height of the cylinder is $\frac{1}{3}$ of height of cone.

25. $x^2 + y^2 = \frac{9}{4}$ ...(*i*)

$y^2 = 4x$ ...(*ii*)

From (*i*) and (*ii*)

$$\left(\frac{y^2}{4}\right)^2 + y^2 = \frac{9}{4}$$



Let $y^2 = t$

$$\frac{t^2}{16} + t = \frac{9}{4}$$

$$t^2 + 16t = 36$$

$$t^2 + 18t - 2t - 36 = 0$$

$$t(t+18) - 2(t+18) = 0$$

$$(t-2)(t+18) = 0$$

$$t = 2, -18$$

$$y^2 = 2$$

$$y = \pm\sqrt{2}$$

Required area $= \int_{-\sqrt{2}}^{\sqrt{2}} (x_2 - x_1)\,dy$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \left(\sqrt{\frac{9}{4} - y^2} - \frac{y^2}{4}\right) dy$$

$$= 2\int_0^{\sqrt{2}} \sqrt{\left(\frac{3}{2}\right)^2 - y^2}\,dy - \frac{2}{4}\int_0^{\sqrt{2}} y^2\,dy$$

$$= 2\left[\frac{y}{2}\sqrt{\frac{9}{4} - y^2} + \frac{9}{8}\sin^{-1}\frac{y}{3/2}\right]_0^{\sqrt{2}} - \frac{1}{2}\left(\frac{y^3}{3}\right)_0^{\sqrt{2}}$$

$$= 2\left[\frac{\sqrt{2}}{2}\sqrt{\frac{9}{4} - 2} + \frac{9}{8}\sin^{-1}\frac{2\sqrt{2}}{3}\right] - \frac{1}{6}\,2\sqrt{2}$$

$$= \frac{1}{\sqrt{2}} + \frac{9}{4} \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) - \frac{\sqrt{2}}{3}$$

$$= \frac{1}{3\sqrt{2}} + \frac{9}{4} \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) \text{ sq. units}$$

26. Let $I = \int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}\, dx$

Put $x = a \cos 2\theta$

$dx = a(-\sin 2\theta)\, 2d\theta$

If $x = a$, then

$\cos 2\theta = 1$

$2\theta = 0$

$\theta = 0$

$x = -a$, $\cos 2\theta = -1$

$2\theta = \pi$

$\theta = \frac{\pi}{2}$

$$\therefore \quad I = \int_{\pi/2}^{0} \sqrt{\frac{a - a\cos 2\theta}{a + a\cos 2\theta}}\, a(-\sin 2\theta)\, 2d\theta$$

$$= \int_{0}^{\pi/2} \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}\, 2a \sin 2\theta\, d\theta$$

$$= 2a\int_0^{\pi/2} 2\sin^2\theta\, d\theta = 2a\int_0^{\pi/2} (1 - \cos 2\theta)\, d\theta$$

$$= 2a\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = 2a\left[\left(\frac{\pi}{2} - \frac{\sin \pi}{2}\right) - \left(0 - \frac{\sin 0}{2}\right)\right]$$

$$= 2a\left[\left(\frac{\pi}{2} - 0\right)\right] = \pi a$$

27. Equation of the plane passing through $(-1, -1, 2)$ is

$a(x+1) + b(y+1) + c(z-2) = 0$ ...(*i*)

(*i*) is perpendicular to $2x + 3y - 3z = 2$

$\therefore \quad 2a + 3b - 3c = 0$ ...(*ii*)

Also (*i*) is perpendicular to $5x - 4y + z = 6$

$\therefore \quad 5a - 4b + c = 0$ ...(*iii*)

From (*ii*) and (*iii*)

$$\frac{a}{3-12} = \frac{b}{-15-2} = \frac{c}{-8-15} = k$$

$$\Rightarrow \quad \frac{a}{-9} = \frac{b}{-17} = \frac{c}{-23} = k$$

$$\Rightarrow \quad a = -9k, \quad b = -17k, \quad c = -23k$$

Putting in equation (i)

$$-9k(x+1)-17k(y+1)-23k(z-2)=0$$

$\Rightarrow$ $$9(x+1)+17(y+1)+23(z-2)=0$$

$\Rightarrow$ $$9x+17y+23z+9+17-46=0$$

$\Rightarrow$ $$9x+17y+23z-20=0$$

$\Rightarrow$ $$9x+17y+23z=20.$$

Which is the required equation of the plane.

OR

Equation of the plane passing through (3, 4, 1) is

$$a(x-3)+b(y-4)+c(z-1)=0 \qquad ...(i)$$

Since this plane passes through (0, 1, 0) also

$\therefore$ $$a(0-3)+b(1-4)+c(0-1)=0$$

or $$-3a-3b-c=0$$

or $$3a+3b+c=0 \qquad ...(ii)$$

Since (i) is parallel to

$$\frac{x+3}{2}=\frac{y-3}{7}=\frac{z-2}{5}$$

$\therefore$ $$2a+7b+5c=0 \qquad ...(iii)$$

From (ii) and (iii)

$$\frac{a}{15-7}=\frac{b}{2-15}=\frac{c}{21-6}=k$$

$\Rightarrow$ $$a=8k,\ b=-13k,\ c=15k$$

Putting in (i), we have

$$8k(x-3)-13k(y-4)+15k(z-1)=0$$

$\Rightarrow$ $$8(x-3)-13(y-4)+15(z-1)=0$$

$\Rightarrow$ $$8x-13y+15z+13=0.$$

Which is the required equation of the plane.

28. Let the owner buys x machines of type A and y machines of type B.

Then

$$1000x+1200y\le 9000 \qquad ...(i)$$

$$12x+8y\le 72 \qquad ...(ii)$$

Objective function is to be maximize $z=60x+40y$

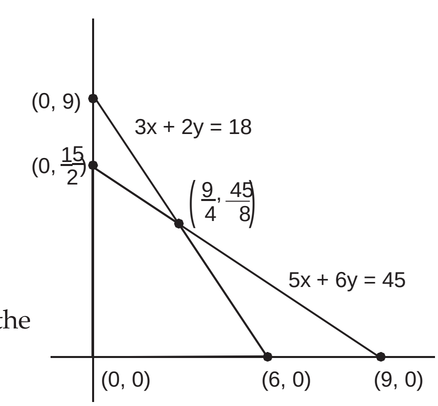
From (i)

$$10x+12y\le 90$$

or $$5x+6y\le 45 \qquad ...(iii)$$

$$3x+2y\le 18 \qquad ...(iv) \quad \text{[from (ii)]}$$

We plot the graph of inequations shaded region in the feasible solutions (iii) and (iv) .



The shaded region in the figure represents the feasible region which is bounded. Let us now evaluate Z at each corner point.

at (0, 0) Z is $60 \times 0 + 40 \times 0 = 0$

Z at $\left(0, \frac{15}{2}\right)$ is $60 \times 0 + 40 \times \frac{15}{2} = 300$

Z at $(6, 0)$ is $60 \times 6 + 40 \times 0 = 360$

Z at $\left(\frac{9}{4}, \frac{45}{8}\right)$ is $60 \times \frac{9}{4} + 40 \times \frac{45}{8} = 135 + 225 = 360.$

$\Rightarrow$ max. $Z = 360$

Therefore there must be

either $x = 6,\ y = 0$ or $x = \frac{9}{4},\ y = \frac{45}{8}$ but second case is not possible as x and y are whole numbers. Hence there must be 6 machines of type A and no machine of type B is required for maximum daily output.

29. Let E_1 be the event that insured person is scooter driver,

E_2 be the event that insured person is car driver,

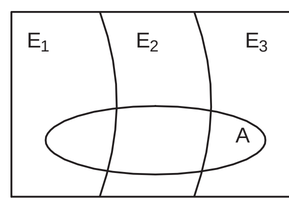
E_3 be the event that insured person is truck driver,

and A be the event that insured person meets with an accident.

$$\therefore \quad P(E_1) = \frac{2,000}{12,000} = \frac{1}{6},\ P\left(\frac{A}{E_1}\right) = 0.01$$

$$P(E_2) = \frac{4,000}{12000} = \frac{1}{3},\ P\left(\frac{A}{E_2}\right) = 0.03$$

$$P(E_3) = \frac{6,000}{12,000} = \frac{1}{2},\ P\left(\frac{A}{E_3}\right) = 0.15$$



$$\therefore \quad P\left(\frac{E_1}{A}\right) = \frac{P(E_1) \,.\, P\left(\frac{A}{E_1}\right)}{P(E_1) \,.\, P\left(\frac{A}{E_1}\right) + P(E_2) \,.\, P\left(\frac{A}{E_2}\right) + P(E_3) \,.\, P\left(\frac{A}{E_3}\right)}$$

$$= \frac{\frac{1}{6} \times 0.01}{\frac{1}{6} \times 0.01 + \frac{1}{3} \times 0.03 + \frac{1}{2} \times 0.15} = \frac{1}{1 + 6 + 45} = \frac{1}{52}$$

Set–II

20. We have,

$$\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{2x + 3x}{1 - (2x) \,.\, (3x)}\right] = \frac{\pi}{4} \quad \left[\text{Using property } \tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x + y}{1 - xy}\right]$$

$\Rightarrow \quad \tan^{-1} \dfrac{5x}{1-6x^2} = \dfrac{\pi}{4}$

$\Rightarrow \quad \dfrac{5x}{1-6x^2} = 1 \quad \Rightarrow \quad 6x^2 + 5x - 1 = 0$

$\Rightarrow \quad 6x^2 + 6x - x - 1 = 0$

$\Rightarrow \quad 6x(x+1) - 1(x+1) = 0$

$\Rightarrow \quad (x+1)(6x-1) = 0$

$\Rightarrow \quad x = -1, \dfrac{1}{6}$ which is the required solution.

21. Let $I = \displaystyle\int_0^{\pi} \frac{x \tan x}{\sec x \operatorname{cosec} x}\, dx$

$\Rightarrow \quad I = \displaystyle\int_0^{\pi} \frac{x \cdot \dfrac{\sin x}{\cos x}}{\dfrac{1}{\cos x} \cdot \dfrac{1}{\sin x}}\, dx$

$\Rightarrow \quad I = \displaystyle\int_0^{\pi} x \sin^2 x\, dx$ ...(*i*)

$\Rightarrow \quad I = \displaystyle\int_0^{\pi} (\pi - x) . \sin^2 (\pi - x)\, dx$ [Using property $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$]

$\Rightarrow \quad I = \displaystyle\int_0^{\pi} (\pi - x) \sin^2 x\, dx$ ...(*ii*)

Adding (*i*) and (*ii*) we have

$2I = \displaystyle\int_0^{\pi} \pi \sin^2 x\, dx$

$\Rightarrow \quad 2I = \pi \displaystyle\int_0^{\pi} \sin^2 x\, dx = \frac{\pi}{2} \int_0^{\pi} (1 - \cos 2x)\, dx \quad \Rightarrow \quad 2I = \frac{\pi}{2} \left[x - \frac{\sin 2x}{2} \right]_0^{\pi}$

$\Rightarrow \quad 2I = \dfrac{\pi}{2} \left[\left(\pi - \dfrac{\sin 2\pi}{2} \right) - \left(0 - \dfrac{\sin 0}{2} \right) \right] \quad \Rightarrow \quad 2I = \dfrac{\pi}{2} [\pi] = \dfrac{\pi^2}{2}$

$\therefore \quad I = \dfrac{\pi^2}{4}.$

Hence $\displaystyle\int_0^{\pi} \frac{x \tan x}{\sec x \cdot \operatorname{cosec} x}\, dx = \frac{\pi^2}{4}.$

22. We have, $y = \sqrt{x^2 + 1} - \log \left(\dfrac{1}{x} + \sqrt{1 + \dfrac{1}{x^2}} \right)$

$\Rightarrow \quad y = \sqrt{x^2 + 1} - \log \left(\dfrac{1 + \sqrt{x^2 + 1}}{x} \right)$

$\Rightarrow \quad y = \sqrt{x^2 + 1} - \log \left(1 + \sqrt{x^2 + 1} \right) + \log x$

On differentiating w.r.t. x, we have

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x^2+1}} \times 2x - \frac{1}{(\sqrt{x^2+1}+1)} \times \frac{1}{2\sqrt{x^2+1}} \times 2x + \frac{1}{x}$$

$$= \frac{x}{\sqrt{x^2+1}} - \frac{x}{\sqrt{x^2+1}\,(\sqrt{x^2+1}+1)} + \frac{1}{x}$$

$$= \frac{x}{\sqrt{x^2+1}} - \frac{x}{\sqrt{x^2+1}\,(\sqrt{x^2+1}+1)} \times \frac{(\sqrt{x^2+1}-1)}{(\sqrt{x^2+1}-1)} + \frac{1}{x}$$

$$= \frac{x}{\sqrt{x^2+1}} - \frac{x\left(\sqrt{x^2+1}-1\right)}{(\sqrt{x^2+1})\,(x^2)} + \frac{1}{x}$$

$$= \frac{x}{\sqrt{x^2+1}} - \frac{(\sqrt{x^2+1}-1)}{x\sqrt{x^2+1}} + \frac{1}{x}$$

$$= \frac{x^2+1-\sqrt{x^2+1}+\sqrt{x^2+1}}{x\sqrt{x^2+1}}$$

$$= \frac{x^2+1}{x\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}}{x}$$

23. Let $\Delta = \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$

Applying $C_1 \to C_1 - b.C_3$ and $C_2 \to C_2 + a.C_3$, we have

$$\Delta = \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b\,(1+a^2+b^2) & -a\,(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$$

Taking out $(1+a^2+b^2)$ from C_1 and C_2, we have

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

Expanding along first row, we have

$$= (1+a^2+b^2)^2\,[1\,.(1-a^2-b^2+2a^2) - 2b\,(-b)]$$

$$= (1+a^2+b^2)^2\,(1+a^2-b^2+2b^2)$$

$$= (1+a^2+b^2)^2\,(1+a^2+b^2) = (1+a^2+b^2)^3.$$

24. Let $I = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx$ ...(*i*)

$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x) \sin(\pi - x)}{1 + \cos^2(\pi - x)}\, dx$ [Using property $\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$]

$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + (-\cos x)^2}\, dx$

$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + \cos^2 x}\, dx$ ...(*ii*)

Adding (*i*) and (*ii*), we have

$$2I = \int_0^{\pi} \frac{\pi \sin x}{1 + \cos^2 x}\, dx = \pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x}\, dx$$

Let $\cos x = t \quad \Rightarrow \quad -\sin x\, dx = dt \quad \Rightarrow \quad \sin x\, dx = -dt$

As $x = 0, t = 1$ and $x = \pi, t = -1$

Now, we have

$$2I = \int_1^{-1} \frac{-dt}{1 + t^2}$$

$$\Rightarrow \quad 2I = \int_{-1}^{1} \frac{dt}{1 + t^2} = [\tan^{-1}(t)]_{-1}^{1}$$

$$\Rightarrow \quad 2I = \tan^{-1}(1) - \tan^{-1}(-1)$$

$$= \frac{\pi}{4} - \left(\frac{-\pi}{4}\right) = \frac{\pi}{2}$$

$$\Rightarrow \quad I = \frac{\pi}{4}.$$

25. The equations of the given curves are

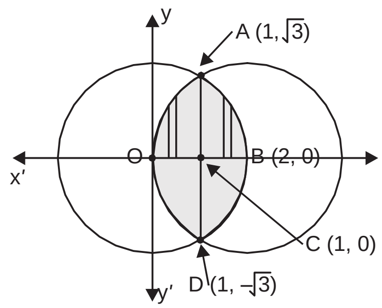
$x^2 + y^2 = 4$ ...(*i*)

and $(x - 2)^2 + y^2 = 4$ ...(*ii*)

Clearly, $x^2 + y^2 = 4$ represents *a* circle with centre (0, 0) and radius 2. Also, $(x - 2)^2 + y^2 = 4$ represents a circle with centre (2, 0) and radius 2. To find the point of intersection of the given curves, we solve (*i*) and (*ii*). Simultaneously, we find the two curves intersect at $A(1, \sqrt{3})$ and $D(1, -\sqrt{3})$.

Since both the curves are symmetrical about x-axis, So, the required area = 2(Area *OABCO*) Now, we slice the area *OABCO* into vertical strips. We observe that the vertical strips change their character at $A(1, \sqrt{3})$. So,

Area *OABCO* = Area *OACO* + Area *CABC*.



When area $OACO$ is sliced in the vertical strips, we find that each strip has its upper end on the circle $(x-2)^2 + (y-0)^2 = 4$ and the lower end on x-axis. So, the approximating rectangle shown in figure has length $= y_1$ width $= \Delta x$ and area $= y_1\ \Delta x$.

As it can move from $x = 0$ to $x = 1$

$\therefore$ Area $OACO = \int_0^1 y_1\ dx$

$\therefore$ Area $OACO = \int_0^1 \sqrt{4-(x-2)^2}\ dx$

Similarly, approximating rectangle in the region $CABC$ has length $= y_2$, width $= \Delta x$ and area $= y_2 \Delta x$.

As it can move from $x = 1$ to $x = 2$

$\therefore$ Area $CABC = \int_1^2 y_2\, dx = \int_1^2 \sqrt{4-x^2}\ dx$

Hence, required area A is given by

$$A = 2\left[\int_0^1 \sqrt{4-(x-2)^2}\ dx + \int_1^2 \sqrt{4-x^2}\ dx\right]$$

$$\Rightarrow A = 2\left[\left[\frac{(x-2)}{2}\cdot\sqrt{4-(x-2)^2} + \frac{4}{2}\sin^{-1}\frac{(x-2)}{2}\right]_0^1 + \left[\frac{x}{2}\cdot\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_1^2\right]$$

$$\Rightarrow A = 2\left[\left\{-\frac{\sqrt{3}}{2} + 2\sin^{-1}\left(-\frac{1}{2}\right) - 2\sin^{-1}(-1)\right\} + \left\{2\sin^{-1}(1) - \frac{\sqrt{3}}{2} - 2\sin^{-1}\frac{1}{2}\right\}\right]$$

$$= 2\left[-\sqrt{3} - 2\left(\frac{\pi}{6}\right) + 2\left(\frac{\pi}{2}\right) + 2\left(\frac{\pi}{2}\right) - 2\left(\frac{\pi}{6}\right)\right] = 2\left(-\sqrt{3} - \frac{2\pi}{3} + 2\pi\right)$$

$$= 2\left(\frac{4\pi}{3} - \sqrt{3}\right) = \left(\frac{8\pi}{3} - 2\sqrt{3}\right) \text{ sq. units.}$$

Set–III

20. We have,

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left\{\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \frac{x-1}{x-2}\cdot\frac{x+1}{x+2}}\right\} = \frac{\pi}{4}$$

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$$\Rightarrow \quad \tan^{-1}\left\{\frac{(x-1)(x+2)+(x-2)(x+1)}{(x-2)(x+2)-(x-1)(x+1)}\right\}=\frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left\{\frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1}\right\}=\frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2x^2-4}{-3}\right)=\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{2x^2-4}{-3}=\tan\frac{\pi}{4} \qquad \Rightarrow \quad \frac{2x^2-4}{-3}=1$$

$$\Rightarrow \quad 2x^2-4=-3$$

$$\Rightarrow \quad 2x^2=1 \qquad \Rightarrow \quad x^2=\frac{1}{2} \qquad \Rightarrow \quad x=\pm\frac{1}{\sqrt{2}}$$

Hence, $x=\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$ are the required values.

21. Given $y=\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$

$$=\cot^{-1}\left[\frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})(\sqrt{1+\sin x}+\sqrt{1-\sin x})}{(\sqrt{1+\sin x}-\sqrt{1-\sin x})(\sqrt{1+\sin x}+\sqrt{1-\sin x})}\right]$$

$$=\cot^{-1}\left[\frac{1+\sin x+1-\sin x+2\sqrt{1-\sin^2 x}}{1+\sin x-1+\sin x}\right]$$

$$=\cot^{-1}\left[\frac{2(1+\cos x)}{2\sin x}\right]=\cot^{-1}\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\cot^{-1}\left(\cot\frac{x}{2}\right)=\frac{x}{2}$$

$$=\frac{dy}{dx}=\frac{1}{2}$$

22. Let $I=\int_0^1 \cot^{-1}(1-x+x^2)\,dx$

$$=\int_0^1 \tan^{-1}\frac{1}{1-x+x^2}\,dx \qquad \left[\because \cot^{-1}x=\tan^{-1}\frac{1}{x}\right]$$

$$= \int_0^1 \tan^{-1} \frac{x + (1-x)}{1 - x(1-x)}\, dx \qquad \text{[Q1 can be written as } x + 1 - x]$$

$$= \int_0^1 [\tan^{-1} x + \tan^{-1}(1-x)]\, dx \qquad \left[\because \tan^{-1}\left\{\frac{a+b}{1-ab}\right\} = \tan^{-1} a + \tan^{-1} b\right]$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}(1-x)\, dx$$

$$= \int_0^1 \tan^{-1} x dx + \int_0^1 \tan^{-1}[1-(1-x)]\, dx \qquad \left[\because \int_0^a f(x) = \int_0^a f(a-x)\, dx\right]$$

$$= 2\int_0^1 \tan^{-1} x dx = 2\int_0^1 \tan^{-1} x.1\, dx, \text{ integrating by parts, we get}$$

$$= 2\left[\{\tan^{-1} x.x\}_0^1 - \int_0^1 \frac{1}{1+x^2}\cdot x dx\right]$$

$$= 2[\tan^{-1} 1 - 0] - \int_0^1 \frac{2x}{1+x^2}\, dx = 2\cdot\frac{\pi}{4} - [\log(1+x^2)]_0^1$$

$$= \frac{\pi}{2} - (\log 2 - \log 1) = \frac{\pi}{2} - \log 2 \qquad [\because \log 1 = 0]$$

23. Let $\Delta = \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we have

$$\Delta = \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

Taking out $2(a+b+c)$ from C_1, we have

$$\Delta = 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

Interchanging row into column, we have

$$\Delta = 2(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ a & b+c+2a & a \\ b & b & c+a+2b \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$, we have

$$\Delta = 2(a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ -(a+b+c) & a+b+c & a \\ 0 & -(a+b+c) & c+a+2b \end{vmatrix}$$

Now expanding along first row, we have

$$2(a+b+c)\,[1\,.\,(a+b+c)^2]$$

$$= 2(a+b+c)^3 \quad = \text{R.H.S.}$$

24. We have, given equations

$$x^2 + y^2 = 8x \qquad \text{...}(i)$$

and $\qquad y^2 = 4x \qquad \text{...}(ii)$

Equation (1) can be written as

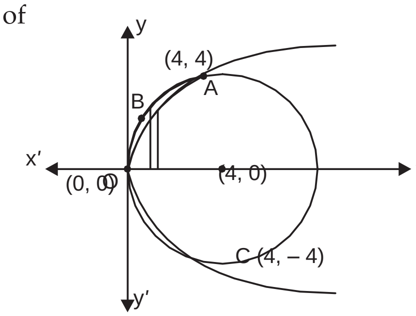
$$(x-4)^2 + y^2 = (4)^2$$

So equation (i) represents a circle with centre (4, 0) and radius 4.

Again, clearly equation (ii) represents parabola with vertex (0, 0) and axis as x-axis.

The curve (i) and (ii) are shown in figure and the required region is shaded.

On solving equation (i) and (ii) we have points of intersection 0(0, 0) and A (4, 4), C(4, – 4)



Now, we have to find the area of region bounded by (i) and (ii) & above x-axis.

So required region is $OBAO$.

Now, area of $OBAO$ is

$$A = \int_0^4 \left(\sqrt{8x - x^2} - \sqrt{4x}\right) dx$$

$$= \int_0^4 \left(\sqrt{(4)^2 - (x-4)^2} - 2\sqrt{x}\right) dx$$

$$= \left[\frac{(x-4)}{2}\sqrt{(4)^2 - (x-4)^2} + \frac{16}{2}\sin^{-1}\frac{(x-4)}{4} - 2 \times \frac{2x^{3/2}}{3}\right]_0^4$$

$$= \left[8\sin^{-1} 0 - \frac{4}{3}(4)^{\frac{3}{2}}\right] - [8\sin^{-1}(-1) - 0]$$

$$= \left(8 \times 0 - \frac{4}{3} \times 8\right) - \left(8 \times -\frac{\pi}{2}\right)$$

$$= -\frac{32}{3} + 4\pi = \left(4\pi - \frac{32}{3}\right) \text{ sq.units}$$

25. Let $I = \int_0^{\pi} \dfrac{x \tan x}{\sec x + \tan x}\, dx \qquad \text{...}(i)$

$$\int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)} \qquad \left[\text{Using property } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{-(\pi - x)\tan x}{-\sec x - \tan x}\, dx$$

$$I = \int \qquad dx$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \tan x}\, dx \qquad \text{...}(ii)$$

Adding (i) and (ii) we have

$$2I = \int_0^{\pi} \frac{\pi \tan x}{\sec x + \tan x}\, dx$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{\tan x}{\sec x + \tan x}\, dx$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{\tan x}{(\sec x + \tan x)} \times \frac{(\sec x - \tan x)}{(\sec x - \tan x)}\, dx$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{\tan x (\sec x - \tan x)}{\sec^2 x - \tan^2 x}\, dx$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} (\tan x \,.\, \sec x - \tan^2 x)\, dx$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} [\sec x \tan x - (\sec^2 x - 1)]\, dx$$

$$\Rightarrow \quad 2I = \pi [\sec x - \tan x + x]_0^{\pi}$$

$$\Rightarrow \quad 2I = \pi [(\sec \pi - \tan \pi + \pi) - (\sec 0 - \tan 0 + 0)]$$

$$\Rightarrow \quad 2I = \pi [(-1 - 0 + \pi) - (1 - 0)]$$

$$\Rightarrow \quad 2I = \pi (\pi - 2)$$

$$\therefore \quad I = \frac{\pi}{2}(\pi - 2)$$

Hence $\displaystyle\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} = \frac{\pi}{2}(\pi - 2)$

EXAMINATION PAPERS – 2008

MATHEMATICS CBSE (All India)
CLASS – XII

Time allowed: 3 hours **Maximum marks: 100**

General Instructions: ***As given in CBSE Examination paper (Delhi)*** **– 2008.**

Set–I

SECTION–A

1. If $f(x)$ is an invertible function, find the inverse of $f(x) = \dfrac{3x-2}{5}$.

2. Solve for x : $\tan^{-1}\dfrac{1-x}{1+x} = \dfrac{1}{2}\tan^{-1} x;\ x > 0$

3. If $\begin{bmatrix} x+3y & y \\ 7-x & 4 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 0 & 4 \end{bmatrix}$, find the values of x and y.

4. Show that the points (1, 0), (6, 0), (0, 0) are collinear.

5. Evaluate : $\displaystyle\int \frac{x+\cos 6x}{3x^2+\sin 6x}\,dx$

6. If $\displaystyle\int (e^{ax}+bx)\,dx = 4e^{4x} + \frac{3x^2}{2}$, find the values of a and b.

7. If $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = 2$ and angle between $\vec{a}$ and $\vec{b}$ is 60°, find $\vec{a}\,.\,\vec{b}$.

8. Find a vector in the direction of vector $\vec{a} = \hat{i} - 2\hat{j}$, whose magnitude is 7.

9. If the equation of a line AB is $\dfrac{x-3}{1} = \dfrac{y+2}{-2} = \dfrac{z-5}{4}$, find the direction ratios of a line parallel to AB.

10. If $\begin{vmatrix} x+2 & 3 \\ x+5 & 4 \end{vmatrix} = 3$, find the value of x.

SECTION–B

11. Let T be the set of all triangles in a plane with R as relation in T given by $R = \{(T_1, T_2) : T_1 \cong T_2\}$. Show that R is an equivalence relation.

12. Prove that $\tan\left(\dfrac{\pi}{4} + \dfrac{1}{2}\cos^{-1}\dfrac{a}{b}\right) + \tan\left(\dfrac{\pi}{4} - \dfrac{1}{2}\cos^{-1}\dfrac{a}{b}\right) = \dfrac{2b}{a}$.

OR

Solve $\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\dfrac{8}{31}$

13. Using properties of determinants, prove that following:

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$

14. Discuss the continuity of the following function at $x=0$:

$$f(x)=\begin{cases} \dfrac{x^4+2x^3+x^2}{\tan^{-1}x}, & x\neq 0 \\ 0, & x=0 \end{cases}$$

OR

Verify Lagrange's mean value theorem for the following function:

$f(x)=x^2+2x+3$, for $[4, 6]$.

15. If $f(x)=\sqrt{\dfrac{\sec x-1}{\sec+1}}$, find $f'(x)$. Also find $f'\left(\dfrac{\pi}{2}\right)$.

OR

If $x\sqrt{1+y}+y\sqrt{1+x}=0$, find $\dfrac{dy}{dx}$.

16. Show that $\int_0^{\pi/2}\sqrt{\tan x}+\sqrt{\cot x}=\sqrt{2}\pi$

17. Prove that the curves $x=y^2$ and $xy=k$ intersect at right angles if $8k^2=1$.

18. Solve the following differential equation:

$x\dfrac{dy}{dx}+y=x\log x;\ \ x\neq 0$

19. Form the differential equation representing the parabolas having vertex at the origin and axis along positive direction of x-axis.

OR

Solve the following differential equation:

$(3xy+y^2)dx+(x^2+xy)dy=0$

20. If $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+5\hat{j}$, $3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ are the position vectors of the points A, B, C and D, find the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Deduce that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

21. Find the equation of the line passing through the point $P(4, 6, 2)$ and the point of intersection of the line $\dfrac{x-1}{3}=\dfrac{y}{2}=\dfrac{z+1}{7}$ and the plane $x+y-z=8$.

22. A and B throw a pair of die turn by turn. The first to throw 9 is awarded a prize. If A starts the game, show that the probability of A getting the prize is $\dfrac{9}{17}$.

SECTION–C

23. Using matrices, solve the following system of linear equations:

$$2x - y + z = 3$$
$$-x + 2y - z = -4$$
$$x - y + 2z = 1$$

OR

Using elementary transformations, find the inverse of the following matrix:

$$\begin{bmatrix} 2 & -1 & 4 \\ 4 & 0 & 2 \\ 3 & -2 & 7 \end{bmatrix}$$

24. Find the maximum area of the isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with its vertex at one end of major axis.

OR

Show that the semi–vertical angle of the right circular cone of given total surface area and maximum volume is $\sin^{-1}\frac{1}{3}$.

25. Find the area of that part of the circle $x^2 + y^2 = 16$ which is exterior to the parabola $y^2 = 6x$.

26. Evaluate: $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\, dx$

27. Find the distance of the point $(-2, 3, -4)$ from the line $\frac{x+2}{3} = \frac{2y+3}{4} = \frac{3z+4}{5}$ measured parallel to the plane $4x + 12y - 3z + 1 = 0$.

28. An aeroplane can carry a maximum of 200 passengers. A profit of Rs. 400 is made on each first class ticket and a profit of Rs. 300 is made on each second class ticket. The airline reserves at least 20 seats for first class. However, at least four times as many passengers prefer to travel by second class then by first class. Determine how many tickets of each type must be sold to maximise profit for the airline. Form an LPP and solve it graphically.

29. A man is known to speak truth 3 out of 4 times. He throws a die and report that it is a 6. Find the probability that it is actually 6.

Set–II

Only those questions, not included in Set I, are given.

20. Using properties of determinants, prove the following:

$$\begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} = 9b^2(a+b)$$

21. Evaluate: $\int_0^{\pi/2} \log \sin x \, dx$

22. Solve the following differential equation:

$$(1+x^2)\frac{dy}{dx} + y = \tan^{-1} x$$

27. Using matrices, solve the following system of linear equations:

$$3x - 2y + 3z = 8$$
$$2x + y - z = 1$$
$$4x - 3y + 2z = 4$$

OR

Using elementary transformations, find the inverse of the following matrix:

$$\begin{bmatrix} 2 & 5 & 3 \\ 3 & 4 & 1 \\ 1 & 6 & 2 \end{bmatrix}$$

28. An insurance company insured 2000 scooter drivers, 3000 car drivers and 4000 truck drivers. The probabilities of their meeting with an accident respectively are 0.04, 0.06 and 0.15. One of the insured persons meets with an accident. Find the probability that he is a car driver.

29. Using integration, find the area bounded by the lines $x + 2y = 2$, $y - x = 1$ and $2x + y = 7$.

Set–III

Only those questions, not included in Set I and Set II are given.

20. If a, b and c are all positive and distinct, show that

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \text{ has a negative value.}$$

21. Evaluate: $\int_0^1 \cot^{-1}(1 - x + x^2)\, dx$

22. Solve the following differential equation:

$$x \log x \frac{dy}{dx} + y = 2 \log x$$

27. Using matrices, solve the following system of linear equations:

$$x + y + z = 4$$
$$2x + y - 3z = -9$$
$$2x - y + z = -1$$

OR

Using elementary transformations, find the inverse of the following matrix:

$$\begin{bmatrix} 2 & 5 & 3 \\ 3 & 4 & 1 \\ 1 & 6 & 3 \end{bmatrix}$$

28. Find the area bounded by the curves $(x-1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$.

29. An insurance company insured 3000 scooter drivers, 5000 car drivers and 7000 truck drivers. The probabilities of their meeting with an accident respectively are 0.04, 0.05 and 0.15 One of the insured persons meets with an accident. Find the probability that he is a car driver.

SOLUTIONS

Set – I

SECTION–A

1. Given $f(x) = \dfrac{3x-2}{5}$

Let $y = \dfrac{3x-2}{5}$

$\Rightarrow \quad 3x - 2 = 5y \quad \Rightarrow \quad x = \dfrac{5y+2}{3}$

$\Rightarrow \quad f^{-1}(x) = \dfrac{5x+2}{3}$

2. $\tan^{-1}\left(\dfrac{1-x}{1+x}\right) = \dfrac{1}{2}\tan^{-1} x$

$\Rightarrow \quad 2\tan^{-1}\left(\dfrac{1-x}{1+x}\right) = \tan^{-1} x$

$\Rightarrow \quad \tan^{-1}\dfrac{2\left(\dfrac{1-x}{1+x}\right)}{1-\left(\dfrac{1-x}{1+x}\right)^2} = \tan^{-1} x$

$\Rightarrow \quad \tan^{-1} 2\left(\dfrac{1-x}{1+x}\right)\dfrac{(1+x)^2}{(1+x)^2-(1-x)^2} = \tan^{-1} x$

$\Rightarrow \quad \tan^{-1}\dfrac{2(1+x)(1-x)}{4x} = \tan^{-1} x$

$\Rightarrow \quad \tan^{-1}\left(\dfrac{1-x^2}{2x}\right) = \tan^{-1} x$

$\Rightarrow \quad \dfrac{1-x^2}{2x} = x \quad \Rightarrow \quad 1 - x^2 = 2x^2$

$\Rightarrow \quad 3x^2 = 1 \qquad \Rightarrow \quad x^2 = \frac{1}{3}$

$\Rightarrow \quad x = \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \qquad x = \frac{1}{\sqrt{3}} \quad (\because x > 0)$

3. Given $\begin{bmatrix} x+3y & y \\ 7-x & 4 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 0 & 4 \end{bmatrix}$

Hence $\quad x + 3y = 4 \qquad$ ...(*i*)

$y = -1 \qquad$ ...(*ii*)

$7 - x = 0 \qquad$ ...(*iii*)

$\Rightarrow \quad x = 7, y = -1$

4. Since $\begin{vmatrix} 1 & 0 & 1 \\ 6 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0$

Hence (1, 0), (6, 0) and (0, 0) are collinear.

5. Let $\quad I = \int \frac{x + \cos 6x}{3x^2 + \sin 6x}\, dx$

Let $\quad 3x^2 + \sin 6x = t$

$\Rightarrow \quad (6x + 6\cos 6x)\, dx = dt$

$\Rightarrow \quad (x + \cos 6x)\, dx = \frac{dt}{6}$

$\therefore \quad I = \int \frac{dt}{6t} = \frac{1}{6} \log|t| + C = \frac{1}{6} \log|3x^2 + \sin 6x| + C$

6. $\int (e^{ax} + bx)\, dx = 4e^{4x} + \frac{3x^2}{2}$

Differentiating both sides, we get

$(e^{ax} + bx) = 16e^{4x} + 3x$

On comparing, we get $b = 3$

But a cannot be found out.

7. $|\vec{a}| = \sqrt{3}, \ |\vec{b}| = 2$

$\vec{a} \cdot \vec{b} = |\vec{a}| . |\vec{b}| \cos\theta$

$= \sqrt{3} . 2 . \cos 60° = \sqrt{3}$

8. $\vec{a} = \hat{i} - 2\hat{j}$

Unit vector in the direction of $\vec{a} = \frac{\hat{i} - 2\hat{j}}{\sqrt{5}}$

Hence a vector in the direction of $\vec{a}$ having magnitude 7 will be $\frac{7}{\sqrt{5}}\hat{i} - \frac{14}{\sqrt{5}}\hat{j}$.

9. The direction ratios of line parallel to AB is 1, –2 and 4.

10. $\begin{vmatrix} x+2 & 3 \\ x+5 & 4 \end{vmatrix} = 3$

$\Rightarrow \quad 4x + 8 - 3x - 15 = 3$

$\Rightarrow \quad x - 7 = 3$

$\Rightarrow \quad x = 10$

SECTION–B

11. (*i*) Reflexive

R is reflexive if $T_1 R T_1 \; \forall T_1$

Since $T_1 \cong T_1$

$\therefore \quad R$ is reflexive.

(*ii*) Symmetric

R is symmetric if $T_1 R T_2 \Rightarrow T_2 R T_1$

Since $T_1 \cong T_2 \Rightarrow T_2 \cong T_1$

$\therefore \quad R$ is symmetric.

(*iii*) Transitive

R is transitive if

$T_1 R T_2$ and $T_2 R T_3 \Rightarrow T_1 R T_3$

Since $T_1 \cong T_2$ and $T_2 \cong T_3 \Rightarrow T_1 \cong T_3$

$\therefore \quad R$ is transitive

From (*i*), (*ii*) and (*iii*), we get

R is an equivalence relation.

12. L.H.S. $= \tan\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right) + \tan\left(\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right)$

$$= \frac{\tan\frac{\pi}{4} + \tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}{1 - \tan\frac{\pi}{4}\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)} + \frac{\tan\frac{\pi}{4} - \tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}{1 + \tan\frac{\pi}{4}\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}$$

$$= \frac{1 + \tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}{1 - \tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)} + \frac{1 - \tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}{1 + \tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}$$

$$= \frac{\left[1 + \tan\left(\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right)\right]^2 + \left[1 - \tan\left(\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right)\right]^2}{1 - \tan^2\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}$$

$$= \frac{2\sec^2\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}{1-\tan^2\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)} = \frac{2\sec^2\theta}{1-\tan^2\theta} = \frac{2(1+\tan^2\theta)}{1-\tan^2\theta} \qquad \left[\text{Let } \frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right) = \theta\right]$$

$$= \frac{2}{\cos 2\theta} = \frac{2}{\cos 2\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)} = \frac{2}{\frac{a}{b}}$$

$$= \frac{2b}{a} \quad = \text{R. H.S.}$$

OR

We have $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\frac{8}{31}$

$$\Rightarrow \qquad \tan^{-1}\left[\frac{(x+1)+(x-1)}{1-(x^2-1)}\right] = \tan^{-1}\frac{8}{31}$$

$$\Rightarrow \qquad \frac{2x}{2-x^2} = \frac{8}{31}$$

$$\Rightarrow \qquad 62x = 16 - 8x^2$$

$$\Rightarrow \qquad 8x^2 + 62x - 16 = 0$$

$$\Rightarrow \qquad 4x^2 + 31x - 8 = 0$$

$$\Rightarrow \qquad x = \frac{1}{4} \text{ and } x = -8$$

As $x = -8$ does not satisfy the equation

Hence $x = \frac{1}{4}$ is only solution..

13. Let $\qquad \Delta = \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

Taking common $2(a+b+c)$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & 0 \\ 1 & 0 & c+a+2b \end{vmatrix} \qquad [\text{by } R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1]$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & a+b+c & 0 \\ 0 & 0 & a+b+c \end{vmatrix}$$

$= 2\,(a+b+\ c)\{(a+b+c)^2 - 0\}$ expanding along C_1.

$= 2(a+b+c)^3 = \text{RHS}$

14. At $x = 0$

L.H.L. $= \lim\limits_{h \to 0} \dfrac{(0-h)^4 + 2\,(0-h)^3 + (0-h)^2}{\tan^{-1}(0-h)}$

$= \lim\limits_{h \to 0} \dfrac{h^4 - 2h^3 + h^2}{-\tan^{-1} h} = \lim\limits_{h \to 0} \dfrac{h^3 - 2h^2 + h}{-\dfrac{\tan^{-1} h}{h}}$

[On dividing numerator and denominator by h.]

$= \dfrac{0}{-1}$ $\left(as \lim\limits_{h \to 0} \dfrac{\tan^{-1} h}{h} = 0\right)$

$= 0$

R.H.L $= \lim\limits_{h \to 0} \dfrac{(0+h)^4 + 2\,(0+h)^3 + (0+h)^2}{\tan^{-1}(0+h)}$

$= \lim\limits_{h \to 0} \dfrac{h^4 + 2h^3 + h^2}{\tan^{-1} h}$

$= \lim\limits_{h \to 0} \dfrac{h^3 + 2h^2 + h}{\dfrac{\tan^{-1} h}{h}}$ (on dividing numerator and denominator by h)

$= \dfrac{0}{1}$ $\left(as \lim\limits_{h \to 0} \dfrac{\tan^{-1} h}{h} = 1\right)$

$= 0$

and $f(0) = 0$ (given)

so, L.H.L = R.H.L = $f(0)$

Hence given function is continuous at $x = 0$

OR

$f(x) = x^2 + 2x + 3$ for $[4, 6]$

(i) Given function is a polynomial hence it is continuous

(ii) $f'(x) = 2x + 2$ which is differentiable

$f(4) = 16 + 8 + 3 = 27$

$f(6) = 36 + 12 + 3 = 51$

$\Rightarrow$ $f(4) \neq f(6)$. All conditions of Mean value theorem are satisfied.

$\therefore$ these exist atleast one real value $C \in (4,6)$

such that $f'(c) = \dfrac{f(6)-f(4)}{6-4} = \dfrac{24}{2} = 12$

$\Rightarrow$ $2c + 2 = 12$ or $c = 5 \in (4,6)$

Hence, Lagrange's mean value theorem is verified

15. $f(x) = \sqrt{\dfrac{\sec x - 1}{\sec x + 1}} = \sqrt{\dfrac{1-\cos x}{1+\cos x} \times \dfrac{1-\cos x}{1-\cos x}}$

$\Rightarrow$ $f(x) = \dfrac{1-\cos x}{\sin x} = \operatorname{cosec} x - \cot x$

$\Rightarrow$ $f'(x) = -\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x$

$\Rightarrow$ $f'(\pi/2) = -1 \times 0 + 1^2$

$\Rightarrow$ $f'(\pi/2) = 1$

OR

We have,

$x\sqrt{1+y} + y\sqrt{1+x} = 0$

$\Rightarrow$ $x\sqrt{1+y} = -y\sqrt{1+x}$

$\Rightarrow$ $\dfrac{x}{y} = -\dfrac{\sqrt{x+1}}{\sqrt{1+y}}$

$\Rightarrow$ $\dfrac{x^2}{y^2} = \dfrac{x+1}{y+1}$

$\Rightarrow$ $x^2y + x^2 = xy^2 + y^2$

$\Rightarrow$ $x^2y - xy^2 + x^2 - y^2 = 0$

$\Rightarrow$ $xy(x-y) + (x-y)(x+y) = 0$

$\Rightarrow$ $(x-y)(xy+x+y) = 0$

but $x \neq y$ $\quad\therefore\quad xy + x + y = 0$

$y(1+x) = -x$ $\quad\therefore\quad y = \dfrac{-x}{1+x}$

$\therefore \dfrac{dy}{dx} = -\left[\dfrac{(1+x).1 - x \times 1}{(1+x)^2}\right] = \dfrac{-1}{(1+x)^2}$

16. $\displaystyle\int_0^{\pi/2} \{\sqrt{\tan x} + \sqrt{\cot x}\}dx$

$\displaystyle\int_0^{\pi/2} \left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right)dx$

$\displaystyle = \sqrt{2}\int_0^{\pi/2} \frac{(\sin x + \cos x)}{\sqrt{2\sin x \cos x}}dx = \sqrt{2}\int_0^{\pi/2} \frac{(\sin x + \cos x)}{\sqrt{\quad 2}}dx$

$1 - (\sin x - \cos x)$

Let $\sin x - \cos x = t$

$(\cos x + \sin x)\, dx = dt$

Now $x = 0 \Rightarrow t = -1$, and $x = \frac{\pi}{2} \Rightarrow t = 1$

$$\therefore \int_0^{\pi/2} \{\sqrt{\tan x} + \sqrt{\cot x}\}\, dx$$

$$= \sqrt{2} \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} = \sqrt{2}\left[\sin^{-1} t\right]_{-1}^{1}$$

$$= \sqrt{2}\,[\sin^{-1} 1 - \sin^{-1}(-1)]$$

$$= \sqrt{2}\,[2 \sin^{-1} 1].$$

$$= 2\sqrt{2}\left(\frac{\pi}{2}\right) = \sqrt{2}\,\pi = \text{RHS}$$

17. Given curves $x = y^2$ ...(*i*)

$xy = k$ ...(*ii*)

Solving (*i*) and (*ii*), $y^3 = k \;\therefore y = k^{1/3},\ x = k^{2/3}$

Differentiating (*i*) w. r. t. x, we get

$$1 = 2y\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2y}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(k^{2/3},\, k^{1/3})} = \frac{1}{2k^{1/3}} = m_1$$

And differentiating (*ii*) w.r.t. x we get

$$x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(k^{2/3},\, k^{1/3})} = -\frac{k^{1/3}}{k^{2/3}} = -k^{-1/3} = m_2$$

$$\therefore \quad m_1 m_2 = -1$$

$$\Rightarrow \quad -\frac{1}{2k^{1/3}}\frac{1}{k^{1/3}} = -1 \quad \Rightarrow \quad k^{2/3} = 1/2 \quad \Rightarrow \quad 8k^2 = 1$$

18. Given $x\frac{dy}{dx} + y = x \log x$ ...(*i*)

$$\frac{dy}{dx} + \frac{y}{x} = \log x$$

This is linear differential equation

Integrating factor I.F. $= e^{\int \frac{1}{x}\,dx} = e^{\log_e x} = x$ Multiplying both sides of (i) by I.F. $= x$, we get

$$x\frac{dy}{dx} + y = x \log x$$

Integrating with respect to x, we get

$$y.\,x = \int x.\log x\, dx$$

$$\Rightarrow \quad xy = \log x.\frac{x^2}{2} - \int \frac{x^2}{2}.\frac{1}{x}\,dx$$

$$\Rightarrow \quad xy = \frac{x^2 \log x}{2} - \frac{1}{2}\frac{x^2}{2} + C$$

$$\Rightarrow \quad y = \frac{x}{2}\left(\log x - \frac{1}{2}\right) + C$$

19. Given $y^2 = 4ax$ ...(i)

$$\Rightarrow \quad 2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad y.\frac{dy}{dx} = 2a \qquad \therefore \qquad y\frac{dy}{dx} = 2.\frac{y^2}{4x} \qquad \text{(from } (i))$$

$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{2x}$ which is the required differential equation

OR

We have, $(3xy - y^2)dx + (x^2 + xy)\,dy = 0$

$$(3xy - y^2)dx = -(x^2 + xy)\,dy$$

$$\frac{dy}{dx} = \frac{y^2 - 3xy}{x^2 + xy}$$

Let $\quad y = Vx$

$$\frac{dy}{dx} = \left(V + x\frac{dV}{dx}\right)$$

$$\therefore \quad \left(V + x\frac{dV}{dx}\right) = \frac{V^2x^2 - 3x.V.x}{x^2 + x.Vx}$$

$$\Rightarrow \quad V + x\frac{dV}{dx} = \frac{V^2 - 3V}{1 + V}$$

$$\Rightarrow \quad x\frac{dV}{dx} = \frac{V^2 - 3V}{1 + V} - V$$

$\Rightarrow \quad x\dfrac{dV}{dx} = \dfrac{V^2 - 3V - V - V^2}{(1+V)} = \dfrac{-4V}{1+V}$

$\Rightarrow \quad \displaystyle\int \frac{1+V}{V}\, dV = -4\int \frac{dx}{x}$

$\Rightarrow \quad \displaystyle\int \frac{1}{V}\, dV + \int dV = -4\int \frac{dx}{x}$

$\Rightarrow \quad \log V + V = -4 \log x + C$

$\log V + \log x^4 + V = C$

$\Rightarrow \quad \log (V.x^4) + V = C$

$\Rightarrow$

$\Rightarrow \quad \log\left(\dfrac{y}{x}x^4\right) + \dfrac{y}{x} = C \quad \text{or } x\log(x^3 y) + y = Cx$

20. Given

$\overrightarrow{OA} = \hat{i} + \hat{j} + \hat{k}$

$\overrightarrow{OB} = 2\hat{i} + 5\hat{j}$

$\overrightarrow{OC} = 3\hat{i} + 2\hat{j} - 3\hat{k}$

$\overrightarrow{OD} = \hat{i} - 6\hat{j} - \hat{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \hat{i} + 4\hat{j} - \hat{k}$

$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = -2\hat{i} - 8\hat{j} + 2\hat{k}$

$\overrightarrow{CD} = -2(\hat{i} + 4\hat{j} - \hat{k})$

$\overrightarrow{CD} = -2\,\overrightarrow{AB}$

Therefore $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel vector so $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear and angle between them is zero.

21. Let $\dfrac{x-1}{3} = \dfrac{y}{2} = \dfrac{z+1}{7} = \lambda$ ...(*i*)

Coordinates of any general point on line (*i*) is of the form $\equiv (1 + 3\lambda, 2\lambda, -1 + 7\lambda)$

For point of intersection

$(1 + 3\lambda) + 2\lambda - (7\lambda - 1) = 8$

$1 + 3\lambda + 2\lambda - 7\lambda + 1 = 8$

$-2\lambda = 6$

$\lambda = -3$

Point of intersection $\equiv (-8, -6, -22)$

$\therefore$ Required equation of line passing through P (4, 6, 2) and Q (– 8, – 6, – 22) is:

$$\frac{x-4}{4+8} = \frac{y-6}{6+6} = \frac{z-2}{2+22}$$

$$\therefore \quad \frac{x-4}{12} = \frac{y-6}{12} = \frac{z-2}{24}. \text{ or } x-4 = y-6 = \frac{z-2}{2}$$

22. Let E be the event that sum of number on two die is 9.

$E = \{(3, 6), (4, 5), (5, 4), (6, 3)\}$

$$P(E) = \frac{4}{36} = \frac{1}{9}$$

$$P(E') = \frac{8}{9}$$

$$P \text{ (A getting the prize } P(A) = \frac{1}{9} + \frac{8}{9}\times\frac{8}{9}\times\frac{1}{9} + \frac{8}{9}\times\frac{8}{9}\times\frac{8}{9}\times\frac{8}{9}\times\frac{1}{9} + \ldots\ldots$$

$$= \frac{1}{9}\left(1 + \left(\frac{8}{9}\right)^2 + \left(\frac{8}{9}\right)^4 + \left(\frac{8}{9}\right)^6 + \ldots.\right)$$

$$= \frac{1}{9}\frac{1}{\left[1-\left(\frac{8}{9}\right)^2\right]} = \frac{1}{9}\cdot\frac{9^2}{(9^2-8^2)} = \frac{9}{17}.$$

SECTION–C

23. Given System of linear equations

$$2x - y + z = 3$$
$$-x + 2y - z = -4$$
$$x - y + 2z = 1$$

we can write these equations as

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad AX = B, \text{ where, } A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad X = A^{-1}B \qquad \ldots(i)$$

Now, $|A| = 2(4-1) - (-1)(-2+1) + 1(1-2)$

$= 6 - 1 - 1 = 4$

Again Co-factors of elements of matrix A are given by

$$C_{11} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 4 - 1 = 3$$

$$C_{12} = -\begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} = -(-2 + 1) = 1$$

$$C_{13} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = (1 - 2) = -1$$

$$C_{21} = -\begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix} = -(-2 + 1) = 1$$

$$C_{22} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = (4 - 1) = 3$$

$$C_{23} = -\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} = -(-2 + 1) = 1$$

$$C_{31} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} = (1 - 2) = -1$$

$$C_{32} = -\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = -(-2 + 1) = 1$$

$$C_{33} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 4 - 1 = 3$$

$$\therefore \quad \text{adj } A = (C)^T = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{adj.\ A}{|A|} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$\therefore$ From (i), we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ -8 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$$\Rightarrow \qquad x = 1,\ y = -2,\ z = -1$$

OR

$$A = I_3 . A$$

$$\begin{bmatrix} 2 & -1 & 4 \\ 4 & 0 & 2 \\ 3 & -2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 2 & -1 & 4 \\ 0 & 2 & -6 \\ 3 & -2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to 1/2R_1$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 2 \\ 0 & 2 & -6 \\ 3 & -2 & 7 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 2 \\ 0 & 2 & -6 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -2 & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 / 2$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 2 \\ 0 & 1 & -3 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & \frac{1}{2} & 0 \\ -\frac{3}{2} & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 + R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} -2 & \frac{1}{2} & 0 \\ -1 & \frac{1}{2} & 0 \\ -\frac{3}{2} & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \to R_3 + 1/2R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -2 & \frac{1}{2} & 1 \\ -1 & \frac{1}{2} & 0 \\ -2 & \frac{1}{4} & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 6R_3$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -\frac{1}{2}\end{bmatrix} = \begin{bmatrix}-2 & \frac{1}{2} & 1\\ 11 & -1 & -6\\ -2 & \frac{1}{4} & 1\end{bmatrix} A$$

Applying $R_3 \to -2R_3$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}-2 & \frac{1}{2} & 1\\ 11 & -1 & -6\\ 4 & -\frac{1}{2} & -2\end{bmatrix} A$$

Hence $A^{-1} = \begin{bmatrix}-2 & \frac{1}{2} & 1\\ 11 & -1 & -6\\ 4 & -\frac{1}{2} & -2\end{bmatrix}$

24. Let ΔABC be an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
Then coordinates of points A and B are given by $(a\cos\theta, b\sin\theta)$ and $(a\cos\theta, -b\sin\theta)$ The area of the isosceles $\Delta ABC = \frac{1}{2} \times AB \times CD$

$\Rightarrow \quad A(\theta) = \frac{1}{2} \times (2b\sin\theta) \times (a - a\cos\theta)$

$\Rightarrow \quad A(\theta) = ab\sin\theta\,(1 - \cos\theta)$

For A_{max}

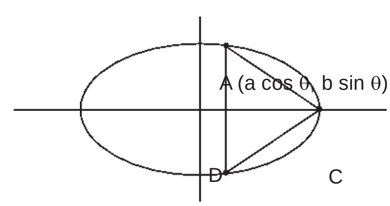
$$\frac{d(A(\theta))}{d\theta} = 0$$

$\Rightarrow \quad ab[\cos\theta(1 - \cos\theta) + \sin^2\theta] = 0$

$\cos\theta - \cos^2\theta + \sin^2\theta = 0$

$\Rightarrow \quad \cos\theta - \cos 2\theta = 0$

$\Rightarrow \quad \theta = \frac{2\pi}{3}$



Now, $\quad \frac{d^2(A(\theta))}{d\theta^2} = ab\,[-\sin\theta + 2\sin 2\theta]$

For $\quad \theta = \frac{2\pi}{3}, \frac{d^2(A(\theta))}{d\theta^2} = ab\left(-\frac{\sqrt{3}}{2} - 2 \times \frac{\sqrt{3}}{2}\right) < 0$

Hence for $\theta = \frac{2\pi}{3}$, A_{max} occurs

$$\therefore \quad A_{max} = ab \sin\frac{2\pi}{3}\left(1-\cos\frac{2\pi}{3}\right) \text{ square units}$$

$$= ab\frac{\sqrt{3}}{2}\left(1+\frac{1}{2}\right) = \frac{3\sqrt{3}}{4}ab \text{ square units}$$

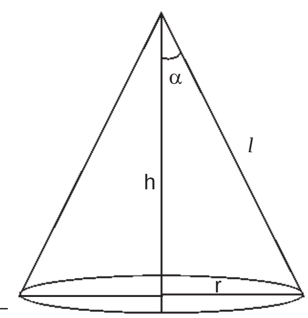
OR

Let r be the radius, l be the slant height and h be the vertical height of a cone of semi - vertical angle α.

Surface area $\quad S = \pi rl + \pi r^2 \quad$ ...(*i*)

or $\quad l = \frac{S - \pi r^2}{\pi r}$

The volume of the cone



$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2\sqrt{l^2 - r^2}$$

$$= \frac{\pi r^2}{3}\sqrt{\frac{(S-\pi r^2)^2}{\pi^2 r^2} - r^2}$$

$$= \frac{\pi r^2}{3}\sqrt{\frac{(S-\pi r^2)^2 - \pi^2 r^4}{\pi^2 r^2}}$$

$$= \frac{\pi r^2}{3}\times\frac{\sqrt{S^2 - 2\pi S r^2 + \pi^2 r^4 - \pi^2 r^4}}{\pi r} = \frac{r}{3}\sqrt{S(S-2\pi r^2)}$$

$$\therefore \quad V^2 = \frac{r^2}{9}S(S-2\pi r^2) = \frac{S}{9}(Sr^2 - 2\pi r^4)$$

$$\frac{dV^2}{dr} = \frac{S}{9}(2Sr - 8\pi r^3)$$

$$\frac{d^2V^2}{dr^2} = \frac{S}{9}(2S - 24\pi r^2) \quad \text{...(ii)}$$

Now $\quad \frac{dV^2}{dr} = 0$

$$\Rightarrow \quad \frac{S}{9}(2Sr - 8\pi r^3) = 0 \quad \text{or} \quad S - 4\pi r^2 = 0 \quad \Rightarrow \quad S = 4\pi r^2$$

Putting $S = 4\pi r^2$ in (*ii*),

$$\frac{d^2V^2}{dr^2} = \frac{4\pi r^2}{9}[8\pi r^2 - 24\pi r^2] < 0$$

$\Rightarrow$ V is maximum when $S = 4\pi r^2$

Putting this value of S in (*i*)

$$4\pi r^2 = \pi rl + \pi r^2$$

or $\quad 3\pi r^2 = \pi rl$

or $\quad \dfrac{r}{l} = \sin\alpha = \dfrac{1}{3}$

$\therefore \quad \alpha = \sin^{-1}\left(\dfrac{1}{3}\right)$

Thus V is maximum, when semi vertical angle is $\sin^{-1}\left(\dfrac{1}{3}\right)$.

25. First finding intersection point by solving the equation of two curves

$$x^2 + y^2 = 16 \qquad \ldots(i)$$

and $$y^2 = 6x \qquad \ldots(ii)$$

$\Rightarrow \quad x^2 + 6x = 16$

$\Rightarrow \quad x^2 + 6x - 16 = 0$

$\Rightarrow \quad x^2 + 8x - 2x - 16 = 0$

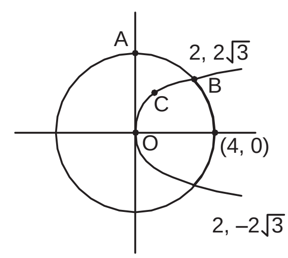
$\Rightarrow \quad x(x+8) - 2(x+8) = 0$

$\Rightarrow \quad (x+8)(x-2) = 0$

$x = -8$ (not possible Q y^2 can not be – ve)

or $\quad x = 2$ (only allowed value)

$\therefore \quad y = \pm 2\sqrt{3}$



Area of $OABCO = \displaystyle\int_0^{2\sqrt{3}} \left(\sqrt{16 - y^2} - \frac{y^2}{6}\right) dy$

$$= \left[\frac{y}{2}\sqrt{16 - y^2} + \frac{16}{2}\sin^{-1}\frac{y}{4} - \frac{y^3}{18}\right]_0^{2\sqrt{3}}$$

$$\left[\int \sqrt{a^2 - x^2} = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \left[\sqrt{3}\,.\,\sqrt{16 - 12} + 8\sin^{-1}\frac{\sqrt{3}}{2} - \frac{3\sqrt{24}}{18}\right]$$

$$= \left[\sqrt{3}\,.\,2 + 8\,\frac{\pi}{3} - \frac{4}{\sqrt{3}}\right] = 2\sqrt{3} - \frac{4}{\sqrt{3}} + \frac{8}{3}\pi = \frac{2}{3}\sqrt{3} + \frac{8}{3}$$

π $\therefore$ Required are $= 2\left(\dfrac{2\sqrt{3}}{3} + \dfrac{8}{3}\pi\right) + \dfrac{1}{2}(\pi 4^2)$

$$= \frac{4\sqrt{3}}{3} + \frac{16}{3}\pi + 8\pi = \frac{4\sqrt{3}}{3} + \frac{40}{3}$$

$\pi = \dfrac{4}{3}(\sqrt{3} + 10\pi)$ sq. units

26. $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\, dx$ ...(*i*)

Using property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, we have

$$\therefore \quad I = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)}\, dx$$

$$I = \int_0^{\pi} \frac{(\pi - x)(-\tan x)}{-\sec x - \tan x}\, dx$$

$$I = \int_0^{\pi} \frac{\pi . \tan x}{\sec x + \tan x}\, dx - \int_0^{\pi} \frac{x . \tan x}{\sec x + \tan x}\, dx \quad \text{...(ii)}$$

Adding (*i*) and (*ii*) we have

$$2I = \pi \int_0^{\pi} \frac{\tan x}{\sec x + \tan x}\, dx$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{\sin x}{1 + \sin x}\, dx$$

$[f(x) = f(2a - x)]$ then $\int_0^{2a} f(x)\,dx = 2 . \int_0^a f(x)\,dx$

$$\Rightarrow \quad 2I = \pi \times 2 \times \int_0^{\pi/2} \frac{\sin x}{1 + \sin x}\, dx$$

$$\Rightarrow \quad I = \pi \int_0^{\pi/2} \frac{\sin x + 1 - 1}{1 + \sin x}\, dx$$

$$\Rightarrow \quad I = \pi \int_0^{\pi/2} dx - \pi \int_0^{\pi/2} \frac{1}{1 + \sin x}\, dx$$

$$\Rightarrow \quad I = \pi \frac{\pi}{2} - \pi \int_0^{\pi/2} \frac{1}{1 + \cos x}\, dx \qquad \left[\text{Using} \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\Rightarrow \quad I = \frac{\pi^2}{2} - \pi \int_0^{\pi/2} \frac{1}{2\cos^2 \frac{x}{2}}\, dx$$

$$\Rightarrow \quad I = \frac{\pi^2}{2} - \frac{\pi}{2} . \int_0^{\pi/2} \sec^2 \frac{x}{2} . dx$$

$$\Rightarrow \quad I = \frac{\pi^2}{2} - \frac{\pi}{2} . \left[\frac{\tan \frac{x}{2}}{\frac{1}{2}}\right]_0^{\pi/2}$$

$$I = \frac{\pi^2}{2} - \frac{\pi}{2} \times 2 \times \left[\tan \frac{\pi}{4} - \tan 0\right]$$

$$I = \frac{\pi^2}{2} - \pi$$

27. Let $\dfrac{x+2}{3} = \dfrac{2y+3}{4} = \dfrac{3z+4}{5} = \lambda$

Any general point on the line is

$$3\lambda - 2, \quad \frac{4\lambda - 3}{2}, \quad \frac{5\lambda - 4}{3}$$

Now, direction ratio if a point on the line is joined to (– 2, 3, – 4) are

$$\Rightarrow \quad 3\lambda, \frac{4\lambda - 9}{2}, \frac{5\lambda + 8}{3}$$

Now the distance is measured parallel to the plane

$$4x + 12y - 3z + 1 = 0$$

$$\therefore \quad 4 \times 3\lambda + 12 \times \left(\frac{4\lambda - 9}{2}\right) - 3 \times \left(\frac{5\lambda + 8}{3}\right) = 0$$

$$\Rightarrow \quad 12\lambda + 24\lambda - 54 - 5\lambda - 8 = 0$$

$$31\lambda - 62 = 0$$

$$\Rightarrow \quad \lambda = 2$$

$\therefore$ The point required is $\left(4, \dfrac{5}{2}, 2\right)$.

$$\therefore \quad \text{Distance} = \sqrt{(4+2)^2 + \left(\frac{5}{2} - 3\right)^2 + (2+4)^2}$$

$$= \sqrt{36 + 36 + \frac{1}{4}} = \sqrt{\frac{289}{4}} = \frac{17}{2} \text{ units}$$

28. Let there be x tickets of first class and y tickets of second class. Then the problem is to

$$\max \ z = 400x + 300y$$

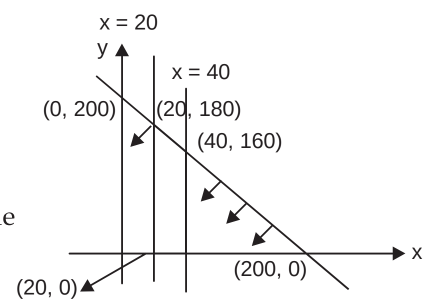
Subject to $x + y \le 200$

$$x \ge 20$$

$$x + 4x \le 200$$

$$5x \le 200$$

$$x \le 40$$



The shaded region in the graph represents the feasible region which is proved.

Le us evaluate the value of z at each corner point

z at (20, 0), $z = 400 \times 20 + 300 \times 0 = 8000$

z at (40, 0) $= 400 \times 40 + 300 \times 0 = 16000$

z at (40, 160) $= 400 \times 40 + 300 \times 160 = 16000 + 48000 = 64000$

z at (20, 180) $= 400 \times 20 + 300 \times 180 = 8000 + 54000 = 62000$

$$\max \ z = 64000 \text{ for } x = 40, y = 160$$

$\therefore$ 40 tickets of first class and 160 tickets of second class should be sold to earn maximum profit of Rs. 64,000.

29. A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six. [CBSE 2005]

Sol. Let E be the event that the man reports that six occurs in the throwing of the die and let S_1 be the event that six occurs and S_2 be the event that six does not occur.

Then $P(S_1)$ = Probability that *six* occurs $= \frac{1}{6}$

$P(S_2)$ = Probability that *six* does not occur $= \frac{5}{6}$

$P(E/S_1)$ = Probability that the man reports that *six* occurs when *six* has actually occurred on the die

= Probability that the man speaks the truth $= \frac{3}{4}$

$P(E/S_2)$ = Probability that the man reports that *six* occurs when *six* has not actually occurred on the die

= Probability that the man does not speak the truth $= 1 - \frac{3}{4} = \frac{1}{4}$.

Thus, by Bayes' theorem, we get

$P(S_1/E)$ = Probability that the report of the man that *six* has occurred is actually a *six*

$$= \frac{P(S_1)\,P(E/S_1)}{P(S_1)\,P(E/S_1) + P(S_2)\,P(E/S_2)} = \frac{\frac{1}{6} \times \frac{3}{4}}{\frac{1}{6} \times \frac{3}{4} + \frac{5}{6} \times \frac{1}{4}} = \frac{3}{8}.$$

Set–II

20. Let $\Delta = \begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3$, we have

$$\Delta = \begin{vmatrix} 3(a+b) & 3(a+b) & 3(a+b) \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix}$$

Taking out $3(a+b)$ from 1st row, we have

$$\Delta = 3(a+b)\begin{vmatrix} 1 & 1 & 1 \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$

$$\Delta = 3(a+b)\begin{vmatrix} 0 & 0 & 1 \\ 2b & -b & a+b \\ -b & 2b & a \end{vmatrix}$$

Expanding along first row, we have

$$\Delta = 3(a+b)\,[1.(4b^2 - b^2)]$$

$$= 3\,(a+b) \times 3b^2 = 9b^2\,(a+b)$$

21. Let $I = \int_0^{\pi/2} \log \sin x \, dx$ ...(*i*)

$\Rightarrow \quad I = \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx$

$\Rightarrow \quad I = \int_0^{\pi/2} \log \cos x \, dx$ ...(*ii*)

Adding (*i*) and (*i*) we have,

$2I = \int_0^{\pi/2} (\log \sin x + \log \cos x)\, dx$

$\Rightarrow \quad 2I = \int_0^{\pi/2} \log \sin x \cos x \, dx$

$\Rightarrow \quad 2I = \int_0^{\pi/2} \log \frac{2 \sin x \cos x}{2} \, dx$

$\Rightarrow \quad 2I = \int_0^{\pi/2} (\log \sin 2\,x - \log 2)\, dx$

$\Rightarrow \quad 2I = \int_0^{\pi/2} \log \sin 2x \, dx - \int_0^{\pi/2} \log 2 dx$

Let $2x = t \quad \Rightarrow \quad dx = \frac{dt}{2}$

When $x = 0, \frac{\pi}{2}, t = 0, \pi$

$\therefore \quad 2I = \frac{1}{2}\int_0^{\pi} \log \sin t \, dt - \log 2.\left(\frac{\pi}{2}\right.$ $\left.\right)$

$- 0 \,|\Rightarrow \quad 2I = I - \frac{\pi}{2} \log 2$ $\left[\because \int_0^a f(x)\, dx = \int_0^a f(t)\, dtx\right]$

$\Rightarrow \quad 2I - I = -\frac{\pi}{2} \log 2$

$\Rightarrow \quad I = -\frac{\pi}{2} \log 2$

22. We have

$$(1 + x^2)\frac{dy}{dx} + y = \tan^{-1} x$$

Dividing each term by $(1 + x^2)$

$$\frac{dy}{dx} + \frac{1}{1+x^2} \cdot y = \frac{\tan^{-1} x}{1+x^2}$$

Clearly, it is linear differential equation of the form $\frac{dy}{dx} + P.y = Q$

So, $\qquad P = \frac{1}{1+x^2}$ and $Q = \frac{\tan^{-1} x}{1+x^2}$

$\therefore$ Integrating factor, I. F. $= e^{\int P\,dx} = e^{\int \frac{1}{1+x^2} dx} = e^{\tan^{-1} x}$

Therefore, solution of given differential equation is

$$y \times I.F. = \int Q \times I.F.\, dx$$

$$\Rightarrow \quad y \,.\, e^{\tan^{-1} x} = \int \frac{\tan^{-1} x}{1+x^2} \,.\, e^{\tan^{-1} x}\, dx$$

Let $\qquad I = \int \frac{\tan^{-1} x \, e^{\tan^{-1} x}}{1+x^2}\, dx$

Let $\quad e^{\tan^{-1} x} = t \qquad \Rightarrow \qquad \frac{e^{\tan^{-1} x}}{1+x^2}\, dx = dt$

Also $\quad \tan^{-1} x = \log t$

$\Rightarrow \qquad I = \int \log t \, dt$

$\Rightarrow \qquad I = t \log t - t + C$ [Integrating by parts]

$\Rightarrow \qquad I = e^{\tan^{-1} x} . \tan^{-1} x - e^{\tan^{-1} x} + C$

Hence required solution is

$$y .\, e^{\tan^{-1} x} = e^{\tan^{-1} x} (\tan^{-1} x - 1) + C$$

$$\Rightarrow \qquad y = (\tan^{-1} x - 1) + C\, e^{-\tan^{-1} x}$$

27. The given system of linear equations.

$$3x - 2y + 3z = 8$$
$$2x + y - z = 1$$
$$4x - 3y + 2z = 4$$

We write the system of linear equation in matrix form

$$\begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

$\Rightarrow \quad A.X = B$, where $A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$

$\Rightarrow \quad X = A^{-1} B$

Now, co-factors of matrix A are

$C_{11} = (-1)^{1+1}.(2-3) = (-1)^2 .(-1) = -1$

$C_{12} = (-1)^{1+2}.(4+4) = (-1)^3 .8 = -8$

$C_{13} = (-1)^{1+3}.(-6-4) = (-1)^4 .(-10) = -10$

$C_{21} = (-1)^{2+1}(-4+9) = (-1)^3 (5) = -5$

$C_{22} = (-1)^{2+2}.(6-12) = (-1)^4 (-6) = -6$

$C_{23} = (-1)^{2+3}(-9+8) = (-1)^5 (-1) = 1$

$C_{31} = (-1)^{3+1}(2-3) = (-1)^4 (-1) = -1$

$C_{32} = (-1)^{3+2}(-3-6) = (-1)^5 .(-9) = 9$

$C_{33} = (-1)^{3+3} (3+4) = (-1)^6\, 7 = 7$

$\therefore \quad \text{adj } A = c^T = \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$ Where c = matrix of co-factors of elements.

and $\quad |A| = \begin{vmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{vmatrix} = 3\,(2-3) + 2\,(4+4) + 3(-6-4)$

$= 3 \times -1 + 2 \times 8 + 3 \times -10 = -3 + 16 - 30 = -17$

$\therefore A^{-1} = \dfrac{adj\,A}{|A|} = -\dfrac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$

Now, $X = A^{-1} B$

$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\dfrac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\dfrac{1}{17}\begin{bmatrix} -8 & -5 & -4 \\ -64 & -6 & +36 \\ -80 & +1 & +28 \end{bmatrix} = -\dfrac{1}{17}\begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\therefore \quad x = 1, y = 2, z = 3$

OR

For elementary transformation we have, $A = IA$

$$\Rightarrow \qquad \begin{bmatrix} 2 & 5 & 3 \\ 3 & 4 & 1 \\ 1 & 6 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 - R_3$

$$\Rightarrow \qquad \begin{bmatrix} 1 & -1 & 1 \\ 3 & 4 & 1 \\ 1 & 6 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 3R_1, \quad R_3 \to R_3 - R_1$

$$\Rightarrow \qquad \begin{bmatrix} 1 & -1 & 1 \\ 0 & 7 & -2 \\ 0 & 7 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ -3 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix} A$$

Applying $R_2 \to \frac{4}{7} R_2$

$$\Rightarrow \qquad \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & \frac{-2}{7} \\ 0 & 7 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ \frac{-3}{7} & \frac{1}{7} & \frac{3}{7} \\ -1 & 0 & 2 \end{bmatrix} A$$

Applying $R_1 \to R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & \frac{5}{7} \\ 0 & 1 & \frac{-2}{7} \\ 0 & 7 & 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{7} & \frac{1}{7} & \frac{-4}{7} \\ \frac{-3}{7} & \frac{1}{7} & \frac{3}{7} \\ -1 & 0 & 2 \end{bmatrix} A$$

Applying $R_3 \to R_3 - 7R_2$

$$\begin{bmatrix} 1 & 0 & \frac{5}{7} \\ 0 & 1 & \frac{-2}{7} \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{4}{7} & \frac{1}{7} & \frac{-4}{7} \\ \frac{-3}{7} & \frac{1}{7} & \frac{3}{7} \\ 2 & -1 & -1 \end{bmatrix} A$$

Applying $R_3 \to \frac{R_3}{3}$

$$\begin{bmatrix} 1 & 0 & \frac{5}{7} \\ 0 & 1 & \frac{-2}{7} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{7} & \frac{1}{7} & \frac{-4}{7} \\ \frac{-3}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{2}{3} & \frac{-1}{3} & \frac{-1}{3} \end{bmatrix} A$$

$R_1 \to R_1 \frac{-5}{7} R_3, \quad R_2 \to R_2 + \frac{2}{7} R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{21} & \frac{8}{21} & \frac{-1}{3} \\ \frac{-5}{21} & \frac{1}{21} & \frac{1}{3} \\ \frac{2}{3} & \frac{-1}{3} & \frac{-1}{3} \end{bmatrix} A$$

$$\therefore \quad A^{-1} = \begin{bmatrix} \frac{2}{21} & \frac{8}{21} & \frac{-1}{3} \\ \frac{-5}{21} & \frac{1}{21} & \frac{1}{3} \\ \frac{2}{3} & \frac{-1}{3} & \frac{-1}{3} \end{bmatrix}$$

$$= \frac{1}{21} \begin{bmatrix} 2 & 8 & -7 \\ -5 & 1 & 7 \\ +14 & -7 & -7 \end{bmatrix}$$

28. Let

S = Event of insurance of scooter driver

C = Event of insurance of Car driver

T = Event of insurance of Truck driver

and A = Event of meeting with an accident

Now, we have, $P(S)$ = Probability of insurance of scooter driver

$\Rightarrow \quad P(S) = \frac{2000}{9000} = \frac{2}{9}$

$P(C)$ = Probability of insurance of car driver

$\Rightarrow \quad P(C) = \frac{3000}{9000} = \frac{3}{9}$

$P(T)$ = Probability of insurance of Truck driver

$\Rightarrow \quad P(T) = \frac{4000}{9000} = \frac{4}{9}$

and, $P(A/S)$ = Probability that scooter driver meet. with an accident

$\Rightarrow \quad P(A/S) = 0.04$

$P(A/C)$ = Probability that car driver meet with an accident

$\Rightarrow \quad P(A/C) = 0.06$

$P(A/T)$ = Probability that Truck driver meet with an accident

$\Rightarrow \quad P(A/T) = 0.15$

By Baye's theorem, we have the required probability

$$P(C / A) = \frac{P(C).P(A / C)}{P(S).P(A / S) + P(C).P(A / C) + P(T).P(A / T)}$$

$$= \frac{\frac{3}{9} \times 0.06}{\frac{2}{9} \times 0.04 + \frac{3}{9} \times 0.06 + \frac{4}{9} \times 0.15}$$

$$= \frac{3 \times 0.06}{2 \times 0.04 + 3 \times 0.06 + 4 \times 0.15} = \frac{0.18}{0.08 + 0.18 + 0.60}$$

$$= \frac{0.18}{0.86} = \frac{18}{86} = \frac{9}{43}$$

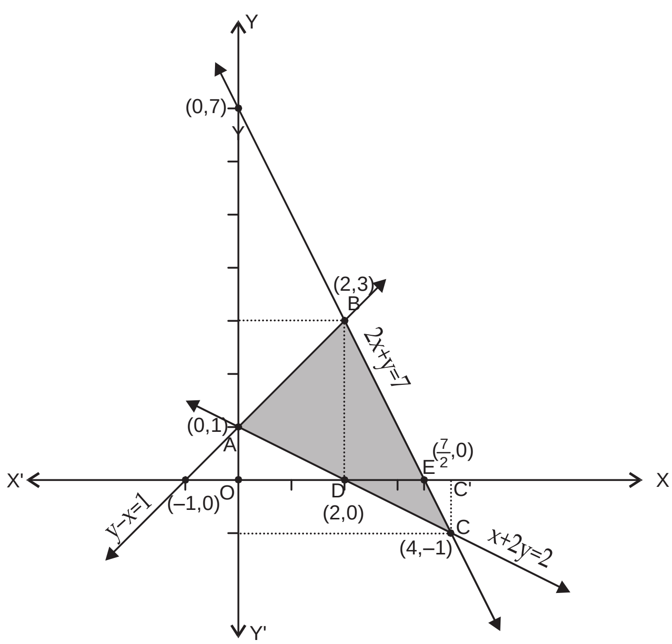
29. Given,

$$x + 2y = 2 \qquad ...(i)$$

$$y - x = 1 \qquad ...(ii)$$

$$2x + y = 7 \qquad ...(iii)$$

On plotting these lines, we have



Area of required region

$$= \int_{-1}^{3} \frac{7 - y}{2} dy - \int_{-1}^{1} (2 - 2y) dy - \int_{1}^{3} (y - 1) dy$$

$$= \frac{1}{2}\left[7y - \frac{y^2}{2}\right]_{-1}^{3} - [2y - y^2]_{-1}^{1} - \left[\frac{y^2}{2} - y\right]_{1}^{3}$$

$$= \frac{1}{2}\left(21 - \frac{9}{2} + 7 + \frac{1}{2}\right) - (2 - 1 + 2 + 1) - \left(\frac{9}{2} - 3 - \frac{1}{2} + 1\right) = 12 - 4 - 2 = 6 \text{ sq. units}$$

Set–III

20. We have

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we have

$$\Delta = \begin{vmatrix} (a+b+c) & b & c \\ (a+b+c) & c & a \\ (a+b+c) & a & b \end{vmatrix}$$

taking out $(a+b+c)$ from Ist column, we have

$$\Delta = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

Interchanging column into row, we have

$$\Delta = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$, we have

$$\Delta = (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ b-c & c-a & a \\ c-a & a-b & b \end{vmatrix}$$

Expanding along Ist row, we have

$$\Delta = (a+b+c)\,[1\,(b-c)\,(a-b) - (c-a)^2]$$

$$= (a+b+c)\,(ba - b^2 - ca + bc - c^2 - a^2 + 2ac)$$

$\Rightarrow$ $\Delta = (a+b+c)\,(ab + bc + ca - a^2 - b^2 - c^2)$

$\Rightarrow$ $\Delta = -\,(a+b+c)\,(a^2 + b^2 + c^2 - ab - bc - ca)$

$\Rightarrow$ $\Delta = -\frac{1}{2}(a+b+c)\,\{(a-b)^2 + (b-c)^2 + (c-a)^2\}$

Here, $(a+b+c)$ is positive as a, b, c are all positive

and it is clear that $(a-b)^2 + (b-c)^2 + (c-a)^2$ is also positive

Hence $\Delta = -\frac{1}{2}(a+b+c)\,[(a-b)^2 + (b-c)^2 + (c-a)^2]$ has negative value.

21. Let $I = \int_0^1 \cot^{-1}(1 - x + x^2)\, dx$

$$= \int_0^1 \tan^{-1} \frac{1}{1 - x + x^2}\, dx \qquad \left[\because \cot^{-1} x = \tan^{-1} \frac{1}{x}\right]$$

$$= \int_0^1 \tan^{-1} \frac{x + (1 - x)}{1 - x(1 - x)}\, dx \qquad [\because 1 \text{ can be written as } x + 1 - x]$$

$$= \int_0^1 [\tan^{-1} x + \tan^{-1}(1 - x)]\, dx \qquad \left[\because \tan^{-1}\left\{\frac{a + b}{1 - ab}\right\} = \tan^{-1} a + \tan^{-1} b\right]$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}(1 - x)\, dx$$

$$= \int_0^1 \tan^{-1} x dx + \int_0^1 \tan^{-1}[1 - (1 - x)]\, dx \qquad \left[\because \int_0^a f(x) = \int_0^a f(a - x)\, dx\right]$$

$$= 2\int_0^1 \tan^{-1} x dx = 2\int_0^1 \tan^{-1} x.1\, dx, \text{ integrating by parts, we get}$$

$$= 2\left[\{\tan^{-1} x.x\}_0^1 - \int_0^1 \frac{1}{1 + x^2} \cdot x dx\right]$$

$$= 2[\tan^{-1} 1 - 0] - \int_0^1 \frac{2x}{1 + x^2}\, dx = 2 \cdot \frac{\pi}{4} - [\log(1 + x^2)]_0^1$$

$$= \frac{\pi}{2} - (\log 2 - \log 1) = \frac{\pi}{2} - \log 2 \qquad [\because \log 1 = 0]$$

22. We have the differential equation

$$x \log x \frac{dy}{dx} + y = 2 \log x$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{1}{x \log x} . y = \frac{2}{x}$$

It is linear differential equation of the from $\frac{dy}{dx} + Py = Q$

So, Here $P = \frac{1}{x \log x}$ and $Q = \frac{2}{x}$

Now, I.F. $= e^{\int p dx} = e^{\int \frac{1}{x \log x} dx} = e^{\log |\log x|}$

$= \log x$

Hence, solution of given differential equation is $y \times I.F. = \int Q \times I.F\, dx$

$$\Rightarrow \quad y \log x = \int \frac{2}{x} . \log x \, dx$$

$$\Rightarrow \quad y \log x = 2\int \frac{1}{x} . \log x \, dx = 2 . \frac{(\log x)^2}{2} + C$$

$$\Rightarrow \quad y \log x = (\log x)^2 + C$$

27. The given system of linear equations is

$$x + y + z = 4$$
$$2x + y - 3z = -9$$
$$2x - y + z = -1$$

We write the system of equation in Matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -3 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -9 \\ -1 \end{bmatrix}$$

$\Rightarrow AX = B$, we have

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -3 \\ 2 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ -9 \\ -1 \end{bmatrix}$$

$\therefore \quad X = A^{-1} B$

Now, co-factors of A

$C_{11} = (-1)^{1+1}(1-3) = -2;$ $\quad C_{12} = (-1)^{1+2}(2+6) = -8$

$C_{13} = (-1)^{1+3}(-2-2) = -4;$ $\quad C_{21} = (-1)^{2+1}(1+1) = -2$

$C_{22} = (-1)^{2+2}(1-2) = -1;$ $\quad C_{23} = (-1)^{2+3}(-1-2) = 3$

$C_{31} = (-1)^{3+1}(-3-1) = -4;$ $\quad C_{32} = (-1)^{3+2}(-3-2) = 5$

$C_{33} = (-1)^{3+3} = (1-2) = -1$

$$\therefore \quad adj \, A = (C)^T = \begin{bmatrix} -2 & -2 & -4 \\ -8 & -1 & 5 \\ -4 & 3 & -1 \end{bmatrix}$$

Now, $|A| = 1(-2) - 1(8) + 1(-4)$

$= -2 - 8 - 4 = -14$

$$\therefore \quad A^{-1} = \frac{adj. A}{|A|}$$

$$= \frac{\begin{bmatrix} -2 & -2 & -4 \\ -8 & -1 & 5 \\ -4 & 3 & -1 \end{bmatrix}}{-14} = \frac{1}{14}\begin{bmatrix} 2 & 2 & 4 \\ 8 & 1 & -5 \\ 4 & -3 & 1 \end{bmatrix}$$

Now, $X = A^{-1}B$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 2 & 2 & 4 \\ 8 & 1 & -5 \\ 4 & -3 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ -9 \\ -1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 8+(-18)+(-4) \\ 32+(-9)+5 \\ 16+27+(-1) \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{14}\begin{bmatrix} -14 \\ 28 \\ 42 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore \quad x = -1$, $y = 2$ and $z = 3$ is the required solution.

OR

Let $\quad A = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 4 & 1 \\ 1 & 6 & 3 \end{bmatrix}$

Therefore, for elementary row transformation, we have

$A = I\,A$

$$\Rightarrow \quad \begin{bmatrix} 2 & 5 & 3 \\ 3 & 4 & 1 \\ 1 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 - R_3$

$$\begin{bmatrix} 1 & -1 & 0 \\ 3 & 4 & 1 \\ 1 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 3R_1$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 7 & 1 \\ 1 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ -3 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 7 & 1 \\ 0 & 7 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ -3 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix} A$$

Applying $R_1 \to R_1 + \frac{1}{7} R_2$

$$\begin{bmatrix} 1 & 0 & \frac{1}{7} \\ 0 & 7 & 1 \\ 0 & 7 & 3 \end{bmatrix} = \begin{bmatrix} \frac{4}{7} & \frac{1}{7} & \frac{-4}{7} \\ -3 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix} A$$

Applying $R_2 \to \frac{R_2}{7}$

$$\begin{bmatrix} 1 & 0 & \frac{1}{7} \\ 0 & 1 & \frac{1}{7} \\ 0 & 7 & 3 \end{bmatrix} = \begin{bmatrix} \frac{4}{7} & \frac{1}{7} & \frac{-4}{7} \\ \frac{-3}{7} & \frac{1}{7} & \frac{3}{7} \\ -1 & 0 & 2 \end{bmatrix} A$$

Applying $R_3 \to R_3 - 7R_2$

$$\begin{bmatrix} 1 & 0 & \frac{1}{7} \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{7} & \frac{1}{7} & \frac{-4}{7} \\ \frac{-3}{7} & \frac{1}{7} & \frac{3}{7} \\ 2 & -1 & -1 \end{bmatrix} A$$

Applying $R_3 \to \frac{R_3}{2}$

$$\begin{bmatrix} 1 & 0 & \frac{1}{7} \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{7} & \frac{1}{7} & \frac{-4}{7} \\ \frac{-3}{7} & \frac{1}{7} & \frac{3}{7} \\ 1 & \frac{-1}{2} & \frac{-1}{2} \end{bmatrix} A$$

Applying $R_1 \to R_1 - \frac{1}{7} R_3, R_2 \to R_2 - \frac{1}{7} R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{7} & \frac{3}{14} & \frac{-1}{2} \\ \frac{-4}{7} & \frac{3}{14} & \frac{1}{2} \\ 1 & \frac{-1}{2} & \frac{-1}{2} \end{bmatrix} A$$

$$\therefore \quad A^{-1} = \begin{bmatrix} \frac{3}{7} & \frac{3}{14} & \frac{-1}{2} \\ \frac{-4}{7} & \frac{3}{14} & \frac{1}{2} \\ 1 & \frac{-1}{2} & \frac{-1}{2} \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 6 & 3 & -7 \\ -8 & 3 & 7 \\ 14 & -7 & -7 \end{bmatrix}$$

28. The equations of the given curves are

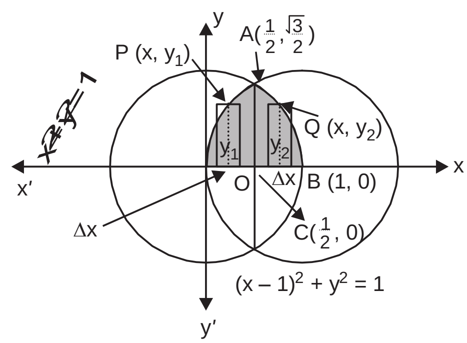
$$x^2 + y^2 = 1 \qquad \text{...}(i)$$

and, $\quad (x-1)^2 + (y-0)^2 = 1 \qquad \ldots(ii)$

Clearly, $x^2 + y^2 = 1$ represents a circle with centre at (0, 0) and radius unity. Also, $(x-1)^2 + y^2 = 1$ represents a circle with centre at (1, 0) and radius unity. To find the points of intersection of the given curves, we solve (1) and (2) simultaneously.

Thus, $\quad 1-(x-1)^2 = 1-x^2$

$\Rightarrow \quad 2x = 1 \quad \Rightarrow \quad x = \dfrac{1}{2}$



We find that the two curves intersect at

$A\,(1/2, \sqrt{3}/2)$ and $D\,(1/2, -\sqrt{3}/2)$.

Since both the curves are symmetrical about x-axis.

So, Required area = 2 (Area $OABCO$)

Now, we slice the area $OABCO$ into vertical strips. We observe that the vertical strips change their character at $A(1/2, \sqrt{3}/2)$. So.

Area $OABCO$ = Area $OACO$ + Area $CABC$.

When area $OACO$ is sliced into vertical strips, we find that each strip has its upper end on the circle $(x-1)^2 + (y-0)^2 = 1$ and the lower end on x-axis. So, the approximating rectangle shown in Fig. has, Length = y_1, Width = Δx and Area = $y_1\,\Delta x$. As it can move from $x = 0$ to $x = 1/2$.

$\therefore \quad$ Area $OACO = \displaystyle\int_0^{1/2} y_1 dx$

$\Rightarrow \quad$ Area $OACO = \displaystyle\int_0^{1/2} \sqrt{1-(x-1)^2}\, dx \qquad \left[\begin{array}{l} \because P(x, y_1) \text{ lies on } (x-1)^2 + y^2 = 1 \\ \therefore (x-1)^2 + y_1^2 = 1 \Rightarrow y_1 = \sqrt{1-(x-1)^2} \end{array}\right]$

Similarly, approximating rectangle in the region $CABC$ has, Length, = y_2, Width Δx and Area = $y_2 \Delta x$. As it can move form $x = \dfrac{1}{2}$ to $x = 1$.

$\therefore \quad$ Area $CABC = \displaystyle\int_{1/2}^{1} y_2\, dx$

$\Rightarrow \quad$ Area $CABC = \displaystyle\int_{1/2}^{1} \sqrt{1-x^2}\, dx \qquad \left[\begin{array}{l} \because Q(x, y_2) \text{ lies on } x^2 + y^2 = 1 \\ \therefore x^2 + y_2^2 = 1 \Rightarrow y_2 = \sqrt{1-x^2} \end{array}\right]$

Hence, required area A is given by

$$A = 2\left[\int_0^{1/2} \sqrt{1-(x-1)^2}\, dx + \int_{1/2}^{1} \sqrt{1-x^2}\, dx\right]$$

$$\Rightarrow \quad A = 2\left[\left[\frac{1}{2}\cdot(x-1)\sqrt{1-(x-1)^2} + \frac{1}{2}\sin^{-1}(x-1)\right]_0^{1/2} \ldots\right]$$

$$+\left[\frac{1}{2}x \qquad 1-x^2+{}^{1}\ \sin^{-1}|\ x\ |\right]^{1}_{1/2}$$

$$\Rightarrow \quad A = \left[\left\{-\frac{\sqrt{3}}{4} + \sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(-1)\right\} + \left\{\sin^{-1}(1) - \frac{\sqrt{3}}{4} - \sin^{-1}\left(\frac{1}{2}\right)\right\}\right]$$

$$\Rightarrow \quad A = -\frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\pi}{2} + \frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\pi}{6} = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) \text{ sq. units}$$

29. Let

S = Event of insuring scooter driver

C = Event of insuring Car driver

T = Event of insuring Truck driver

and A = Event of meeting with an accident.

Now, we have

$P(S)$ = Probability of insuring scooter driver $= \frac{3000}{15000} = \frac{3}{15}$

$P(C)$ = Probability of insuring car driver $= \frac{5000}{15000} = \frac{5}{15}$

$P(T)$ = Probability of insuring Truck driver $= \frac{7000}{15000} = \frac{7}{15}$

and, $P(A / S)$ = Probability that scooter driver meet with an accident = 0.04

$P(A / C)$ = Probability that car driver meet with an accident = 0.05

$P(A / T)$ = Probability that Truck driver meet with an accident = 0.15

By Baye's theorem, we have

Required probability $= P(C / A) = \dfrac{P(C)\cdot P(A / C)}{P(S) \,.\, P(A / S) + P(C) \,.P(A / C) + P(T). \,P(A / T)}$

$$= \frac{\frac{5}{15} \times 0.05}{\frac{3}{15} \times 0.04 + \frac{5}{15} \times 0.05 + \frac{7}{15} \times 0.15}$$

$$= \frac{5 \times 0.05}{3 \times 0.04 + 5 \times 0.05 + 7 \times 0.15}$$

$$= \frac{0.25}{0.12 + 0.25 + 1.05}$$

$$= \frac{0.25}{1.42} = \frac{25}{142}$$

EXAMINATION PAPERS – 2009

MATHEMATICS CBSE (Delhi)

CLASS – XII

Time allowed: 3 hours **Maximum marks: 100**

General Instructions:

1. ***All*** *questions are compulsory.*
2. *The question paper consists of* ***29*** *questions divided into three Sections* ***A, B*** *and* ***C.*** *Section A comprises of* ***10*** *questions of* ***one*** *mark each, Section B comprises of* ***12*** *questions of* ***four*** *marks each and Section C comprises of* ***7*** *questions of* ***six*** *marks each.*
3. ***All*** *questions in Section A are to be answered in* ***one*** *word,* ***one*** *sentence or as per the exact requirement of the question.*
4. *There is no overall choice. However, internal choice has been provided in* ***4*** *questions of* ***four*** *marks each and* ***2*** *questions of* ***six*** *marks each. You have to attempt only* ***one*** *of the alternatives in all such questions.*
5. *Use of calculators is* ***not*** *permitted.*

Set–I

SECTION–A

1. Find the projection of $\vec{a}$ on $\vec{b}$ if $\vec{a} \cdot \vec{b} = 8$ and $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$.

2. Write a unit vector in the direction of $\vec{a} = 2\hat{i} - 6\hat{j} + 3\hat{k}$.

3. Write the value of p, for which $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are parallel vectors.

4. If matrix $A = (1\ 2\ 3)$, write AA', where A' is the transpose of matrix A.

5. Write the value of the determinant $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$.

6. Using principal value, evaluate the following:

$$\sin^{-1}\left(\sin \frac{3\pi}{5}\right)$$

7. Evaluate : $\int \frac{\sec^2 x}{3 + \tan x}\, dx$.

8. If $\int_0^1 (3x^2 + 2x + k)\, dx = 0$, find the value of k.

9. If the binary operation * on the set of integers Z, is defined by $a * b = a + 3b^2$, then find the value of $2 * 4$.

10. If A is an invertible matrix of order 3 and $|A| = 5$, then find $|adj.\,A|$.

SECTION–B

11. If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$ show that $\vec{a} - \vec{d}$ is parallel to $\vec{b} - \vec{c}$, where $\vec{a} \neq \vec{d}$ and $\vec{b} \neq \vec{c}$.

12. Prove that: $\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{16}{65}\right) = \frac{\pi}{2}$

OR

Solve for x : $\tan^{-1} 3x + \tan^{-1} 2x = \frac{\pi}{4}$

13. Find the value of λ so that the lines

$$\frac{1-x}{3} = \frac{7y-14}{2\lambda} = \frac{5z-10}{11} \text{ and } \frac{7-7x}{3\lambda} = \frac{y-5}{1} = \frac{6-z}{5}.$$

are perpendicular to each other.

14. Solve the following differential equation:

$$\frac{dy}{dx} + y = \cos x - \sin x$$

15. Find the particular solution, satisfying the given condition, for the following differential equation:

$$\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\left(\frac{y}{x}\right) = 0;\ y = 0 \text{ when } x = 1$$

16. By using properties of determinants, prove the following:

$$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$$

17. A die is thrown again and again until three sixes are obtained. Find the probability of obtaining the third six in the sixth throw of the die.

18. Differentiate the following function w.r.t. x :

$$x^{\sin x} + (\sin x)^{\cos x}.$$

19. Evaluate : $\int \frac{5-4e^{x}-e^{2x}}{\sqrt{}}\,dx$

OR

Evaluate : $\int \frac{(x-4)e^x}{(x-2)^3}\,dx$

20. Prove that the relation R on the set $A = \{1, 2, 3, 4, 5\}$ given by $R = \{(a, b) : |a-b| \text{ is even}\}$, is an equivalence relation.

21. Find $\frac{dy}{dx}$ if $(x^2 + y^2)^2 = xy$.

OR

If $y = 3\cos(\log x) + 4\sin(\log x)$, then show that $x^2 . \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$.

22. Find the equation of the tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $4x - 2y + 5 = 0$.

OR

Find the intervals in which the function f given by $f(x) = x^3 + \frac{1}{x^3}$, $x \neq 0$ is

(i) increasing (ii) decreasing.

SECTION–C

23. Find the volume of the largest cylinder that can be inscribed in a sphere of radius r.

OR

A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs Rs. 70 per sq. metre for the base and Rs. 45 per sq. metre for sides, what is the cost of least expensive tank?

24. A diet is to contain at least 80 units of Vitamin A and 100 units of minerals. Two foods F_1 and F_2 are available. Food F_1 costs Rs. 4 per unit and F_2 costs Rs. 6 per unit. One unit of food F_1 contains 3 units of Vitamin A and 4 units of minerals. One unit of food F_2 contains 6 units of Vitamin A and 3 units of minerals. Formulate this as a linear programming problem and find graphically the minimum cost for diet that consists of mixture of these two foods and also meets the minimal nutritional requirements.

25. Three bags contain balls as shown in the table below:

Bag	Number of White balls	Number of Black balls	Number of Red balls
I	1	2	3
II	2	1	1
III	4	3	2

A bag is chosen at random and two balls are drawn from it. They happen to be white and red. What is the probability that they came from the III bag?

26. Using matrices, solve the following system of equations:

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3$$

27. Evaluate: $\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} dx$.

OR

Evaluate: $\int_0^{\pi/2} (2\log\sin x - \log\sin 2x)\,dx$.

28. Using the method of integration, find the area of the region bounded by the lines

$2x + y = 4,\quad 3x - 2y = 6$ and $x - 3y + 5 = 0$.

29. Find the equation of the plane passing through the point (-1, 3, 2) and perpendicular to each of the planes $x + 2y + 3z = 5$ and $3x + 3y + z = 0$.

Set–II

Only those questions, not included in Set I, are given.

2. Evaluate: $\int \sec^2(7 - x)\,dx$

7. Write a unit vector in the direction of $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$.

11. Differentiate the following function w.r.t. x :

$y = (\sin x)^x + \sin^{-1}\sqrt{x}$.

18. Find the value of λ so that the lines $\dfrac{1-x}{3} = \dfrac{y-2}{2\lambda} = \dfrac{z-3}{2}$ and $\dfrac{x-1}{3\lambda} = \dfrac{y-1}{1} = \dfrac{6-z}{7}$ are perpendicular to each other.

19. Solve the following differential equation :

$(1 + x^2)\dfrac{dy}{dx} + y = \tan^{-1} x$.

21. Using the properties of determinants, prove the following:

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3 + b^3 + c^3 - 3abc.$$

23. Two groups are competing for the position on the Board of Directors of a corporation. The probabilities that the first and the second groups will win are 0.6 and 0.4 respectively. Further, if the first group wins, the probability of introducing a new product is 0.7 and the corresponding probability is 0.3, if the second group wins. Find the probability that the new product was introduced by the second group.

26. Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by $x = 0,\ x = 4,\ y = 4$ and $y = 0$ into three equal parts.

Set–III

Only those questions, not included in Set I and Set II, are given.

4. Evaluate : $\int \dfrac{(1 + \log x)^2}{x}\,dx$

9. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes 1 and 2 respectively and when $|\vec{a} \times \vec{b}| = \sqrt{3}$.

15. Using properties of determinants, prove the following:

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$$

17. Differentiate the following function w.r.t. x :

$$(x)^{\cos x} + (\sin x)^{\tan x}$$

19. Solve the following differential equation:

$$x \log x \frac{dy}{dx} + y = 2 \log x .$$

20. Find the value of λ so that the following lines are perpendicular to each other.

$$\frac{x-5}{5\lambda+2} = \frac{2-y}{5} = \frac{1-z}{-1}; \quad \frac{x}{1} = \frac{2y+1}{4\lambda} = \frac{1-z}{-3} .$$

24. Find the area of the region enclosed between the two circles $x^2 + y^2 = 9$ and $(x-3)^2 + y^2 = 9$.

27. There are three coins. One is a two headed coin (having head on both faces), another is a biased coin that comes up tail 25% of the times and the third is an unbiased coin. One of the three coins is chosen at random and tossed, it shows heads, what is the probability that it was the two headed coin?

SOLUTIONS

Set–I

SECTION–A

1. Given $\quad \vec{a} \cdot \vec{b} = 8$

$$\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$$

We know projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

$$= \frac{8}{\sqrt{4+36+9}} = \frac{8}{7}$$

2. Given $\quad \vec{a} = 2\hat{i} - 6\hat{j} + 3\hat{k}$

Unit vector in the direction of $\vec{a} = \dfrac{\vec{a}}{|\vec{a}|} = \hat{a}$

$\Rightarrow \qquad \hat{a} = \dfrac{2\hat{i} - 6\hat{j} + 3\hat{k}}{\sqrt{4+36+9}}$

$\Rightarrow \qquad \hat{a} = \dfrac{2}{7}\hat{i} - \dfrac{6}{7}\hat{j} + \dfrac{3}{7}\hat{k}$

3. Since $\vec{a} \parallel \vec{b}$, therefore $\vec{a} = \lambda\,\vec{b}$

$\Rightarrow \qquad 3\hat{i} + 2\hat{j} + 9\hat{k} = \lambda(\hat{i} + p\hat{j} + 3\hat{k})$

$\Rightarrow \qquad \lambda = 3, 2 = \lambda p, 9 = 3\lambda$

or $\qquad \lambda = 3, p = \dfrac{2}{3}$

4. Given $A = (1\ \ 2\ \ 3)$

$$A' = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$AA' = (1\times 1 + 2\times 2 + 3\times 3) = (14)$$

5. Given determinant $|A| = \begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$

$\Rightarrow \qquad |A| = 3x\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 2 & 3 & 4 \end{vmatrix} = 0 \qquad (\because R_1 = R_3)$

6. $\dfrac{3\pi}{5} = \pi - \dfrac{2\pi}{5}$

$\therefore \quad \sin^{-1}\left(\sin\dfrac{3\pi}{5}\right)$

$$= \sin^{-1}\left[\sin\left(\pi - \frac{2\pi}{5}\right)\right]$$

$$= \sin^{-1}\left[\sin\frac{2\pi}{5}\right] = \frac{2\pi}{5} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

7. $\displaystyle\int \frac{\sec^2 x}{3 + \tan x}\,dx$

Let $3 + \tan x = t$

$\sec^2 x\, dx = dt$

$\therefore \quad \displaystyle\int \frac{\sec^2 x}{3+\tan x}\,dx = \int \frac{dt}{t}$

$$= \log|t| + c$$

$$= \log|3 + \tan x| + c$$

8. $\int_0^1 (3x^2 + 2x + k)\, dx = 0$

$$\Rightarrow \qquad \left[\frac{3x^3}{3} + \frac{2x^2}{2} + kx\right]_0^1 = 0$$

$$\Rightarrow \qquad 1 + 1 + k = 0 \qquad \Rightarrow \qquad k = -2$$

9. Given $a * b = a + 3b^2 \quad \forall\ a, b \in z$

$\therefore \quad 2 * 4 = 2 + 3 \times 4^2 = 2 + 48 = 50$.

10. Given $|A| = 5$

We know $\quad |adj.\ A| = |A|^2$

$\therefore \qquad |adj.\ A| = 5^2 = 25$

SECTION–B

11. $\vec{a} - \vec{d}$ will be parallel to $\vec{b} - \vec{c}$, if $(\vec{a} - \vec{d}) \times (\vec{b} - \vec{c}) = \vec{0}$

Now $\quad (\vec{a} - \vec{d}) \times (\vec{b} - \vec{c}) = \vec{a} \times \vec{b} - \vec{a} \times \vec{c} - \vec{d} \times \vec{b} + \vec{d} \times \vec{c}$

$$= \vec{a} \times \vec{b} - \vec{a} \times \vec{c} + \vec{b} \times \vec{d} - \vec{c} \times \vec{d}$$

$$= 0 \qquad [\because \text{ given } \vec{a} \times \vec{b} = \vec{c} \times \vec{d} \text{ and } \vec{a} \times \vec{c} = \vec{b} \times \vec{d}\,]$$

$\therefore \qquad (\vec{a} - \vec{d}) \parallel (\vec{b} - \vec{c})$

12. We know

$$\sin^{-1} x + \sin^{-1} y = \sin^{-1}(x\sqrt{1 - y^2} + y\sqrt{1 - x^2})$$

$$\therefore \quad \sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{16}{65}\right)$$

$$= \sin^{-1}\left(\frac{4}{5}\sqrt{1 - \frac{25}{169}} + \frac{5}{13}\sqrt{1 - \frac{16}{25}}\right) + \sin^{-1}\left(\frac{16}{65}\right)$$

$$= \sin^{-1}\left(\frac{4}{5} \times \frac{12}{13} + \frac{5}{13} \times \frac{3}{5}\right) + \sin^{-1}\left(\frac{16}{65}\right)$$

$$= \sin^{-1}\left(\frac{63}{65}\right) + \sin^{-1}\left(\frac{16}{65}\right) \qquad \ldots (i)$$

Let $\sin^{-1} \dfrac{63}{65} = \theta$

$$\Rightarrow \quad \frac{63}{65} = \sin\theta \qquad \Rightarrow \qquad \frac{63^2}{65^2} = \sin^2\theta$$

$$\Rightarrow \quad \cos^2\theta = 1 - \frac{63^2}{65^2} = \frac{65^2 - 63^2}{65^2} = \frac{(65 + 63)(65 - 63)}{65^2}$$

$\Rightarrow \quad \cos^2\theta = \frac{256}{65^2} \qquad \therefore \quad \cos\theta = \frac{16}{65}$

$\therefore$ Equation (i) becomes

$$\sin^{-1}\left(\frac{63}{65}\right) + \sin^{-1}\left(\frac{16}{65}\right) = \cos^{-1}\left(\frac{63}{65}\right) + \sin^{-1}\left(\frac{16}{65}\right)$$

$$= \frac{\pi}{2} \qquad \left[\because \sin^{-1}A + \cos^{-1}A = \frac{\pi}{2}\right]$$

OR

Given, $\quad \tan^{-1}3x + \tan^{-1}2x = \frac{\pi}{4}$

$\Rightarrow \quad \tan^{-1}\left(\frac{3x+2x}{1-3x\times 2x}\right) = \frac{\pi}{4} \qquad \left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$

$\Rightarrow \quad \frac{5x}{1-6x^2} = 1$

$\Rightarrow \quad 5x = 1 - 6x^2$

$\Rightarrow \quad 6x^2 + 5x - 1 = 0$

$\Rightarrow \quad 6x^2 + 6x - x - 1 = 0$

$6x(x+1) - 1(x+1) = 0$

$\Rightarrow \quad (6x-1)(x+1) = 0$

$x = \frac{1}{6}$ or $x = -1.$

$\Rightarrow$

$\therefore \quad = \quad =$

13. The given lines

$$\frac{1-x}{3} \quad \frac{7y-14}{2\lambda} \quad \frac{5z-10}{11}$$

and $\frac{7-7x}{3\lambda} = \frac{y-5}{1} = \frac{6-z}{5}$ are rearranged to get

$$\frac{x-1}{-3} = \frac{y-2}{2\lambda/7} = \frac{z-2}{11/5} \qquad \text{... }(i)$$

$$\frac{x-1}{-3\lambda/7} \quad \frac{y-5}{1} \quad \frac{z-6}{-5} \qquad \text{... }(ii)$$

Direction ratios of lines are

$$-3, \frac{2\lambda}{7}, \frac{11}{5} \text{ and } \frac{-3\lambda}{7}, 1, -5$$

As the lines are perpendicular

$\therefore \quad -3\left(\frac{-3\lambda}{7}\right) + \frac{2\lambda}{7}\times 1 + \frac{11}{5}(-5) = 0$

$\Rightarrow \quad \frac{9\lambda}{7} + \frac{2\lambda}{7} - 11 = 0$

$\Rightarrow \qquad \dfrac{11}{7}\lambda = 11$

$\Rightarrow \qquad \lambda = 7$

14. Given differential equation

$\dfrac{dy}{dx} + y = \cos x - \sin x$ is a linear differential equation of the type $\dfrac{dy}{dx} + Py = Q$.

Here $\qquad I.F = e^{\int 1.dx} = e^{x}$

Its solution is given by

$\Rightarrow \qquad y\,e^{x} = \int e^{x}(\cos x - \sin x)\,dx$

$\Rightarrow \qquad y\,e^{x} = \int e^{x}\cos x\,dx - \int e^{x}\sin x\,dx$

Integrate by parts

$\Rightarrow \qquad y\,e^{x} = e^{x}\cos x - \int -\sin x\,e^{x}\,dx - \int e^{x}\sin dx$

$\therefore \qquad y\,e^{x} = e^{x}\cos x + C$

$\Rightarrow \qquad y = \cos x + C\,e^{-x}$

15. $\dfrac{dy}{dx} - \dfrac{y}{x} + \text{cosec}\left(\dfrac{y}{x}\right) = 0 \qquad \ldots (i)$

It is a homogeneous differential equation,

Let $\qquad \dfrac{y}{x} = v \quad \Rightarrow \quad y = vx$

$\dfrac{dy}{dx} = v + \dfrac{xdv}{dx}$

(Substituting in equation (*i*))

$\Rightarrow \qquad v + x\dfrac{dv}{dx} = v - \text{cosec}\,v$

$\Rightarrow \qquad x\dfrac{dv}{dx} = -\,\text{cosec}\,v$

$\Rightarrow \qquad \dfrac{dv}{\text{cosec}\,v} = -\dfrac{dx}{x} \qquad \Rightarrow \qquad \sin v\,dv = -\dfrac{dx}{x}$

Integrating both sides

$\int \sin v\,dv = -\int \dfrac{dx}{x} \qquad \Rightarrow \qquad -\cos v = -\log|x| + C$

$\Rightarrow \qquad \cos v = \log|x| + C$

or $\qquad \cos\dfrac{y}{x} = \log|x| + C$

Given $y = 0$, when $x = 1$

$\Rightarrow \qquad \cos 0 = \log|1| + C$

$\Rightarrow \qquad 1 = \text{C}$

Hence, solution of given differential equation is $\cos\dfrac{y}{x} = \log|x| + 1$.

16. Let $|A| = \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$

Apply $C_1 \to C_1 + C_2 + C_3$

$$|A| = \begin{vmatrix} 5x+4 & 2x & 2x \\ 5x+4 & x+4 & 2x \\ 5x+4 & 2x & x+4 \end{vmatrix}$$

Take $5x + 4$ common from C_1

$$|A| = (5x+4)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x+4 & 2x \\ 1 & 2x & x+4 \end{vmatrix}$$

Apply $R_2 \to R_2 - R_1;\ R_3 \to R_3 - R_1$

$$|A| = (5x+4)\begin{vmatrix} 1 & 2x & 2x \\ 0 & 4-x & 0 \\ 0 & 0 & 4-x \end{vmatrix}$$

Expanding along C_1, we get

$$|A| = (5x+4)(4-x)^2 = \text{R.H.S.}$$

17. If there is third 6 in 6th throw, then five earlier throws should result in two 6.

Hence taking $n = 5$, $p = \frac{1}{6}$, $q = \frac{5}{6}$

$\therefore$ $P(2 \text{ sixes}) = P(5, 2) = {}^5C_2 p^2 q^3$

$\Rightarrow$ $P(2 \text{ sixes}) = \frac{5!}{2!\,3!}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3 = \frac{10 \times 125}{6^5}$

$\therefore$ $P(3 \text{ sixes in 6 throws}) = \frac{10 \times 125}{6^5} \times \frac{1}{6} = \frac{1250}{6^6} = \frac{625}{3 \times 6^5}$

18. Let $y = x^{\sin x} + (\sin x)^{\cos x}$

Let $u = x^{\sin x}$ and $v = (\sin x)^{\cos x}$

Then, $y = u + v$

$\Rightarrow$ $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ...(i)

Now, $u = x^{\sin x}$

Taking log both sides, we get

$\Rightarrow$ $\log u = \sin x \log x$

Differentiating w.r.t. x

$\Rightarrow$ $\frac{1}{u}\frac{du}{dx} = \frac{\sin x}{x} + \log x \,.\, \cos x$

$$\Rightarrow \quad \frac{du}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \log x \cdot \cos x\right]$$

Similarly taking log on $v = (\sin x)^{\cos x}$

$$\log v = \cos x \log \sin x$$

Differentiating w. r. t. x

$$\frac{1}{v}\frac{dv}{dx} = \cos x \cdot \frac{\cos x}{\sin x} + \log \sin x \cdot (-\sin x)$$

$$\frac{dv}{dx} = (\sin x)^{\cos x}[\cos x \cdot \cot x - \sin x \cdot \log \sin x]$$

Form (i), we have

$$\frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \log x \cdot \cos x\right] + (\sin x)^{\cos x}[\cos x \cdot \cot x - \sin x \cdot \log \sin x]$$

19. Let $I = \int \frac{e^x}{\sqrt{5 - 4e^x - e^{2x}}}\, dx$

Suppose $e^x = t \quad \Rightarrow \quad e^x dx = dt$

$$\Rightarrow \quad I = \int \frac{dt}{\sqrt{5 - 4t - t^2}} = \int \frac{dt}{\sqrt{-(t^2 + 4t - 5)}}$$

$$\Rightarrow \quad I = \int \frac{dt}{\sqrt{-(t^2 + 4t + 4 - 9)}}$$

$$\Rightarrow \quad I = \int \frac{dt}{\sqrt{3^2 - (t+2)^2}} = \sin^{-1}\frac{t+2}{3} + C$$

$$\Rightarrow \quad I = \sin^{-1}\left(\frac{e^x + 2}{3}\right) + C$$

OR

Let $I = \int \frac{(x-4)e^x}{(x-2)^3}\, dx$

$$= \int \left[\frac{(x-2) - 2}{(x-2)^3}\right] e^x\, dx$$

$$= \int \frac{e^x\, dx}{(x-2)^2} - 2\int \frac{e^x\, dx}{(x-2)^3}$$

$$= \frac{e^x}{(x-2)^2} + 2\int \frac{e^x\, dx}{(x-2)^3} - 2\int \frac{e^x\, dx}{(x-2)^3}$$

$$= \frac{e^x}{(x-2)^2} + C$$

20. The relation given is

$R = \{(a, b) : |a - b| \text{ is even}\}$ where

$a, b \in A = \{1, 2, 3, 4, 5\}$

To check: Reflexivity

Let $a \in A$

Then $\quad aRa$ as $|a - a| = 0$ which is even.

$\therefore \quad (a, a) \in R$. Hence R is reflexive.

To check: Symmetry

Let $(a, b) \in R \qquad |a - b|$ is even

$\Rightarrow \Rightarrow \qquad |b - a|$ is even

$\Rightarrow \qquad (b - a) \in R.$

Hence R is symmetric.

To check: Transitivity

Let $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow \quad |a - b|$ is even and $|b - c|$ is also even.

Then,

$$|a - c| = |(a - b) + (b - c)| \le \underbrace{|a - b|}_{\text{even}} + \underbrace{|b - c|}_{\text{even}}$$

$\therefore \quad |a - c| = \text{even}$

So, $(a, c) \in R$.

It is transitive.

As R is reflexive, symmetric as well as transitive, it is an equivalence relation.

21. Given equation is

$$(x^2 + y^2)^2 = xy$$

Differentiating w.r.t. x

$$\Rightarrow \quad 2(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) = x\frac{dy}{dx} + y$$

$$\Rightarrow \quad 2(x^2 + y^2).2y\frac{dy}{dx} - x\frac{dy}{dx} = y - 4(x^2 + y^2)x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y - 4x(x^2 + y^2)}{4y(x^2 + y^2) - x}$$

OR

$$y = 3\cos(\log x) + 4\sin(\log x)$$

Differentiating w.r.t. x

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-3\sin(\log x)}{x} + \frac{4\cos(\log x)}{x}$$

$$\Rightarrow \quad x\frac{dy}{dx} = -3\sin(\log x) + 4\cos(\log x)$$

Differentiating again w.r.t. x

$$\Rightarrow \quad x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{-3\cos(\log x)}{x} - \frac{4\sin(\log x)}{x}$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

22. Given curve is $y = \sqrt{3x-2}$ ...(i)

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1 \times 3}{2\sqrt{3x-2}}$$

Since tangent is parallel to line

$$4x - 2y + 5 = 0$$

$$\Rightarrow \quad \frac{-4}{-2} = \text{slope of line} = \frac{3}{2\sqrt{3x-2}}$$

$$\Rightarrow \quad 4 = \frac{9}{4(3x-2)}$$

$$\Rightarrow \quad 48x - 32 = 9 \quad \Rightarrow \quad x = \frac{41}{48}$$

Substituting value of x in (i)

$$y = \sqrt{3 \times \frac{41}{48} - 2} = \sqrt{\frac{9}{16}} = \frac{3}{4}$$

Thus point of tangency is $\left(\frac{41}{48}, \frac{3}{4}\right)$

$\therefore$ Equation of tangent is

$$y - \frac{3}{4} = 2\left(x - \frac{41}{48}\right)$$

$$\Rightarrow \quad \frac{4y-3}{4} = \frac{48x-41}{24}$$

$$\Rightarrow \quad 24y - 18 = 48x - 41$$

$\Rightarrow \quad 48x - 24y - 23 = 0$ is the equation of tangent.

OR

Given $f(x) = x^3 + \frac{1}{x^3}$

$$f'(x) = 3x^2 - \frac{3}{x^4}$$

$$= \frac{3(x^6-1)}{x^4} = \frac{3(x^2-1)(x^4+x^2+1)}{x^4}$$

But $x^4 + x^2 + 1, x^4$ are always > 0

$\therefore \quad f'(x) = 0 \Rightarrow x = \pm 1$

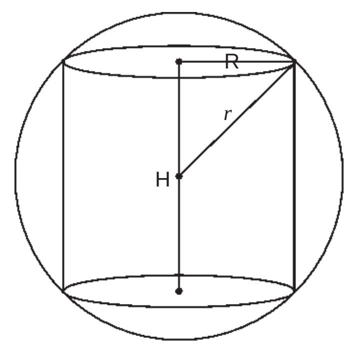
Intervals	$x - 1$	$x + 1$	sign of $f'(x)$
$x < -1$	–ve	–ve	+ve
$-1 < x < 1$	–ve	+ve	–ve
$x > 1$	+ve	+ve	+ve

$\therefore$ Given function is increasing $\forall\, x \in (-\infty, 1) \cup (1, \infty)$ and is decreasing $\forall\, x \in (-1, 1)$.

SECTION–C

23. Let a right circular cylinder of radius 'R' and height 'H' is inscribed in the sphere of given radius 'r'.

$\therefore \quad R^2 + \dfrac{H^2}{4} = r^2$



Let V be the volume of the cylinder.

Then, $V = \pi R^2 H$

$\Rightarrow \quad V = \pi\left(r^2 - \dfrac{H^2}{4}\right)H \qquad ...(i)$

$\Rightarrow \quad V = \pi r^2 H - \dfrac{\pi}{4}H^3$

Differentiating both sides w.r.t. H

$$\frac{dV}{dH} = \pi r^2 - \frac{3\pi H^2}{4} \qquad \ldots (ii)$$

For maximum volume $\dfrac{dV}{dH} = 0$

$\Rightarrow \quad \dfrac{3\pi H^2}{4} = \pi r^2 \quad \Rightarrow \quad H^2 = \dfrac{4r^2}{3} \quad$ or $\quad H = \dfrac{2}{\sqrt{3}}r$

Differentiating (ii) again w.r.t. H

$$\frac{d^2V}{dH^2} = -\frac{6\pi H}{4} \Rightarrow \left.\frac{d^2V}{dH^2}\right]_{H=\frac{2}{\sqrt{3}}r} = \frac{-6\pi}{4} \times \frac{2}{\sqrt{3}}r < 0$$

$\therefore \quad$ Volume is maximum when height of the cylinder is $\dfrac{2}{\sqrt{3}}r$.

Substituting $H = \dfrac{2}{\sqrt{3}}r$ in (i), we get

$$V_{\max} = \pi\left(r^2 - \frac{4r^2}{4 \times 3}\right).\frac{2}{\sqrt{3}}r = \frac{\pi 2r^2}{3}.\frac{2r}{\sqrt{3}}$$

$$= \frac{4\pi r^3}{3\sqrt{3}} \text{ cubic units.}$$

OR

Let the length and breadth of the tank are L and B.

$\therefore$ Volume $= 8 = 2\,LB \Rightarrow B = \dfrac{4}{L}$ ... (i)

The surface area of the tank, S = Area of Base + Area of 4 Walls

$= LB + 2(B + L)\cdot 2$

$= LB + 4B + 4L$

The cost of constructing the tank is

$C = 70(LB) + 45(4B + 4L)$

$= 70\left(L\cdot\dfrac{4}{L}\right) + 180\left(\dfrac{4}{L} + L\right)$

$\Rightarrow$ $C = 280 + 180\left(\dfrac{4}{L} + L\right)$ ... (ii)

Differentiating both sides w.r.t. L

$\dfrac{dC}{dL} = -\dfrac{720}{L^2} + 180$ ... (iii)

For minimisation $\dfrac{dC}{dL} = 0$

$\Rightarrow$ $\dfrac{720}{L^2} = 180$

$\Rightarrow$ $L^2 = \dfrac{720}{180} = 4$

$\Rightarrow$ $L = 2$

Differentiating (iii) again w.r.t. L

$\dfrac{d^2C}{dL^2} = \dfrac{1440}{L^3} > 0 \;\forall\, L > 0$

$\therefore$ Cost is minimum when $L = 2$

From (i), $B = 2$

Minimum cost $= 280 + 180\left(\dfrac{4}{2} + 2\right)$ (from (ii))

$= 280 + 720$

$=$ Rs 1000

24. Let x units of food F_1 and y units of food F_2 are required to be mixed.

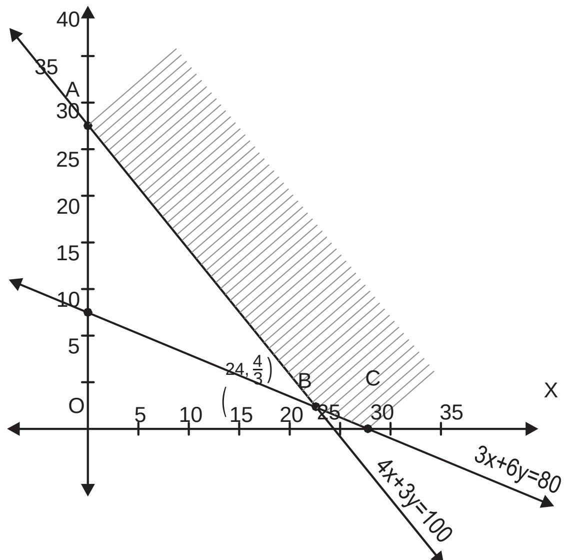
Cost $= Z = 4x + 6y$ is to be minimised subject to following constraints.

$3x + 6y \ge 80$

$4x + 3y \ge 100$

$x \ge 0, y \ge 0$

To solve the LPP graphically the graph is plotted as shown.



The shaded regions in the graph is the feasible solution of the problem. The corner points are $A\left(0, \frac{100}{3}\right)$, $B\left(24, \frac{4}{3}\right)$ and $C\left(\frac{80}{3}, 0\right)$. The cost at these points will be

$$Z]_A = 4 \times 0 + 6 \times \frac{100}{3} = \text{Rs } 200$$

$$Z]_B = 4 \times 24 + 6 \times \frac{4}{3} = \text{Rs } 104$$

$$Z]_C = 4 \times \frac{80}{3} + 0 = \text{Rs } \frac{320}{3} = \text{Rs } 106.67$$

Thus cost will be minimum if 24 units of F_1 and 4/3 units of F_2 are mixed. The minimum cost is Rs 104.

25. The distribution of balls in the three bags as per the question is shown below.

Bag	Number of white balls	Number of black balls	Number of red balls	Total balls
I	1	2	3	6
II	2	1	1	4
III	4	3	2	9

As bags are randomly choosen

$\therefore \quad P(\text{bag I}) = P(\text{bag II}) = P(\text{bag III}) = \frac{1}{3}$

Let E be the event that one white and one red ball is drawn.

$$P(E/\text{bag I}) = \frac{{}^1C_1 \times {}^3C_1}{{}^6C_2} = \frac{3 \times 2}{6 \times 5} = \frac{1}{5}$$

$$P(E/\text{bag II}) = \frac{{}^2C_1 \times {}^1C_1}{{}^4C_2} = \frac{2 \times 2}{4 \times 3} = \frac{1}{3}$$

$$P(E/\text{bag III}) = \frac{{}^4C_1 \times {}^2C_1}{{}^9C_2} = \frac{4 \times 2 \times 2}{9 \times 8} = \frac{2}{9}$$

Now, required probability

$$= P(\text{bag III}/E) = \frac{P(\text{bag III}).P(E/\text{bag III})}{P(\text{bag I}).P(E/\text{bag I}) + P(\text{bag II}).P(E/\text{bag II}) + P(\text{bagIII}).P(E/\text{bagIII})}$$

$$= \frac{\frac{1}{3} \times \frac{2}{9}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{9}} = \frac{\frac{1}{3} \times \frac{2}{9}}{\frac{1}{3}\left(\frac{1}{5} + \frac{1}{3} + \frac{2}{9}\right)}$$

$$= \frac{\frac{2}{9}}{\frac{9 + 15 + 10}{45}} = \frac{2}{9} \times \frac{45}{34} = \frac{5}{17}$$

26. Given system of equations is

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3$$

The equations can be expressed as matrix equation $AX = B$

$$\Rightarrow \begin{pmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ -5 \\ -3 \end{pmatrix}$$

$\therefore \quad X = A^{-1} B$

Now, $\quad |A| = 2(-4 + 4) + 3(-6 + 4) + 5(3 - 2)$

$$= -6 + 5 = -1 \neq 0 \Rightarrow A^{-1} \text{ exists.}$$

The cofactors of elements of A are

$C_{11} = 0 \quad C_{12} = 2 \quad C_{13} = 1$

$C_{21} = -1 \quad C_{22} = -9 \quad C_{23} = -5$

$C_{31} = 2 \quad C_{32} = 23 \quad C_{33} = 13$

$$\text{Matrix of cofactors} = \begin{pmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{pmatrix}$$

$$\therefore \quad \text{Adj } A = \begin{pmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{pmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{23}\begin{pmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{pmatrix} \qquad \left(\because A^{-1} = \frac{1}{|A|}(\text{Adj}\, A)\right)$$

$$\therefore \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{23}\begin{pmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{pmatrix}\begin{pmatrix} 11 \\ -5 \\ -3 \end{pmatrix} = -\frac{1}{23}\begin{pmatrix} 0+5-6 \\ 22+45-69 \\ 11+25-39 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Hence solution of given equations is $x = 1, y = 2, z = 3.$

27. Let $I = \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\, dx$ ...(*i*)

$$\Rightarrow \quad I = \int_0^{\pi} \frac{e^{\cos(\pi - x)}}{e^{\cos(\pi-x)} + e^{-\cos(\pi - x)}}\, dx \qquad \left(\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right)$$

$$= \int_0^{\pi} \frac{e^{-\cos x}}{e^{-\cos x} + e^{\cos x}}\, dx \qquad ...(ii)$$

Adding (*i*) and (*ii*), we get

$$2I = \int_0^{\pi} \frac{e^{\cos x} + e^{-\cos x}}{e^{\cos x} + e^{-\cos x}}\, dx = \int_0^{\pi} dx = x\Big]_0^{\pi} = \pi \qquad \Rightarrow \qquad I = \frac{\pi}{2}$$

OR

Let $I = \int_0^{\frac{\pi}{2}} (2 \log \sin x - \log \sin 2x) dx$ … (*i*)

$$\Rightarrow \quad I = \int_0^{\frac{\pi}{2}} \left[2 \log \sin\left(\frac{\pi}{2} - x\right) - \log \sin 2\left(\frac{\pi}{2} - x\right)\right] dx \qquad \left(\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right)$$

$$\Rightarrow \quad I = \int_0^{\frac{\pi}{2}} (2 \log \cos x - \log \sin 2x) dx \qquad … (ii)$$

Adding (*i*) and (*ii*), we get

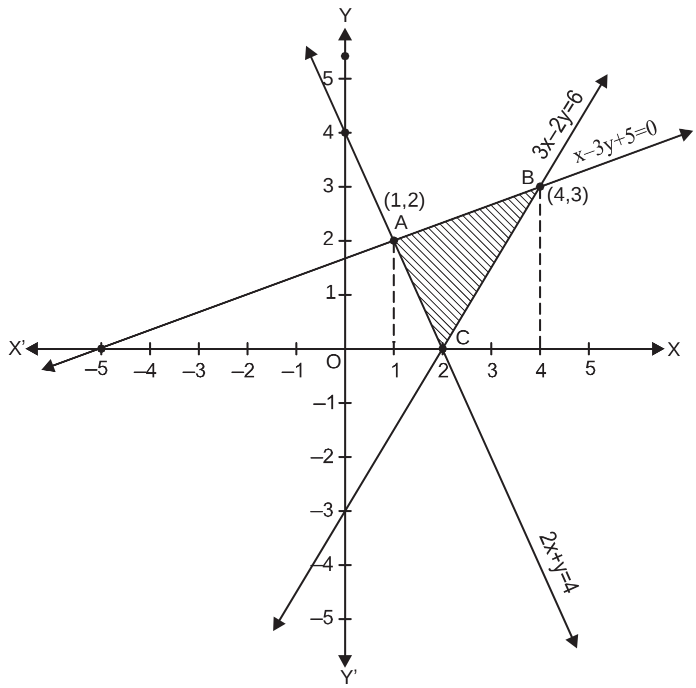
$$2I = \int_0^{\frac{\pi}{2}} 2 \log \sin x + 2 \log \cos x - 2 \log \sin 2x$$

$$\Rightarrow \quad 2I = \int_0^{\frac{\pi}{2}} 2\, [\log \sin x + \log \cos x - \log \sin 2x] dx$$

$$\Rightarrow \quad I = \int_0^{\frac{\pi}{2}} \log \frac{\sin x \cos x}{2 \sin x \cos x}\, dx = \log \frac{1}{2} \int_0^{\frac{\pi}{2}} dx = \log \frac{1}{2} \cdot x \Big]_0^{\frac{\pi}{2}}$$

$$\Rightarrow \quad I = \frac{\pi}{2} \log \frac{1}{2}$$

28. The lines are plotted on the graph as shown.



$$\text{Area of } \Delta ABC = \int_1^4 \frac{x+5}{3}\, dx - \int_1^2 (4-2x)\, dx - \int_2^4 \frac{3x-6}{2}\, dx$$

$$= \frac{1}{3}\left(\frac{x^2}{2} + 5x\right)\Bigg]_1^4 - \left(4x - \frac{2x^2}{2}\right)\Bigg]_1^2 - \frac{1}{2}\left(\frac{3x^2}{2} - 6x\right)\Bigg]_2^4$$

$$= \frac{1}{3}\left(8 + 20 - \frac{1}{2} - 5\right) - (8 - 4 - 4 + 1) - \frac{1}{2}(24 - 24 - 6 + 12)$$

$$= \frac{1}{3}\left(\frac{45}{2}\right) - 1 - \frac{1}{2}(6)$$

$$= \frac{15}{2} - 1 - 3 = \frac{15}{2} - 4 = \frac{7}{2} \text{ square units.}$$

29. The equation of plane through $(-1, 3, 2)$ can be expressed as

$$A(x+1) + B(y-3) + C(z-2) = 0 \qquad \ldots (i)$$

As the required plane is perpendicular to $x + 2y + 3z = 5$ and $3x + 3y + z = 0$, we get

$$A + 2B + 3C = 0$$
$$3A + 3B + C = 0$$

$$\Rightarrow \frac{A}{2-9} = \frac{B}{9-1} = \frac{C}{3-6} \Rightarrow \frac{A}{-7} = \frac{B}{8} = \frac{C}{-3}$$

$\therefore$ Direction ratios of normal to the required plane are $-7, 8, -3$.

Hence equation of the plane will be

$$-7(x+1) + 8(y-3) - 3(z-2) = 0$$

$$\Rightarrow -7x - 7 + 8y - 24 - 3z + 6 = 0$$

or $7x - 8y + 3z + 25 = 0$

Set–II

2. Let $I = \int \sec^2(7-x)dx$

$$= \frac{\tan(7-x)}{-1} + C$$

$$= -\tan(7-x) + C$$

7. Given $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$

Unit vector in the direction of $\vec{b} = \dfrac{\vec{b}}{|\vec{b}|} = \hat{b}$

$$\therefore \hat{b} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$$

11. Let $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

Suppose $z = (\sin x)^x$

Taking log on both sides

$$\log z = x \log \sin x$$

Differentiating both sides w.r.t. x

$$\frac{1}{z}\frac{dz}{dx} = x.\frac{\cos x}{\sin x} + \log \sin x$$

$$\Rightarrow \frac{dz}{dx} = (\sin x)^x (x \cot x + \log \sin x)$$

$$\therefore \frac{dy}{dx} = (\sin x)^x [x \cos x + \log \sin x] + \frac{1}{\sqrt{1-x}}\frac{1}{2\sqrt{x}}$$

$$= (\sin x)^x (x \cos x + \log \sin x) + \frac{1}{2\sqrt{x(1-x)}}$$

18. The given lines can be expressed as

$$\frac{x-1}{-3}=\frac{y-2}{2\lambda}=\frac{z-3}{2} \text{ and}$$

$$\frac{x-1}{3\lambda}=\frac{y-1}{1}=\frac{z-6}{-7}$$

The direction ratios of these lines are respectively –3, 2λ, 2 and 3λ, 1, –7.

Since the lines are perpendicular, therefore

$$-3(3\lambda)+2\lambda(1)+2(-7)=0$$

$$\Rightarrow \quad -9\lambda+2\lambda-14=0$$

$$\Rightarrow \quad -7\lambda=14 \Rightarrow \lambda=-2$$

19. Given differential equation is

$$(1+x^2)\frac{dy}{dx}+y=\tan^{-1}x$$

The equation can be expressed as

$$\frac{dy}{dx}+\frac{y}{1+x^2}=\frac{\tan^{-1}x}{1+x^2}$$

This is a linear differential equation of the type $\frac{dy}{dx}+Py=Q$

Here $I.F=e^{\int\frac{dx}{1+x^2}}=e^{\tan^{-1}x}$

Its solution is given by

$$y\,e^{\tan^{-1}x}=\int e^{\tan^{-1}x}.\frac{\tan^{-1}x}{1+x^2}dx \qquad \ldots (i)$$

$$\text{Suppose } I=\int e^{\tan^{-1}x}\frac{\tan^{-1}x}{1+x^2}dx$$

Let $\tan^{-1}x=t$

$$\frac{1}{1+x^2}dx=dt$$

$$\Rightarrow \quad I=\int e^t.t\,dt$$

Integrating by parts, we get

$$I=t\,e^t-\int e^t\,dt$$

$$\Rightarrow \quad I=t\,e^t-e^t+C'$$

$$\Rightarrow \quad I=e^{\tan^{-1}x}(\tan^{-1}x-1)+C'$$

From (i)

$$y\,e^{\tan^{-1}x}=e^{\tan^{-1}x}(\tan^{-1}x-1)+C$$

$\Rightarrow \quad y=\tan^{-1}x-1+C\,e^{-\tan^{-1}x}$ which is the solution of given differential equation.

21. Let $|A| = \begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$ Apply $C_1 \to C_1 + C_2 + C_3$

$$|A| = \begin{vmatrix} a+b+c & b & c \\ 0 & b-c & c-a \\ 2(a+b+c) & c+a & a+b \end{vmatrix}$$

Taking $(a+b+c)$ common from C_1

$$|A| = (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 2 & c+a & a+b \end{vmatrix}$$

Apply $R_3 \to R_3 - 2R_1$

$$|A| = (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 0 & c+a-2b & a+b-2c \end{vmatrix}$$

Expand along C_1 to get

$$|A| = (a+b+c)[(b-c)(a+b-2c) - (c+a-2b)(c-a)]$$
$$= (a+b+c)[ab + b^2 - 2bc - ac - cb + 2c^2 - (c^2 - ac + ac - a^2 - 2bc + 2ab)]$$
$$= (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$$
$$= a^3 + b^3 + c^3 - 3abc = \text{RHS}$$

23. $P(G_I) = 0.6$ $\qquad$ $P(G_{II}) = 0.4$

Let E is the event of introducing new product then

$P(E/G_I) = 0.7$ $\qquad$ $P(E/G_{II}) = 0.3$

To find P(G_{II}/E)

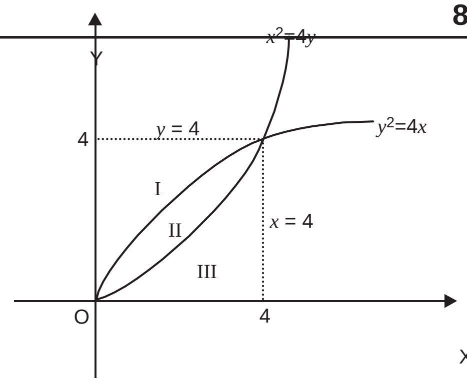
Using Baye's theorem we get

$$P(G_{II}/E) = \frac{P(G_{II}).P(E/G_{II})}{P(G_I).P(E/G_I) + P(G_{II}).P(E/G_{II})}$$
$$= \frac{0.4 \times 0.3}{0.6 \times 0.7 + 0.4 \times 0.3}$$
$$= \frac{0.12}{0.42 + 0.12}$$
$$= \frac{12}{54} = \frac{2}{9}$$

26. We plot the curves $y^2 = 4x$ and $x^2 = 4y$ and also the various areas of the square.

To show that area of regions I = II = III

Area of region $I = \int_0^4 4dx - \int_0^4 2\sqrt{x}dx$



$$= 4x - 2\frac{x^{3/2}}{3/2}\Bigg]_0^4$$

$$= 16 - \frac{4}{3} \times 8 = \frac{16}{3} \text{ square units}$$

Area of Region II $= 2\int_0^4 \sqrt{x}\, dx - \int_0^4 \frac{x^2}{4}\, dx$

$$= 2.\frac{2}{3}x^{3/2} - \frac{x^3}{12}\Bigg]_0^4$$

$$= \frac{4}{3} \times 8 - \frac{64}{12} - 0 = \frac{128 - 64}{12} = \frac{64}{12} = \frac{16}{3} \text{ square units}$$

Area of Region III $= \int_0^4 \frac{x^2}{4}\, dx$

$$= \frac{x^3}{12}\Bigg]_0^4 = \frac{64}{12} = \frac{16}{3} \text{ square units.}$$

Thus, the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of given square into three equal parts.

Set–III

4. Let $I = \int \frac{(1 + \log x)^2}{x}\, dx$

Let $1 + \log x = t$

$$\frac{1}{x}\, dx = dt$$

$$\therefore \quad I = \int t^2\, dt = \frac{t^3}{3} + C$$

$$= \frac{(1 + \log x)^3}{3} + C$$

9. Given $|\vec{a} \times \vec{b}| = \sqrt{3}$

$\Rightarrow \quad ab \sin\theta = \sqrt{3}$

$\Rightarrow \quad 1 \times 2 \sin\theta = \sqrt{3}$ $\qquad$ ($\because a = 1,\ b = 2$)

$$\sin\theta = \frac{\sqrt{3}}{2}$$

$\Rightarrow \quad \theta = \frac{\pi}{3}$ radians.

15. Let $|A| = \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$

Apply $R_1 \to R_1 + bR_3$

$$|A| = \begin{vmatrix} 1+a^2+b^2 & 0 & -b-ba^2-b^3 \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

Taking $1+a^2+b^2$ common from R_1

$$|A| = (1+a^2+b^2)\begin{vmatrix} 1 & 0 & -b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

Apply $R_2 \to R_2 - aR_3$

$$|A| = (1+a^2+b^2)\begin{vmatrix} 1 & 0 & -b \\ 0 & 1+a^2+b^2 & a+a^3+ab^2 \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

Taking $1+a^2+b^2$ common from R_2

$$|A| = (1+a^2+b^2)^2\begin{vmatrix} 1 & 0 & -b \\ 0 & 1 & a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

Apply $R_3 \to R_3 - 2bR_1$

$$|A| = (1+a^2+b^2)^2\begin{vmatrix} 1 & 0 & -b \\ 0 & 1 & a \\ 0 & -2a & 1-a^2+b^2 \end{vmatrix}$$

Expanding along C_1, we get

$$|A| = (1+a^2+b^2)^2[1(1-a^2+b^2+2a^2)]$$

$$= (1+a^2+b^2)^3 = \text{RHS}$$

17. Let $y = x^{\cos x} + (\sin x)^{\tan x}$ … (i)

Let $u = x^{\cos x}, v = (\sin x)^{\tan x}$

Taking log on both side

$\log u = \cos x . \log x, \log v = \tan x \log \sin x$

Differentiating w.r.t. x

$$\frac{1}{u}\frac{du}{dx} = \cos x . \frac{1}{x} + \log x(-\sin x), \frac{1}{v}\frac{dv}{dx} = \frac{\tan x . \cos x}{\sin x} + \log \sin x . \sec^2 x$$

$$\frac{du}{dx} = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \log x\right), \frac{dv}{dx} = (\sin x)^{\tan x}(1 + \sec^2 x \log \sin x)$$

$\therefore$ From (i) we get

$$\frac{dy}{dx} = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \log x\right) + (\sin x)^{\tan x}[1 + \sec^2 x \log \sin x]$$

19. Given differential equation is

$$x \log x \frac{dy}{dx} + y = 2 \log x$$

This can be rearranged as

$$\frac{dy}{dx} + \frac{y}{x \log x} = \frac{2}{x}$$

It is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$

Now, $IF = e^{\int \frac{1}{x \log x} dx} = e^{\log(\log x)} = \log x$

Its solution is given by

$$y \log x = \int \log x \frac{2}{x}\, dx$$

$\Rightarrow$ $y \log x = 2\frac{(\log x)^2}{2} + C$ $\qquad \because \int f(x).f'(x)\, dx = [f(x)]^2 + C$

$\Rightarrow$ $y = \log x + \frac{C}{\log x}$ which is the solution of the given differential equation

20. The given lines on rearrangement are expressed as

$$\frac{x-5}{5\lambda + 2} = \frac{y-2}{-5} = \frac{z-1}{1} \text{ and } \frac{x}{1} = \frac{y + 1/2}{2\lambda} = \frac{z-1}{3}$$

The direction ratios of the two lines are respectively

$5\lambda + 2, -5, 1$ and $1, 2\lambda, 3$

As the lines are perpendicular,

$\therefore$ $(5\lambda + 2) \times 1 - 5(2\lambda) + 1(3) = 0$

$\Rightarrow$ $5\lambda + 2 - 10\lambda + 3 = 0$

$\Rightarrow$ $-5\lambda = -5 \Rightarrow \lambda = 1$

Hence $\lambda = 1$ for lines to be perpendicular.

24. The two circles are re-arranged and expressed as

$y^2 = 9 - x^2$ $\qquad \ldots (i)$

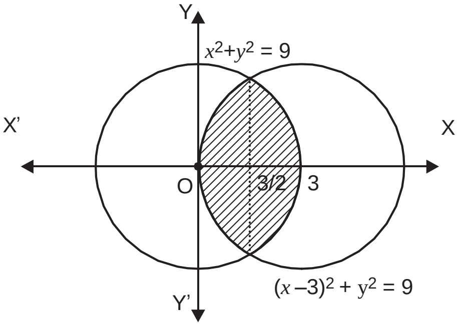
$y^2 = 9 - (x-3)^2$ $\qquad \ldots (ii)$

To find the point of intersection of the circles we equate y^2

$\Rightarrow$ $9 - x^2 = 9 - (x-3)^2$

$\Rightarrow$ $9 - x^2 = 9 - x^2 - 9 + 6x$

$\Rightarrow \qquad x = \frac{3}{2}$



The circles are shown in the figure and the shaded area is the required area.

Now, area of shaded region

$$= 2\left[\int_0^{\frac{3}{2}} \sqrt{9-(x-3)^2}\,dx + \int_{\frac{3}{2}}^{3} \sqrt{9-x^2}\,dx\right]$$

$$= 2\left[\frac{x-3}{2}\sqrt{9-(x-3)^2} + \frac{9}{2}\sin^{-1}\frac{x-3}{3}\right]_0^{3/2} + 2\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right]_{\frac{3}{2}}^{3}$$

$$= 2\left[\frac{-3}{4}\sqrt{9-\frac{9}{4}} + \frac{9}{2}\sin^{-1}\left(-\frac{1}{2}\right) - \frac{9}{2}\sin^{-1}(-1)\right] + 2\left[\frac{9}{2}\sin^{-1}1 - \frac{3}{4}\sqrt{9-\frac{9}{4}} - \frac{9}{2}\sin^{-1}\frac{1}{2}\right]$$

$$= 2\left[\frac{-3}{4}\cdot\frac{3\sqrt{3}}{2} - \frac{9}{2}\cdot\frac{\pi}{6} + \frac{9}{2}\cdot\frac{\pi}{2}\right] + 2\left[\frac{9}{2}\cdot\frac{\pi}{2} - \frac{3}{4}\cdot\frac{3\sqrt{3}}{2} - \frac{9}{2}\cdot\frac{\pi}{6}\right]$$

$$= 2\left[-\frac{9\sqrt{3}}{8} - \frac{3\pi}{4} + \frac{9\pi}{4} + \frac{9\pi}{4} - \frac{9\sqrt{3}}{8} - \frac{3\pi}{4}\right]$$

$$= 2\left[-\frac{9\sqrt{3}}{4} + \frac{12\pi}{4}\right] = 6\pi - \frac{9\sqrt{3}}{2} \text{ square units.}$$

27. The three coins C_1, C_2 and C_3 are choosen randomly.

$\therefore \quad P(C_1) = P(C_2) = P(C_3) = \frac{1}{3}$

Let E be the event that coin shows head.

Then, $\quad P(E/C_1) = 1$

$P(E/C_2) = \frac{75}{100} = \frac{3}{4} \qquad P(E/C_3) = \frac{1}{2}$

To find: $P(C_1/E)$

From Baye's theorem, we have

$$P(C_1/E) = \frac{P(C_1).P(E/C_1)}{P(C_1).P(E/C_1) + P(C_2)P(E/C_2) + P(C_3).P(E/C_3)}$$

$$= \frac{\frac{1}{3}\times 1}{\frac{1}{3}\times 1 + \frac{1}{3}\times\frac{3}{4} + \frac{1}{3}\times\frac{1}{2}} = \frac{\frac{1}{3}}{\frac{1}{3}\left(1+\frac{3}{4}+\frac{1}{2}\right)}$$

$$= \frac{1}{1+\frac{3}{4}+\frac{1}{2}} = \frac{4}{4+3+2} = \frac{4}{9}$$

Thus, probability of getting head from the two headed coin is $\frac{4}{9}$.

EXAMINATION PAPERS – 2009

MATHEMATICS CBSE (All India)

CLASS – XII

Time allowed: 3 hours **Maximum marks: 100**

General Instructions: ***As given in CBSE Examination paper (Delhi)*** **– 2009.**

Set–I

SECTION–A

1. Find the value of x, if $\begin{pmatrix} 3x+y & -y \\ 2y-x & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -5 & 3 \end{pmatrix}$.

2. Let * be a binary operation on N given by $a * b =$ HCF (a, b) $a, b, \in N$. Write the value of 22 * 4.

3. Evaluate : $\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-x^2}}\, dx.$

4. Evaluate : $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx.$

5. Write the principal value of, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$.

6. Write the value of the following determinant : $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

7. Find the value of x, from the following: $\begin{vmatrix} x & 4 \\ 2 & 2x \end{vmatrix} = 0$

8. Find the value of p, if $(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + 3\hat{j} + p\hat{k}) = \vec{0}$.

9. Write the direction cosines of a line equally inclined to the three coordinate axes.

10. If $\vec{p}$ is a unit vector and $(\vec{x} - \vec{p}).(\vec{x} + \vec{p}) = 80$, then find $|\vec{x}|$.

SECTION–B

11. The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When $x = 8$ cm and $y = 6$ cm, find the rate of change of (a) the perimeter, (b) the area of the rectangle.

OR

Find the intervals in which the function f given by $f(x) = \sin x + \cos x$, $0 \le x \le 2\pi$, is strictly increasing or strictly decreasing.

12. If $\sin y = x \sin(a + y)$, prove that $\dfrac{dy}{dx} = \dfrac{\sin^2(a + y)}{\sin a}$.

OR

If $(\cos x)^y = (\sin y)^x$, find $\dfrac{dy}{dx}$.

13. Let $f : N \to N$ be defined by $f(n) = \begin{cases} \dfrac{n+1}{2}, & \text{if } n \text{ is odd} \\ \dfrac{n}{2}, & \text{if } n \text{ is even} \end{cases}$ for all $n \in N$.

Find whether the function f is bijective.

14. Evaluate : $\displaystyle\int \frac{dx}{\sqrt{5 - 4x - 2x^2}}$

OR

Evaluate : $\displaystyle\int x \sin^{-1} x \, dx$

15. If $y = \dfrac{\sin^{-1} x}{\sqrt{1 - x^2}}$, show that $(1 - x^2)\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} - y = 0$.

16. On a multiple choice examination with three possible answers (out of which only one is correct) for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing?

17. Using properties of determinants, prove the following : $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix} = 1$

18. Solve the following differential equation : $x\dfrac{dy}{dx} = y - x\tan\left(\dfrac{y}{x}\right)$

19. Solve the following differential equation : $\cos^2 x\dfrac{dy}{dx} + y = \tan x$.

20. Find the shortest distance between the following two lines :

$$\vec{r} = (1 + \lambda)\hat{i} + (2 - \lambda)\hat{j} + (\lambda + 1)\hat{k};$$

$$\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k}).$$

21. Prove the following : $\cot^{-1}\left(\dfrac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right) = \dfrac{x}{2}, \; x \in \left(0, \dfrac{\pi}{4}\right)$

OR

Solve for $x : 2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

22. The scalar product of the vector $\hat{i}+\hat{j}+\hat{k}$ with the unit vector along the sum of vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ.

SECTION–C

23. Find the equation of the plane determined by the points $A(3,-1,2)$, $B(5,2,4)$ and $C(-1,-1,6)$. Also find the distance of the point $P(6,5,9)$ from the plane.

24. Find the area of the region included between the parabola $y^2=x$ and the line $x+y=2$.

25. Evaluate : $\displaystyle\int_0^{\pi}\frac{x\,dx}{a^2\cos^2 x+b^2\sin^2 x}$

26. Using matrices, solve the following system of equations :

$$\begin{aligned} x+y+z&=6\\ x+2z&=7\\ 3x+y+z&=12 \end{aligned}$$

OR

Obtain the inverse of the following matrix, using elementary operations : $A=\begin{bmatrix}3&0&-1\\2&3&0\\0&4&1\end{bmatrix}$.

27. Coloured balls are distributed in three bags as shown in the following table :

Bag	**Colour of the ball**		
	Black	**White**	**Red**
I	1	2	3
II	2	4	1
III	4	5	3

A bag is selected at random and then two balls are randomly drawn from the selected bag. They happen to be black and red. What is the probability that they came from bag I?

28. A dealer wishes to purchase a number of fans and sewing machines. He has only Rs. 5,760 to invest and has a space for at most 20 items. A fan costs him Rs. 360 and a sewing machine Rs. 240. His expectation is that he can sell a fan at a profit of Rs. 22 and a sewing machine at a profit of Rs. 18. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximize the profit? Formulate this as a linear programming problem and solve it graphically.

29. If the sum of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\frac{\pi}{3}$.

OR

A manufacturer can sell x items at a price of Rs. $\left(5-\frac{x}{100}\right)$ each. The cost price of x items is Rs. $\left(\frac{x}{5}+500\right)$. Find the number of items he should sell to earn maximum profit.

Set–II

Only those questions, not included in Set I, are given

2. Evaluate : $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

5. Find the value of y, if $\begin{pmatrix} x-y & 2 \\ x & 5 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 3 & 5 \end{pmatrix}$.

11. If $y = 3e^{2x} + 2e^{3x}$, prove that

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

18. Find the shortest distance between the following two lines:

$$\vec{r} = (1+2\lambda)\hat{i} + (1-\lambda)\hat{j} + \lambda\hat{k};$$

$$\vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k}).$$

19. Form the differential equation of the family of circles touching the y axis at origin.

21. Using properties of determinants, prove the following:

$$\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3$$

25. Find the area of the region included between the parabola $4y = 3x^2$ and the line $3x - 2y + 12 = 0$.

29. Coloured balls are distributed in three bags as shown in the following table:

Bag	Colour of the ball		
	Black	White	Red
I	2	1	3
II	4	2	1
III	5	4	3

A bag is selected at random and then two balls are randomly drawn from the selected bag. They happen to be white and red. What is the probability that they came from bag II?

Set–III

Only those questions, not included in Set I and Set II are given

7. Evaluate : $\int \frac{\sec^2\sqrt{x}}{\sqrt{x}}\,dx$

10. Find the value of x from the following :

$$\begin{pmatrix} 2x-y & 5 \\ 3 & y \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 3 & -2 \end{pmatrix}.$$

13. Find the shortest distance between the following two lines:

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k});$$

$$\vec{r} = (4 + 2\mu)\hat{i} + (5 + 3\mu)\hat{j} + (6 + \mu)\hat{k}.$$

14. Form the differential equation representing the family of curves given by $(x - a)^2 + 2y^2 = a^2$, where a is an arbitrary constant.

16. Using properties of determinants, prove the following:

$$\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = xyz + xy + yz + zx.$$

18. If $y = e^x (\sin x + \cos x)$, then show that

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

23. Find the area of the region bounded by the curves $y^2 = 4ax$ and $x^2 = 4ay$.

26. A man is known to speak the truth 3 out of 5 times. He throws a die and reports that it is a number greater than 4. Find the probability that it is actually a number greater than 4.

SOLUTIONS

Set – I

SECTION–A

1. Given,

$$\begin{pmatrix} 3x+y & -y \\ 2y-x & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -5 & 3 \end{pmatrix}$$

Using equality of two matrices, we have

$$3x + y = 1, \qquad -y = 2$$
$$\Rightarrow y = -2$$

Substituting the values of y, we get

$$3x + (-2) = 1 \quad \Rightarrow \quad x = 1$$

2. Given $a * b = \text{HCF}\ (a, b),\ a, b \in N$

$\Rightarrow$ $22 * 4 = \text{HCF}\ (22, 4) = 2$

3. $$\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1 - x^2}}\, dx = \left[\sin^{-1} x\right]_0^{1/\sqrt{2}}$$

$$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1} 0 = \frac{\pi}{4}$$

4. Let $I = \int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$ Let $\sqrt{x} = t$

$\frac{1}{2\sqrt{x}}\,dx = dt$

$\Rightarrow \quad I = \int \cos t.2\,dt$

$\Rightarrow \quad I = 2\sin t + C$

$I = 2\sin\sqrt{x} + C$

5. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

$= \cos^{-1}\left(\cos\left(\pi + \frac{\pi}{6}\right)\right)$

$= \cos^{-1}\left(-\cos\frac{\pi}{6}\right)$

$= \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \pi - \frac{\pi}{6}$

$= \frac{5\pi}{6}$

6. Given determinant is

$$|A| = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

Use the transformation $C_1 \to C_1 + C_2 + C_3$

$$|A| = \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} = 0$$

7. We are given that

$$\begin{vmatrix} x & 4 \\ 2 & 2x \end{vmatrix} = 0$$

$\Rightarrow \quad 2x^2 - 8 = 0$

$\Rightarrow \quad 2x^2 = 8$

$\Rightarrow \quad x^2 = 4$

$\Rightarrow \quad x = \pm 2$

8. $(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + 3\hat{j} + p\hat{k}) = \vec{0}$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 27 \\ 1 & 3 & p \end{vmatrix} = \vec{0}$$

$\Rightarrow\quad (6p-81)\hat{i}-(2p-27)\hat{j}+0\hat{k}=\vec{0}$

$\Rightarrow\quad 6p=81$

$\Rightarrow\quad p=\dfrac{81}{6}=\dfrac{27}{2}.$

9. Any line equally inclined to co-ordinate axes will have direction cosines l, l, l

$\therefore\quad l^2+l^2+l^2=1$

$3l^2=1 \qquad \Rightarrow \qquad l=\pm\dfrac{1}{\sqrt{3}}$

$\therefore$ Direction cosines are $+\dfrac{1}{\sqrt{3}},+\dfrac{1}{\sqrt{3}},+\dfrac{1}{\sqrt{3}}$ or $-\dfrac{1}{\sqrt{3}},-\dfrac{1}{\sqrt{3}},-\dfrac{1}{\sqrt{3}}$

10. Given $(\vec{x}-\vec{p}).(\vec{x}+\vec{p})=80$

$\Rightarrow\quad |\vec{x}|^2-|\vec{p}|^2=80$

$\Rightarrow\quad |\vec{x}|^2-1=80$

$\Rightarrow\quad |\vec{x}|^2=81 \qquad$ or $\qquad \vec{x}=9$

SECTION–B

11. Given $\dfrac{dx}{dt}=-5$ cm/min $\qquad \dfrac{dy}{dt}=4$ cm/min

where x = length of rectangle and y = breadth of rectangle.

Perimeter of rectangle is given by

$P=2(x+y)$

$\therefore$ Rate of change of P is

$\dfrac{dP}{dt}=2.\dfrac{dx}{dt}+2\dfrac{dy}{dt}$

$\Rightarrow\quad \dfrac{dP}{dt}=2(-5)+2(4)=-2$

$\Rightarrow\quad \left.\dfrac{dP}{dt}\right|_{(8,6)}=-2$

$x=8$ cm $=-2$ cm/min

$y=6$ cm.

i.e., the perimeter is decreasing at the rate of 2 cm/min.

Now, Area of rectangle is given by

$A=xy$

$\Rightarrow\quad \dfrac{dA}{dt}=x\dfrac{dy}{dx}+y\dfrac{dx}{dt}$

$=4x-5y$

$\Rightarrow \left.\dfrac{dP}{dt}\right|_{(8,6)} = 32 - 30 = 2$

i.e., the area is increasing at the rate of 2 cm^2/min.

OR

Given function $f(x) = \sin x + \cos x \quad 0 \le x \le 2\pi$

$f'(x) = \cos x - \sin x$

For the critical points of the function over the interval $0 \le x \le 2\pi$ is given by

$$f'(x) = 0 \Rightarrow \cos x - \sin x = 0$$
$$\Rightarrow \cos x = \sin x$$
$$\Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$$

Possible intervals are $\left(0, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{5\pi}{4}\right), \left(\frac{5\pi}{4}, 2\pi\right)$

If $0 < x < \frac{\pi}{4}$, $f'(x) = \cos x - \sin x > 0 \quad \because \cos x > \sin x$

$$\Rightarrow f'(x) > 0$$
$\Rightarrow$ $f(x)$ is strictly increasing.

If $\frac{\pi}{4} < x < \frac{5\pi}{4}$, $f'(x) = \cos x - \sin x < 0 \quad \because \cos x < \sin x$

$\Rightarrow$ $f(x)$ is strictly decreasing.

If $\frac{5\pi}{4} < x < 2\pi \Rightarrow f'(x) = \cos x - \sin x > 0 \because \cos x > \sin x$

$\Rightarrow$ $f(x)$ is again strictly increasing.

$\therefore$ Given function $f(x) = \sin x + \cos x\,[0, 2\pi]$ is strictly increasing $\forall x \in \left(0, \frac{\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right)$ while it is strictly decreasing $\forall\, x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

12. If $\sin y = x \sin(a + y)$

$$\Rightarrow \frac{\sin y}{\sin(a+y)} = x$$

Differentiating both sides w.r.t. x

$$\Rightarrow \frac{\sin(a+y).\cos y\dfrac{dy}{dx} - \sin y \cos(a+y).\dfrac{dy}{dx}}{\sin^2(a+y)} = 1$$

$$\Rightarrow \frac{\dfrac{dy}{dx}\left[\sin(a+y)\cos y - \sin y.\cos(a+y)\right]}{\sin^2(a+y)} = 1$$

$$\Rightarrow \frac{dy}{dx}\left[\sin(a+y-y)\right] = \sin^2(a+y)$$

$$\therefore \quad \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

OR

Given $(\cos x)^y = (\sin y)^x$

Taking log on both sides

$\therefore \quad \log (\cos x)^y = \log(\sin y)^x$

$\Rightarrow \quad y \log(\cos x) = x \log(\sin y)$

Differentiating both sides w.r.t. x, we get

$$y \frac{1}{\cos x} \cdot \frac{d}{dx} \cos x + \log(\cos x) \cdot \frac{dy}{dx} = x \cdot \frac{1}{\sin y} \cdot \frac{d}{dx} \sin y + \log \sin y . 1$$

$$\Rightarrow \quad -y \frac{\sin x}{\cos x} + \log(\cos x) \cdot \frac{dy}{dx} = x \frac{\cos y}{\sin y} \frac{dy}{dx} + \log \sin y$$

$$\Rightarrow \quad -y \tan x + \log(\cos x) \frac{dy}{dx} = x \cot y \frac{dy}{dx} + \log \sin y$$

$$\Rightarrow \quad \log(\cos x) \cdot \frac{dy}{dx} - x \cot y \frac{dy}{dx} = \log \sin y + y \tan x$$

$$\Rightarrow \quad \frac{dy}{dx} [\log(\cos x) - x \cot y] = \log \sin y + y \tan x$$

$$\therefore \quad \frac{dy}{dx} = \frac{\log \sin y + y \tan x}{\log \cos x - x \cot y}$$

13. Given $f: N \to N$ defined such that $f(n) = \begin{cases} \dfrac{n+1}{2}, & \text{if } n \text{ is odd} \\ \dfrac{n}{2}, & \text{if } n \text{ is even} \end{cases}$

Let $x, y \in N$ and let they are odd then

$$f(x) = f(y) \Rightarrow \frac{x+1}{2} = \frac{y+1}{2} \Rightarrow x = y$$

If $x, y \in N$ are both even then also

$$f(x) = f(y) \Rightarrow \frac{x}{2} = \frac{y}{2} \Rightarrow x = y$$

If $x, y \in N$ are such that x is even and y is odd then

$$f(x) = \frac{x+1}{2} \text{ and } f(y) = \frac{y}{2}$$

Thus, $x \neq y$ for $f(x) = f(y)$

Let $x = 6$ and $y = 5$

We get $f(6) = \frac{6}{2} = 3, \qquad f(5) = \frac{5+1}{2} = 3$

$\therefore \qquad f(x) = f(y)$ but $x \neq y$ ...(i)

So, $f(x)$ is not one-one.

Hence, $f(x)$ is not bijective.

14. Let $I = \int \frac{dx}{\sqrt{5 - 4x - 2x^2}}$

$$\Rightarrow \quad I = \int \frac{dx}{\sqrt{-2\left(x^2 + 2x - \frac{5}{2}\right)}}$$

$$\Rightarrow \quad I = \int \frac{dx}{\sqrt{-2\left[(x+1)^2 - \frac{7}{2}\right]}}$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(\sqrt{\frac{7}{2}}\right)^2 - (x+1)^2}} = \frac{1}{\sqrt{2}} \sin^{-1} \frac{\sqrt{2}(x+1)}{\sqrt{7}} + C$$

OR

Let $I = \int \underset{II}{x} \underset{I}{\sin^{-1} x}\, dx$

$$I = \sin^{-1} x . \frac{x^2}{2} - \int \frac{x^2}{2\sqrt{1 - x^2}} dx \qquad \text{(using integration by parts)}$$

$$\Rightarrow \quad I = \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \frac{1 - x^2 - 1}{\sqrt{1 - x^2}} dx$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \sqrt{1 - x^2}\, dx - \frac{1}{2} \sin^{-1} x$$

$$= \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \sin^{-1} x + \frac{1}{2}\left[\frac{x}{2}\sqrt{1 - x^2} + \frac{1}{2} \sin^{-1} x\right] + C$$

$$= \frac{x^2}{2} \sin^{-1} x - \frac{1}{4} \sin^{-1} x + \frac{1}{4} x\sqrt{1 - x^2} + C$$

$$= \frac{1}{4}\left[(2x^2 - 1) \sin^{-1} x + x\sqrt{1 - x^2}\right] + C$$

15. If $y = \frac{\sin^{-1} x}{\sqrt{1 - x^2}}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{1 - x^2} . \frac{1}{\sqrt{1 - x^2}} - \sin^{-1} x . \frac{-2x}{2\sqrt{1 - x^2}}}{1 - x^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1 + xy}{1 - x^2} \qquad \text{...}(i)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{(1 - x^2)\left(x\frac{dy}{dx} + y\right) + 2x(1 + xy)}{(1 - x^2)^2}$$

$$\Rightarrow \quad (1-x^2)^2 \frac{d^2y}{dx^2} = (1-x^2)x.\frac{dy}{dx} + y(1-x^2) + 2x(1+xy)$$

$$\Rightarrow \quad (1-x^2)^2 \frac{d^2y}{dx^2} = (1-x^2)x.\frac{dy}{dx} + y(1-x^2) + 2x.(1-x^2)\frac{dy}{dx} \qquad \text{(using } (i))$$

$$\Rightarrow \quad (1-x^2)^2 \frac{d^2y}{dx^2} = 3x(1-x^2)\frac{dy}{dx} + y(1-x^2)$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} = 3x\frac{dy}{dx} + y$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0$$

16. Let p = probability of correct answer $= \frac{1}{3}$

$\Rightarrow \quad q$ = probability of incorrect answer $= \frac{2}{3}$

Here total number of questions = 5

P(4 or more correct) = P(4 correct) + P(5 correct)

$$= {}^5C_4 p^4 q^1 + {}^5C_5 p^5 q^0 \quad \text{using } P(r \text{ success}) = {}^nC_r\, p^r\, q^{n-r}$$

$$= 5 \times \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right) + 1 \times \left(\frac{1}{3}\right)^5$$

$$= 5 \times \frac{1}{81} \times \frac{2}{3} + \frac{1}{243}$$

$$= \frac{11}{243}$$

17. Let $|A| = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix}$

Using the transformation $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$

$$|A| = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & -1+p \\ 0 & 3 & -2+3p \end{vmatrix}$$

Using $R_3 \to R_3 - 3R_2$

$$\Rightarrow \quad |A| = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & -1+p \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along column C_1, we get

$$|A| = 1$$

18. Given differential equation is

$$x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \tan\frac{y}{x}$$

It is a homogeneous differential equation.

Let $y = xt$

$$\Rightarrow \frac{dy}{dx} = x.\frac{dt}{dx} + t$$

$$\therefore \quad x\frac{dt}{dx} + t = t - \tan t$$

$$\Rightarrow x\frac{dt}{dx} = -\tan t$$

$$\Rightarrow \frac{dt}{\tan t} = -\frac{dx}{x}$$

$$\Rightarrow \cot t.dt = -\frac{dx}{x}$$

Integrating both sides

$$\therefore \int \cot t.dt = -\int \frac{dx}{x}$$

$$\Rightarrow \log|\sin t| = -\log|x| + \log C$$

$$\Rightarrow \log\left|\sin\left(\frac{y}{x}\right)\right| + \log x = \log C$$

$$\Rightarrow \log\left|x.\sin\left(\frac{y}{x}\right)\right| = \log C$$

Hence $x.\sin\frac{y}{x} = C$

19. Given differential equation is

$$\cos^2 x.\frac{dy}{dx} + y = \tan x$$

$$\Rightarrow \frac{dy}{dx} + y\sec^2 x = \tan x.\sec^2 x$$

Given differential equation is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$

I.F. $= e^{\int Pdx} = e^{\int \sec^2 xdx} = e^{\tan x}$

$\therefore$ Solution is given by

$$e^{\tan x}y = \int \tan x.\sec^2 x.e^{\tan x}dx$$

Let $I = \int \tan x.\sec^2 x.e^{\tan x}dx$

Let $\tan x = t, \quad \sec^2 x dx = dt$

$\Rightarrow \quad I = \int t\, e^t\, dt$

Integrating by parts

$\therefore \quad I = te^t - \int e^t\, dt = t\, e^t - e^t + C,$

$\Rightarrow \quad I = \tan x\, e^{\tan x} - e^{\tan x} + C,$

Hence $e^{\tan x} y = e^{\tan x}(\tan x - 1) + C$

$\Rightarrow \quad y = \tan x - 1 + C\, e^{-\tan x}$

20. The given equation of the lines can be re-arranged as given below.

$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and

$\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$

Thus $\quad \vec{a}_1 = \hat{i} + 2\hat{j} + \hat{k}, \quad \vec{b}_1 = \hat{i} - \hat{j} + \hat{k},$

$\vec{a}_2 = 2\hat{i} - \hat{j} - \hat{k}, \quad \vec{b}_2 = 2\hat{i} + \hat{j} + 2\hat{k}$

The given lines are not parallel

$\therefore$ Shortest distance between lines $= \left| \dfrac{(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right|$

We have $\vec{a}_2 - \overleftarrow{a}_1 = \hat{i} - 3\hat{j} - 2\hat{k}$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix} = -3\hat{i} + 0\hat{j} + 3\hat{k}$$

$|\vec{b}_1 \times \vec{b}_2| = \sqrt{9+9} = 3\sqrt{2}$

$\therefore$ Shortest distance $= \left| \dfrac{(\hat{i} - 3\hat{j} - 2\hat{k}).(-3\hat{i} + 3\hat{k})}{3\sqrt{2}} \right| = \left| \dfrac{-3-6}{3\sqrt{2}} \right|$

$$= \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2} \text{ units.}$$

21. $\cot^{-1}\left[\dfrac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right]$ where $x \in \left(0, \dfrac{\pi}{4}\right)$

$$= \cot^{-1}\left[\frac{\sqrt{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2} + \sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2} - \sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}}\right]$$

$$x\Big) = \cot^{-1}\left(\frac{\cos\frac{x}{2} + \sin\frac{x}{2} + \cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2} - \cos\frac{x}{2} + \sin\frac{x}{2}}\right)$$

$$= \cot^{-1}\left[\cot\frac{x}{2}\right] = \frac{x}{2}$$

OR

Given $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}\left(\frac{2}{\sin x}\right) \qquad \because 2\tan^{-1}A = \tan^{-1}\left(\frac{2A}{1-A^2}\right)$$

$$\Rightarrow \quad \frac{2\cos x}{\sin^2 x} = \frac{2}{\sin x}$$

$$\Rightarrow \quad \cot x = 1$$

$$\therefore \quad x = \frac{\pi}{4}$$

22. Let sum of vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\lambda\hat{i} + 2\hat{j} + 3\hat{k} = \vec{a}$

$$\vec{a} = (2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(2+\lambda)^2 + 36 + 4}}$$

Hence $(\hat{i} + \hat{j} + \hat{k}) \cdot \hat{a} = (\hat{i} + \hat{j} + \hat{k}) \cdot \dfrac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(2+\lambda)^2 + 40}} = 1$

$$\Rightarrow \quad (2+\lambda) + 6 - 2 = \sqrt{(2+\lambda)^2 + 40}$$

$$\Rightarrow \quad (\lambda + 6)^2 = (2+\lambda)^2 + 40$$

$$\Rightarrow \quad \lambda^2 + 36 + 12\lambda = 4 + \lambda^2 + 4\lambda + 40$$

$$\Rightarrow \quad 8\lambda = 8 \Rightarrow \lambda = 1.$$

SECTION–C

23. The equation of the plane through three non-collinear points A(3, –1, 2), B(5, 2, 4) and (–1, –1, 6) can be expressed as

$$\begin{vmatrix} x-3 & y+1 & z-2 \\ 5-3 & 2+1 & 4-2 \\ -1-3 & -1+1 & 6-2 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x-3 & y+1 & z-2 \\ 2 & 3 & 2 \\ -4 & 0 & 4 \end{vmatrix} = 0$$

$\Rightarrow \quad 12(x-3)-16(y+1)+12(z-2)=0$

$\Rightarrow \quad 12x-16y+12z-76=0 \Rightarrow 3x-4y+3z-19=0$ is the required equation.

Now distance of $P(6, 5, 9)$ from the plane is given by

$$=\left|\frac{3\times 6-4(5)+3(9)-19}{\sqrt{9+16+9}}\right| = \left|\frac{6}{\sqrt{34}}\right| = \frac{6}{\sqrt{34}} \text{ units.}$$

24. Plot the two curves $y^2 = x$ … (*i*)

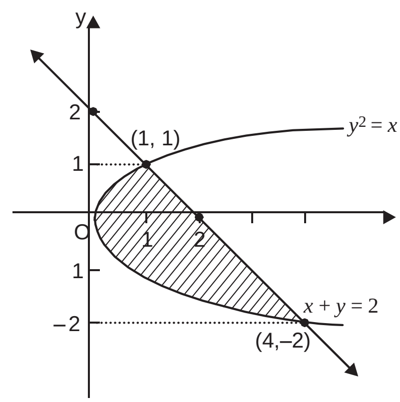
and $x + y = 2$ … (*ii*)

Solving (*i*) and (*ii*), we have

$y^2 + y = 2$

$\Rightarrow \; (y+2)(y-1)=0$

$\Rightarrow \; y=-2, 1 \quad \therefore \quad x = 4, 1$



We have to determine the area.of the shaded region.

Required Area $= \int_{-2}^{1}(2-y)dy - \int_{-2}^{1} y^2 dy$

$$= 2y - \frac{y^2}{2} - \frac{y^3}{3}\Big]_{-2}^{1}$$

$$= \left(2-\frac{1}{2}-\frac{1}{3}\right)-\left(-4-\frac{4}{2}+\frac{8}{3}\right) = \frac{9}{2} \text{ square units.}$$

25. Let $I = \int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2 \sin^2 x}$ … (*i*)

$\Rightarrow \; I = \int_0^{\pi} \frac{\pi - x}{a^2\cos^2(\pi - x) + b^2\sin^2(\pi - x)}$ [using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$]

$\Rightarrow \; I = \int_0^{\pi} \frac{\pi - x}{a^2\cos^2 x + b^2\sin^2 x}$ … (*ii*)

Adding (*i*) and (*ii*)

$$2I = \int_0^{\pi} \frac{\pi}{a^2\cos^2 x + b^2\sin^2 x}dx$$

$$I = \frac{\pi}{2}\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$$

Divide numerator and denominator by $\cos^2 x$

$I = \frac{\pi}{2}\int_0^{\pi} \frac{\sec^2 x\,dx}{a^2 + b^2\tan^2 x} \Rightarrow I = \pi\int_0^{\frac{\pi}{2}} \frac{\sec^2 x\,dx}{a^2 + b^2\tan^2 x}$ [using $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$]

Let $b \tan x = t$

$$b \sec^2 x \, dx = dt$$

When $x = 0, \quad t = 0$

$$x = \frac{\pi}{2} \quad t = \infty$$

$$I = \frac{\pi}{b}\int_0^\infty \frac{dt}{a^2 + t^2} = \frac{\pi}{b}\cdot\frac{1}{a}\tan^{-1}\frac{t}{a}\Big]_0^\infty$$

$$I = \frac{\pi}{ab}(\tan^{-1}\infty - \tan^{-1}0) = \frac{\pi}{ab}\cdot\frac{\pi}{2}$$

$$I = \frac{\pi^2}{2ab}$$

26. The given system of equation are

$$x + y + z = 6$$
$$x + 2z = 7$$
$$3x + y + z = 12$$

In matrix form the equation can be written as $AX = B$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$

$|A| = 1(0-2) - 1(1-6) + 1(1-0) = 4 \neq 0 \Rightarrow A^{-1}$ exists.

To find Adj A we have

$$C_{11} = -2 \quad C_{12} = 5 \quad C_{13} = 1$$
$$C_{21} = 0 \quad C_{22} = -2 \quad C_{23} = 2$$
$$C_{31} = 2 \quad C_{32} = -1 \quad C_{33} = -1$$

$\therefore$ Matrix of co-factors of elements $= \begin{bmatrix} -2 & 5 & 1 \\ 0 & -2 & 2 \\ 2 & -1 & -1 \end{bmatrix}$

$$\text{Adj } A = \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{Adj}\,A}{|A|} = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$

$$\Rightarrow \quad X = A^{-1}B = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -12+24 \\ 30-14-12 \\ 6+14-12 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 12 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore$ Solution of the equations is $x = 3, y = 1, z = 2$

OR

Given matrix is $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}$

We know $A = IA$

$$\therefore \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Apply $R_1 \to R_1 - R_2$

$$\Rightarrow \begin{bmatrix} 1 & -3 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Apply $R_2 \to R_2 - 2R_1$

$$\Rightarrow \begin{bmatrix} 1 & -3 & -1 \\ 0 & 9 & 2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Apply $R_2 \to R_2 - 2R_3$

$$\Rightarrow \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix} A$$

Apply $R_1 \to R_1 + 3R_2$, $R_3 \to R_3 - 4R_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 8 & -6 \\ -2 & 3 & -2 \\ 8 & -12 & 9 \end{bmatrix} A$$

Apply $R_1 \to R_1 + R_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 8 & -12 & 9 \end{bmatrix} A$$

$$\Rightarrow \quad A^{-1} = \begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 8 & -12 & 9 \end{bmatrix}$$

27. Given distribution of the balls is shown in the table

Bag	**Colour of the ball**		
	Black	**White**	**Red**
I	1	2	3
II	2	4	1
III	4	5	3

As bags are selected at random $P(\text{bag } I) = \frac{1}{3} = P(\text{bag } II) = P(\text{bag } III)$

Let E be the event that 2 balls are 1 black and 1 red.

$$P(E/\text{bag I}) = \frac{{}^1C_1 \times {}^3C_1}{{}^6C_2} = \frac{1}{5} \qquad P(E/\text{bag II}) = \frac{{}^2C_1 \times {}^1C_1}{{}^7C_2} = \frac{2}{21}$$

$$P(E/\text{bag III}) = \frac{{}^4C_1 \times {}^3C_1}{{}^{12}C_2} = \frac{2}{11}$$

We have to determine

$$P(\text{bag I}/E) = \frac{P(\text{bag I}).P(\text{E / bag I})}{\sum_{i=\text{I}}^{\text{III}} P(\text{bag } i)\,.P(\text{E / bag } i)}$$

$$= \frac{\frac{1}{3} \times \frac{1}{5}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{2}{21} + \frac{1}{3} \times \frac{2}{11}} = \frac{\frac{1}{3} \times \frac{1}{5}}{\left(\frac{1}{5} + \frac{2}{21} + \frac{2}{11}\right)\frac{1}{3}}$$

$$= \frac{\frac{1}{5}}{\frac{1}{5} + \frac{2}{21} + \frac{2}{11}} = \frac{231}{551}$$

28. Let the no. of fans purchased by the dealer $= x$

and number of sewing machines purchased $= y$

then the L.P.P. is formulated as

$Z = 22x + 18y$ to be maximised subject to constrains

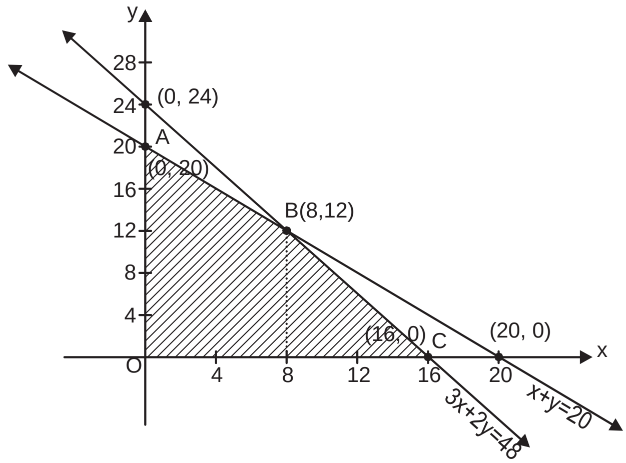
$x + y \le 20$ … (i) [space only for 20 items]

$360x + 240y \le 5760$

$\Rightarrow$ $3x + 2y \le 48$ … (ii)

$x \ge 0, y \ge 0$ … (iii)

We plot the graph of the constraints.



As per the constraints the feasible solution is the shaded region.

Possible points for maximising Z are $A(0, 20)$, $B(8, 12)$, $C(16, 0)$

$$Z]_A = 22 \times 0 + 18 \times 20 = 360$$
$$Z]_B = 22 \times 8 + 18 \times 12 = 392$$
$$Z]_C = 22 \times 16 + 18 \times 0 = 352$$

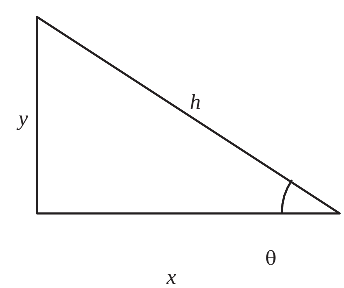
Hence profit is maximum of Rs 392 when the dealer purchases 8 fans and 12 sewing machines.

29. Let the hypotenuse and one side of the right triangle be h and x respectively.

Then $\quad h + x = k \quad$ (given as constant)

Let the third side of the triangle be y

$$y^2 + x^2 = h^2 \quad \text{(using Pythagoras theorem)}$$



$$\Rightarrow \quad y = \sqrt{h^2 - x^2}$$

$$\Rightarrow \quad A = \text{Area of } \Delta = \frac{1}{2} yx = \frac{1}{2} x\sqrt{h^2 - x^2}$$

$$\therefore \quad A = \frac{x}{2}\sqrt{(k - x)^2 - x^2}$$

$$A = \frac{x}{2}\sqrt{k^2 - 2kx}$$

Squaring both sides

$$A^2 = \frac{x^2}{4}(k^2 - 2kx)$$

For maxima we find $\frac{dA}{dx}$

$$2A\frac{dA}{dx} = \frac{xk^2}{2} - \frac{3kx^2}{2} \qquad \ldots(i)$$

If $\frac{dA}{dx} = 0 \Rightarrow \frac{xk^2}{2} = \frac{3kx^2}{2} \qquad \Rightarrow \qquad \frac{k}{3} = x$

Differentiating (i) again w.r.t. x we get

$$2\left(\frac{dA}{dx}\right)^2 + 2.A\frac{d^2A}{dx^2} = \frac{k^2}{2} - 3kx$$

$$\Rightarrow \qquad 2 \times 0 + 2 \cdot A \cdot \frac{d^2A}{dx^2} = \frac{k^2}{2} - 3k.\frac{k}{3}\Bigg] \text{at } x = \frac{k}{3}$$

$$\Rightarrow \qquad \frac{d^2A}{dx^2} = -\frac{k^2}{2}.\frac{1}{2A} < 0$$

$\therefore$ Area is maximum $x = k/3$

$\Rightarrow \qquad h = 2k/3$

In the right triangle, $\cos\theta = \frac{x}{h} = \frac{k/3}{2k/3} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

OR

Selling price of x items $= SP = \left(5 - \frac{x}{100}\right)x$

Cost price of x items $= CP = \frac{x}{5} + 500$

Let profit $= P = 5x - \frac{x^2}{100} - \frac{x}{5} - 500$

$$P = \frac{24x}{5} - \frac{x^2}{100} - 500$$

To find maximisation of profit function $\frac{dP}{dx} = 0$

$$\Rightarrow \qquad \frac{dP}{dx} = \frac{24}{5} - \frac{x}{50} = 0 \qquad \ldots(i)$$

$$\Rightarrow \qquad \frac{24}{5} - \frac{x}{50} = 0 \qquad \Rightarrow \qquad \frac{24}{5} = \frac{x}{50}$$

$\Rightarrow \qquad x = 240$ items.

Differentiating (i) again w.r.t. x

$$\frac{d^2P}{dx^2} = \frac{-1}{50} < 0$$

$\therefore$ Profit is maximum if manufacturer sells 240 items

Set–II

2. To find $I = \int \frac{\sin\sqrt{x}}{\sqrt{x}}dx$

Let $\sqrt{x} = t \qquad \therefore \quad \frac{1}{2\sqrt{x}}\,dx = dt$

$$I = 2\int \sin t\, dt \qquad [\text{Let } \sqrt{x} = t \ \therefore \ \frac{1}{2\sqrt{x}}\, dx = dt]$$

$$= -2\cos t + c = -2\cos\sqrt{x} + C$$

5. Using equality of two matrices, we have

$x - y = 2$ equating a_{11} elements of two sides

$x = 3$ equating a_{21} elements of two sides

$\Rightarrow \ 3 - y = 2 \Rightarrow -y = -1 \ \therefore \ y = 1$

11. Given

$$y = 3e^{2x} + 2e^{3x} \qquad \dots (i)$$

Differentiating w.r.t. x

$$\frac{dy}{dx} = 3.2e^{2x} + 2.3e^{3x} = 6e^{2x} + 6e^{3x}$$

$$\Rightarrow \quad \frac{dy}{dx} = 6e^{2x} + \frac{6(y - 3e^{2x})}{2} \qquad \text{(using (i))}$$

$$\Rightarrow \quad \frac{dy}{dx} = 6e^{2x} + 3y - 9e^{2x} = -3e^{2x} + 3y \qquad \dots (ii)$$

Differentiating again w.r.t. x

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 3.\frac{dy}{dx} - 6e^{2x} \qquad \dots (iii)$$

From (ii) $\frac{dy}{dx} - 3y = -3e^{2x}$

$$\Rightarrow \quad \frac{\frac{dy}{dx} - 3y}{-3} = e^{2x}$$

Substitute in (iii)

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 3.\frac{dy}{dx} - 6\left(\frac{\frac{dy}{dx} - 3y}{-3}\right)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 3\frac{dy}{dx} + 2\frac{dy}{dx} - 6y$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - \frac{5dy}{dx} + 6y = 0$$

18. Given lines are

$$\vec{r} = (1 + 2\lambda)\hat{i} + (1 - \lambda)\hat{j} + \lambda\hat{k} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$$

$$\vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$$

$$\therefore \quad \left.\begin{aligned} \vec{a}_1 &= \hat{i} + \hat{j} \\ \vec{a}_2 &= 2\hat{i} + \hat{j} - \hat{k} \end{aligned}\right\} \Rightarrow \vec{a}_2 - \vec{a}_1 = \hat{i} - \hat{k}$$

$$\left.\begin{array}{l}\vec{b}_1 = 2\hat{i} - \hat{j} + \hat{k} \\ \vec{b}_2 = 3\hat{i} - 5\hat{j} + 2\hat{k}\end{array}\right\} \Rightarrow \text{lines are not parallel}$$

$\therefore$ Shortest distance $= \left|\dfrac{(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right|$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix} = 3\hat{i} - \hat{j} - 7\hat{k}$$

$\Rightarrow \; |\vec{b}_1 \times \vec{b}_2| = \sqrt{9 + 1 + 49} = \sqrt{59}$

$\therefore$ Shortest distance $= \left|\dfrac{(\hat{i} - \hat{k}).(3\hat{i} - \hat{j} - 7\hat{k})}{\sqrt{59}}\right|$

$= \dfrac{10}{\sqrt{59}}$ units

19. As the circle touches y axis at origin, x axis is its diameter. Centre lies on x axis *i.e.*, centre is $(r, 0)$. Hence equations of circle will be

$$(x - r)^2 + (y - 0)^2 = r^2 \qquad \ldots (i)$$

$\Rightarrow \; x^2 + y^2 - 2rx = 0$

Differentiating w.r.t. 'x' we get

$$2x + 2y\frac{dy}{dx} - 2r = 0 \Rightarrow r = x + y\frac{dy}{dx}$$

Putting value of r in (i) we get

$$\left(x - x - y\frac{dy}{dx}\right)^2 + y^2 = \left(x + y\frac{dy}{dx}\right)^2$$

$$\Rightarrow \; y^2\left(\frac{dy}{dx}\right)^2 + y^2 = x^2 + y^2\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx}$$

$\Rightarrow \; 2xy\dfrac{dy}{dx} + x^2 - y^2 = 0$ which is the required differential equation.

21. Given determinant is

$$\begin{vmatrix} x + y & x & x \\ 5x + 4y & 4x & 2x \\ 10x + 8y & 8x & 3x \end{vmatrix}$$

Taking x common from both C_2 and C_3 we get

$$x^2\begin{vmatrix} x+y & 1 & 1 \\ 5x+4y & 4 & 2 \\ 10x+8y & 8 & 3 \end{vmatrix}$$

Apply $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$ we get

$$x^2\begin{vmatrix} x+y & 1 & 1 \\ 3x+2y & 2 & 0 \\ 7x+5y & 5 & 0 \end{vmatrix}$$

Expanding along C_3 we get

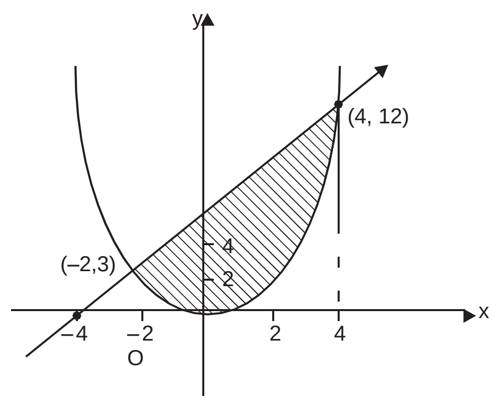
$$x^2[(15x+10y-14x-10y)] = x^3 = \text{RHS}$$

25. Given the equation of parabola $4y = 3x^2 \Rightarrow y = \dfrac{3x^2}{4}$ ... (*i*)

and the line $3x - 2y + 12 = 0$

$\Rightarrow \quad \dfrac{3x+12}{2} = y$ ... (*ii*)

The line intersect the parabola at (–2, 3) and (4, 12). Hence the required area will be the shaded region.



Required Area $= \displaystyle\int_{-2}^{4} \frac{3x+12}{2}\,dx - \int_{-2}^{4} \frac{3x^2}{4}\,dx$

$= \left[\dfrac{3}{4}x^2 + 6x - \dfrac{x^3}{4}\right]_{-2}^{4}$

$= (12 + 24 - 16) - (3 - 12 + 2)$

$= 20 + 7 = 27$ square units.

29. From the given distribution of balls in the bags.

Bag	Colour of the ball		
	Black	**White**	**Red**
I	2	1	3
II	4	2	1
III	5	4	3

As bags are randomly selected

$P(\text{bag } I) = 1/3 = P(\text{bag } II) = P(\text{bag } III)$

Let E be the event that the two balls are 1 white + 1 Red

$$P(E/\text{bag I}) = \frac{{}^1C_1 \times {}^3C_1}{{}^6C_2} = \frac{1}{5} \quad P(E/\text{bag II}) = \frac{{}^2C_1 \times {}^1C_1}{{}^7C_2} = \frac{2}{21}$$

$$P(E/\text{bag III}) = \frac{{}^4C_1 \times {}^3C_1}{{}^{12}C_2} = \frac{2}{11}$$

$$\therefore \quad P(\text{bag II}/E) = \frac{P(\text{bag II}) \cdot P(E/\text{bag II})}{\sum_{i=I}^{III} P(\text{bag } i) \cdot P(E/\text{bag } i)}$$

$$= \frac{\frac{1}{3} \times \frac{2}{21}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{2}{21} + \frac{1}{3} \times \frac{2}{11}} = \frac{\frac{1}{3} \times \frac{2}{21}}{\frac{1}{3}\left(\frac{1}{5} + \frac{2}{21} + \frac{2}{\ }\right)}$$

$$= \frac{\frac{2}{21}}{\frac{1}{5} + \frac{2}{21} + \frac{2}{11}} = \frac{110}{551}$$

Set–III

7. Let $I = \int \frac{\sec^2 \sqrt{x}}{\sqrt{x}}\, dx$

Let $\sqrt{x} = t \Rightarrow \frac{1}{\sqrt{x}}\, dx = 2dt$

$\therefore \quad I = 2\int \sec^2 t\, dt = 2 \tan t + C$

$\Rightarrow \quad I = 2 \tan \sqrt{x} + C$

10. Using equality of two matrices

$$\begin{bmatrix} 2x - y & 5 \\ 3 & y \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 3 & -2 \end{bmatrix}$$

$\Rightarrow \qquad 2x - y = 6$

$\boxed{y = -2}$ equating a_{11}

$\therefore$ $\boxed{x = 2}$ equating a_{22}

13. The given lines are

$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$ … (i)

$\Rightarrow \quad a_1 = \hat{i} + 2\hat{j} + 3\hat{k}, \qquad \vec{b_1} = \hat{i} - 3\hat{j} + 2\hat{k}$

$\vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu(2\hat{i} + 3\hat{j} + \hat{k})$ … (ii) [by rearranging given equation]

$\vec{a_2} = 4\hat{i} + 5\hat{j} + 6\hat{k} \qquad \vec{b_2} = 2\hat{i} + 3\hat{j} + \hat{k}$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix} = -9\hat{i} + 3\hat{j} + 9\hat{k}$$

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{81+9+81} = \sqrt{171} = 3\sqrt{19}$$

$$\vec{a}_2 - \vec{a}_1 = 3\hat{i} + 3\hat{j} + 3\hat{k}$$

As lines (*i*) and (*ii*) are not parallel, the shortest distance

$$= \left| \frac{(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right| = \left| \frac{(3\hat{i} + 3\hat{j} + 3\hat{k}) \cdot (-9\hat{i} + 3\hat{j} + 9\hat{k})}{3\sqrt{19}} \right|$$

Shortest distance $= \left| \dfrac{-27+9+27}{3\sqrt{19}} \right| = \dfrac{3}{\sqrt{19}}$ units

14. Equation of family of curves is

$$(x-a)^2 + 2y^2 = a^2 \qquad \ldots (i)$$

$$\Rightarrow \quad x^2 + 2y^2 - 2ax = 0 \qquad \ldots (ii)$$

Differentiating w.r.t. 'x'

$$2x + 4y\frac{dy}{dx} - 2a = 0$$

$$\Rightarrow \quad a = x + 2yy_1$$

Substituting value of 'a' in (*ii*)

$$x^2 + 2y^2 - 2(x + 2yy_1).x = 0$$

$\Rightarrow \quad 2y^2 - x^2 - 4xyy_1 = 0$ which is required differential equation.

16. Given determinant is

$$|A| = \begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix}$$

Apply $C_2 \to C_2 - C_3$

$$|A| = \begin{vmatrix} 1+x & 0 & 1 \\ 1 & y & 1 \\ 1 & -z & 1+z \end{vmatrix}$$

Apply $C_1 \to C_1 - C_3$

$$|A| = \begin{vmatrix} x & 0 & 1 \\ 0 & y & 1 \\ -z & -z & 1+z \end{vmatrix}$$

Apply $C_1 \to C_1 - xC_3$

$$|A| = \begin{vmatrix} 0 & 0 & 1 \\ -x & y & 1 \\ -z-x-xz & -z & 1+z \end{vmatrix}$$

Expand along R_1

$$|A| = 1(xz + yz + xy + xyz) = \text{RHS}$$

18. Given equation is

$$y = e^x(\sin x + \cos x)$$

Differentiating w.r.t. 'x' we get

$$\frac{dy}{dx} = e^x(\cos x - \sin x) + e^x(\sin x + \cos x)$$

$$\Rightarrow \frac{dy}{dx} = e^x(\cos x - \sin x) + y$$

Differentiating again w.r.t. 'x' we get

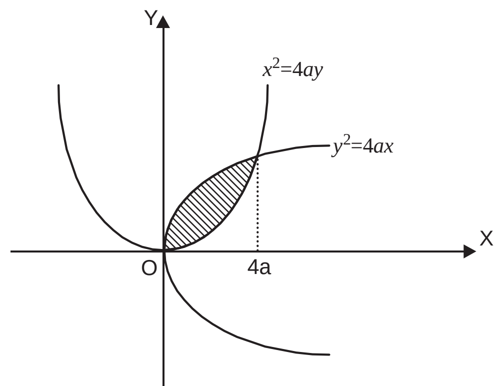
$$\Rightarrow \frac{d^2y}{dx^2} = e^x(-\sin x - \cos x) + e^x(\cos x - \sin x) + \frac{dy}{dx} = -y + \frac{dy}{dx} - y + \frac{dy}{dx}$$

$$\therefore \boxed{\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0}$$

23. The curves $y^2 = 4ax$ and $x^2 = 4ay$ intersects at points where $\left(\frac{x^2}{4a}\right)^2 = 4ax$

$$\Rightarrow \frac{x^4}{16a^2} = 4ax \quad \Rightarrow \quad x^4 = 64a^3x$$

$$\Rightarrow x(x^3 - 64a^3) = 0 \quad \Rightarrow \quad x = 0 \text{ or } x = 4a$$



We plot the curves on same system of axes to get the required region.

The enclosed area $= \int_0^{4a}\left(\sqrt{4ax} - \frac{x^2}{4a}\right)dx$

$$= 2\sqrt{a}\left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{12a}\right]_0^{4a}$$

$$= \frac{4}{3}\sqrt{a}(4a)^{\frac{3}{2}} - \frac{(4a)^3}{12a} - 0 = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3} \text{ square units.}$$

26. Let E_1 be event getting number > 4

E_2 be event getting number ≤ 4

$$P(E_1) = \frac{2}{6} = \frac{1}{3} \qquad P(E_2) = \frac{4}{6} = \frac{2}{3}$$

Let E be the event that man reports getting number > 4.

$$P(E/E_1) = \frac{3}{5} \qquad P(E/E_2) = \frac{2}{5}$$

By Baye's theorem

$$P(E_1/E) = \frac{P(E_1)\cdot P(E/E_1)}{P(E_1)\cdot P(E/E_1) + P(E_2).P(E/E_2)} = \frac{\frac{1}{3}\times\frac{3}{5}}{\frac{1}{3}\times\frac{3}{5} + \frac{2}{3}\times\frac{2}{5}} = \frac{3}{3+4} = \frac{3}{7}$$

EXAMINATION PAPERS – 2009

MATHEMATICS CBSE (Foreign)

CLASS – XII

Time allowed: 3 hours **Maximum marks: 100**

General Instructions: *As given in CBSE Examination paper (Delhi)* **– 2009.**

Set–I

SECTION–A

1. Evaluate: $\int \frac{1}{x + x\log x}\,dx$.

2. Evaluate: $\int_0^1 \frac{1}{\sqrt{2x+3}}\,dx$.

3. If the binary operation *, defined on Q, is defined as $a * b = 2a + b - ab$, for all $a,\ b \in Q$, find the value of $3 * 4$.

4. If $\begin{pmatrix} y + 2x & 5 \\ -x & 3 \end{pmatrix} = \begin{pmatrix} 7 & 5 \\ -2 & 3 \end{pmatrix}$, find the value of y.

5. Find a unit vector in the direction of $\vec{a} = 2\hat{i} - 3\hat{j} + 6\hat{k}$.

6. Find the direction cosines of the line passing through the following points:

$(-2, 4, -5),\ (1, 2, 3)$.

7. If $A = (a_{ij}) = \begin{pmatrix} 2 & 3 & -5 \\ 1 & 4 & 9 \\ 0 & 7 & -2 \end{pmatrix}$ and $B = (b_{ij}) = \begin{pmatrix} 2 & 1 & -1 \\ -3 & 4 & 4 \\ 1 & 5 & 2 \end{pmatrix}$, then find $a_{22} + b_{21}$.

8. If $|\vec{a}| = \sqrt{3},\ |\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = \sqrt{3}$, find the angle between $\vec{a}$ and $\vec{b}$.

9. If $A = \begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix}$, then find the value of k if $|2A| = k|A|$.

10. Write the principal value of $\tan^{-1}\left[\tan \frac{3\pi}{4}\right]$.

SECTION–B

11. Evaluate: $\int \frac{\cos x}{(2+\sin x)(3+4\sin x)}\,dx$

OR

Evaluate: $\int x^2 . \cos^{-1} x\, dx$

12. Show that the relation R in the set of real numbers, defined as $R = \{(a,\ b):\ a \le b^2\}$ is neither reflexive, nor symmetric, nor transitive.

13. If $\log(x^2 + y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$, then show that $\frac{dy}{dx} = \frac{x+y}{x-y}$.

OR

If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, then find $\frac{d^2y}{dx^2}$.

14. Find the equation of the tangent to the curve $y = \sqrt{4x-2}$ which is parallel to the line $4x - 2y + 5 = 0$.

OR

Using differentials, find the approximate value of $f(2 \cdot 01)$, where $f(x) = 4x^3 + 5x^2 + 2$.

15. Prove the following:

$$\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right) = \frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right).$$

OR

Solve the following for x:

$$\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right) + \tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{2\pi}{3}.$$

16. Find the angle between the line $\frac{x+1}{2} = \frac{3y+5}{9} = \frac{3-z}{-6}$ and the plane $10x + 2y - 11z = 3$.

17. Solve the following differential equation:

$$(x^3 + y^3)dy - x^2 y dx = 0$$

18. Find the particular solution of the differential equation.

$$\frac{dy}{dx} + y\cot x = 4x \operatorname{cosec} x,\ (x \ne 0),\ \text{given that } y = 0 \text{ when } x = \frac{\pi}{2}.$$

19. Using properties of determinants, prove the following:

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2.$$

20. The probability that A hits a target is $\frac{1}{3}$ and the probability that B hits it is $\frac{2}{5}$. If each one of A and B shoots at the target, what is the probability that

(i) the target is hit?

(ii) exactly one-of-them-hits the target?

21. Find $\frac{dy}{dx}$, if $y^x + x^y = a^b$, where a, b are constants.

22. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a} . \vec{b} = \vec{a} . \vec{c}$ and $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$, $\vec{a} \neq 0$, then show that $\vec{b} = \vec{c}$.

SECTION–C

23. One kind of cake requires 200 g of flour and 25 g of fat, and another kind of cake requires 100 g of flour and 50 g of fat. Find the maximum number of cakes which can be made from 5 kg of flour and 1 kg of fat assuming that there is no shortage of the other integredients used in making the cakes. Formulate the above as a linear programming problem and solve graphically.

24. Using integration, find the area of the region:

$$\{(x, y): 9x^2 + y^2 \leq 36 \text{ and } 3x + y \geq 6\}$$

25. Show that the lines $\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5}$; $\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5}$ are coplanar. Also find the equation of the plane containing the lines.

26. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

OR

Show that the total surface area of a closed cuboid with square base and given volume is minimum, when it is a cube.

27. Using matrices, solve the following system of linear equations:

$$3x - 2y + 3z = 8$$
$$2x + y - z = 1$$
$$4x - 3y + 2z = 4$$

28. Evaluate: $\int \frac{x^4 dx}{(x-1)(x^2+1)}$

OR

Evaluate: $\int_1^4 [|x-1| + |x-2| + |x-4|] dx$

29. Two cards are drawn simultaneously (or successively without replacement) from a well suffled pack of 52 cards. Find the mean and variance of the number of red cards.

Set–II

Only those questions, not included in Set I, are given.

7. Evaluate:

$$\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\, dx$$

10. If $\begin{pmatrix} 3x - 2y & 5 \\ x & -2 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ -3 & -2 \end{pmatrix}$, find the value of y.

13. Find the angle between the line $\frac{x-2}{3} = \frac{2y-5}{4} = \frac{3-z}{-6}$ and the plane $x + 2y + 2z - 5 = 0$.

15. Solve the following differential equation:

$$(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{2}{x^2 - 1}$$

16. Using properties of determinants, prove the following:

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1 - x^3)^2.$$

18. If $y = a\cos(\log x) + b\sin(\log x)$, then show that

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

26. A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the mean and variance of the number of successes.

28. Using integration, find the area of the region:

$$\{(x, y) : 25x^2 + 9y^2 \le 225 \text{ and } 5x + 3y \ge 15\}$$

Set–III

Only those questions, not included in Set I and Set II are given.

1. If $\begin{pmatrix} 7y & 5 \\ 2x - 3y & -3 \end{pmatrix} = \begin{pmatrix} -21 & 5 \\ 11 & -3 \end{pmatrix}$, find the value of x.

4. Evaluate:

$$\int \frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}\, dx$$

15. If $\sqrt{1 - x^2} + \sqrt{1 - y^2} = a(x - y)$, show that $\frac{dy}{dx} = \sqrt{\frac{1 - y^2}{1 - x^2}}$.

17. Using properties of determinants, prove the following:

$$\begin{vmatrix} a+bx & c+dx & p+qx \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} = (1-x^2)\begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix}$$

18. For the differential equation $xy\dfrac{dy}{dx} = (x+2)(y+2)$, find the solution curve passing through the point (1, –1).

20. Find the equation of the perpendicular drawn from the point (1, –2, 3) to the plane $2x - 3y + 4z + 9 = 0$. Also find the co-ordinates of the foot of the perpendicular.

24. Using integration, find the area of the triangle ABC with vertices as $A(-1, 0)$, $B(1, 3)$ and $C(3, 2)$.

27. From a lot of 30 bulbs which includes 6 defectives, a sample of 4 bulbs is drawn at random with replacement. Find the mean and variance of the number of defective bulbs.

SOLUTIONS

Set – I

SECTION–A

1. Let $I = \int \frac{1}{x + x\log x}\,dx = \int \frac{dx}{x(1+\log x)}$

Let $1 + \log x = t$

$\frac{1}{x}\,dx = dt$

$\therefore \quad I = \int \frac{dt}{t} = \log|t| + C$

$= \log\left|1 + \log x\right| + C$

2. $\int_0^1 \frac{1}{\sqrt{2x+3}}\,dx = \int_0^1 (2x+3)^{-\frac{1}{2}}\,dx$

$= \left[\frac{(2x+3)^{\frac{1}{2}}}{\frac{1}{2}\times 2}\right]_0^1$

$= 5^{\frac{1}{2}} - 3^{\frac{1}{2}} = \sqrt{5} - \sqrt{3}$

3. Given binary operation is

$a * b = 2a + b - ab$

$\therefore \quad 3 * 4 = 2 \times 3 + 4 - 3 \times 4$

$\Rightarrow \quad 3 * 4 = -2$

4. Using equality of two matrices in

$\begin{pmatrix} y+2x & 5 \\ -x & 3 \end{pmatrix} = \begin{pmatrix} 7 & 5 \\ -2 & 3 \end{pmatrix}$

We get

$y + 2x = 7$

$-x = -2 \quad \Rightarrow \quad x = 2$

$\therefore \quad y + 2(2) = 7 \quad \Rightarrow \quad y = 3$

5. Given $\vec{a} = 2\hat{i} - 3\hat{j} + 6\hat{k}$

$\Rightarrow \quad |\vec{a}| = \sqrt{4 + 9 + 36} = 7$

$\therefore \quad \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7}$

$\Rightarrow \quad \hat{a} =$ Unit vector in direction of $\vec{a}$

$$= \frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$$

6. Direction ratios of the line passing through (–2, 4, –5) and (1, 2, 3) are 1 – (–2), 2 – 4, 3 – (–5) = 3, – 2, 8

$\therefore$ Direction cosines are $= \frac{3}{\sqrt{9+4+64}}, \frac{-2}{\sqrt{9+4+64}}, \frac{8}{\sqrt{9+4+64}}$

$$= \frac{3}{\sqrt{77}}, \frac{-2}{\sqrt{77}}, \frac{8}{\sqrt{77}}$$

7. $a_{22} = 4, \quad b_{21} = -3$

$a_{22} + b_{21} = 4 - 3 = 1$

8. Given $|\vec{a}| = \sqrt{3}, \quad |\vec{b}| = 2, \quad \vec{a}\cdot\vec{b} = \sqrt{3}$

We know

$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta \Rightarrow \sqrt{3} = \sqrt{3}(2)\cos\theta$

$\Rightarrow \frac{1}{2} = \cos\theta \Rightarrow \theta = \frac{\pi}{3}$

9. Given $A = \begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix}$

$\Rightarrow \quad 2A = \begin{pmatrix} 2 & 4 \\ 8 & 4 \end{pmatrix}$

$\therefore \quad |2A| = 8 - 32 = -24$

$|A| = 2 - 8 = -6$

$\Rightarrow \quad -24 = k(-6)$

$4 = k$

10. $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}\left(\tan\left(\pi - \frac{\pi}{4}\right)\right)$

$= \tan^{-1}(-1) = -\frac{\pi}{4}$

$\therefore$ Principal value of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \frac{-\pi}{4}$.

SECTION–B

11. Let $I = \int \cos x\,dx$

$$(2+\sin x)(3+4\sin x)$$

Let $\sin x = t$

$\cos x\, dx = dt$

$\therefore \quad I = \int \frac{dt}{(2+t)(3+4t)}$

Let $\frac{1}{(2+t)(3+4t)} = \frac{A}{2+t} + \frac{B}{3+4t}$

$\Rightarrow \quad 1 = A(3+4t) + B(2+t)$

$\Rightarrow \quad 3A + 2B = 1$

$4A + B = 0 \quad \Rightarrow \quad B = -4A$

$\therefore \quad 3A - 8A = 1$

$A = -\frac{1}{5} \quad \Rightarrow \quad B = \frac{4}{5}$

$\Rightarrow \quad I = \int \frac{dt}{(2+t)(3+4t)} = \frac{-1}{5}\int \frac{dt}{2+t} + \frac{4}{5}\int \frac{dt}{3+4t}$

$= \frac{-1}{5}\log|2+t| + \frac{4}{5}\frac{\log|3+4t|}{4} + C$

$= \frac{-1}{5}\log|2+\sin x| + \frac{1}{5}\log|3+4\sin x| + C$

$= \frac{1}{5}\log\left|\frac{3+4\sin x}{2+\sin x}\right| + C$

OR

Let $I = \int x^2 \cos^{-1} x\, dx$

$= \cos^{-1} x . \frac{x^3}{3} - \int \frac{-1}{\sqrt{1-x^2}} \times \frac{x^3}{3}\, dx$ [Integrating by parts]

$= \frac{x^3}{3}\cos^{-1} x + \frac{1}{3}\int \frac{x^3\, dx}{\sqrt{1-x^2}}$

$= \frac{x^3}{3}\cos^{-1} x + \frac{1}{3} I_1$

In I_1, let $1 - x^2 = t$ so that $-2x dx = dt$

$\therefore \quad I_1 = -\frac{1}{2}\int \frac{1-t}{\sqrt{t}}\, dt = -\frac{1}{2}\int\left(\frac{1}{\sqrt{t}} - \sqrt{t}\right)dt$

$= -\frac{1}{2}\left(2\sqrt{t} - \frac{2}{3}t^{3/2}\right) + C'$

$= -\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{3/2} + C'$

$\therefore \quad I = \frac{x^3}{3}\cos^{-1} x - \frac{1}{3}\sqrt{1-x^2} + \frac{1}{9}(1-x^2)^{3/2} + C$

12. Given relation is $R = \{(a, b) : a \le b^2\}$

Reflexivity:

Let $a \in$ real numbers.

$aRa \Rightarrow a \le a^2$

but if $a < 1$ Let $a = \frac{1}{2}$ $\Rightarrow$ $a^2 = \frac{1}{4}$

$a \not\le a^2$

Hence R is not reflexive.

Symmetry

Let $a, b \in$ real numbers.

$aRb \Rightarrow a \le b^2$

But then $b \le a^2$ is not true.

$\therefore$ $aRb \not\Rightarrow bRa$

For example, let $a = 2, b = 5$

then $2 \le 5^2$ but $5 \le 2^2$ is not true.

Hence R is not symmetric.

Transitivity

Let $a, b, c \in$ real numbers

$aRb \Rightarrow a \le b^2$ and

$bRc \Rightarrow b \le c^2$

Considering aRb and bRc

$\Rightarrow$ $a \le c^4 \not\Rightarrow aRc$

Hence R is not transitive

e.g., if $a = 2$, $b = -3$, $c = 1$

$aRb \Rightarrow 2 \le 9$

$bRc \Rightarrow -3 \le 1$

$aRc \Rightarrow 2 \le 1$ is not true.

13. Given $\log(x^2 + y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$ Differentiating w.r.t. x

$$\frac{2x + 2y\frac{dy}{dx}}{x^2 + y^2} = \frac{2}{1 + \frac{y^2}{x^2}} \cdot \frac{x\frac{dy}{dx} - y}{x^2}$$

$\Rightarrow \quad 2x + 2y\frac{dy}{dx} = 2x^2\left(\frac{x\frac{dy}{dx} - y}{x^2}\right)$

$\Rightarrow \quad x + y\frac{dy}{dx} = x\frac{dy}{dx} - y$

$\Rightarrow \quad x + y = x\frac{dy}{dx} - y\frac{dy}{dx}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{x+y}{x-y}$

OR

Given $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$

$\Rightarrow \quad \frac{dx}{dt} = a(-\sin t + t\cos t + \sin t) = at\cos t, \frac{dy}{dt} = a(\cos t + t\sin t - \cos t) = at\sin t$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{at\sin t}{at\cos t} = \tan t$

Differentiating w.r.t.x again

$\Rightarrow \quad \frac{d^2y}{dx^2} = \sec^2 t.\frac{dt}{dx}$

$= \sec^2 t.\frac{1}{at\cos t}$

$= \frac{\sec^3 t}{at}$

14. Given curve is $y = \sqrt{4x-2}$ … (i)

Differentiating w.r.t.x

$\frac{dy}{dx} = \frac{4}{2\sqrt{4x-2}} = \frac{2}{\sqrt{4x-2}}$

The tangent is parallel to the line $4x - 2y + 5 = 0$.

The slope of this line is $= \frac{-4}{-2} = 2$

$\therefore$ Slope of tangent $= \frac{2}{\sqrt{4x-2}} = 2$

$\Rightarrow \quad 1 = \sqrt{4x-2}$

$\Rightarrow \quad 1 = 4x - 2 \quad \Rightarrow \quad x = \frac{3}{4}$

Put value of x in (i)

$y = \sqrt{4 \times \frac{3}{4} - 2} = 1$

$\therefore$ Equation of tangent will be

$$y - 1 = 2\left(x - \frac{3}{4}\right)$$

$$\Rightarrow \quad y - 1 = 2x - \frac{3}{2}$$

or $\quad 2y - 2 = 4x - 3$

Hence equation of tangent is

$$4x - 2y - 1 = 0$$

OR

Given $f(x) = 4x^3 + 5x^2 + 2$

$$\Rightarrow \quad f'(x) = 12x^2 + 10x$$

We know for finding approximate values

$$f(x + \Delta x) = f(x) + f'(x).\Delta x$$

$$\therefore \quad f(2.01) = f(2) + f'(2)(0.01)$$

$$= [4(2)^3 + 5(2)^2 + 2] + [12(2)^2 + 10(2)](0.01)$$

$$= [4 \times 8 + 5 \times 4 + 2] + [12 \times 4 + 20](0.01)$$

$$= 54 + (68)(0.01)$$

$$= 54.68$$

15. LHS of given equation $= \tan^{-1}\frac{1}{4} + \tan^{-1}\left(\frac{2}{9}\right)$

$$= \tan^{-1}\left(\frac{\frac{1}{4} + \frac{2}{9}}{1 - \frac{1}{4}\cdot\frac{2}{9}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{17}{36}}{\frac{34}{36}}\right)$$

$$= \tan^{-1}\frac{1}{2} = \frac{1}{2}\left(2\tan^{-1}\frac{1}{2}\right)$$

Using $2\tan^{-1} A = \cos^{-1}\frac{1 - A^2}{1 + A^2}$

$$= \frac{1}{2}\cos^{-1}\left(\frac{1 - \frac{1}{4}}{1 + \frac{1}{4}}\right)$$

$$= \frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right) = \text{R.H.S.}$$

OR

Given $\cos^{-1}\left(\dfrac{x^2-1}{x^2+1}\right)+\tan^{-1}\left(\dfrac{2x}{x^2-1}\right)=\dfrac{2\pi}{3}$

$$\Rightarrow \quad \cos^{-1}\left(\frac{-(1-x^2)}{1+x^2}\right)+\tan^{-1}\left(-\frac{2x}{1-x^2}\right)=\frac{2\pi}{3}$$

$$\Rightarrow \quad \pi-\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)-\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{2\pi}{3}$$

[Using $\cos^{-1}(-A)=\pi-\cos^{-1}A$ and $\tan^{-1}(-A)=-\tan^{-1}A$]

$$\Rightarrow \quad \pi-2\tan^{-1}x-2\tan^{-1}x=\frac{2\pi}{3}$$

$$\Rightarrow \quad \pi-\frac{2\pi}{3}=4\tan^{-1}x$$

$$\Rightarrow \quad \frac{\pi}{12}=\tan^{-1}x \quad \Rightarrow \qquad x=\tan\frac{\pi}{12}=\tan\left(\frac{\pi}{4}-\frac{\pi}{6}\right)$$

$$\therefore \quad x=\frac{\tan\frac{\pi}{4}-\tan\frac{\pi}{6}}{1+\tan\frac{\pi}{4}\times\tan\frac{\pi}{6}}=\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}$$

$$\Rightarrow \quad x=\frac{\sqrt{3}-1}{\sqrt{3}+1} \quad \Rightarrow \qquad x=\frac{(\sqrt{3}-1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)}$$

$$x=\frac{3+1-2\sqrt{3}}{2}=2-\sqrt{3}$$

16. Given line can be rearranged to get

$$\frac{x-(-1)}{2}=\frac{y-(-5/3)}{3}=\frac{z-3}{6}$$

Its direction ratios are 2, 3, 6.

Direction ratios of normal to the plane $10x+2y-11z=3$ are 10, 2, – 11

Angle between the line and plane

$$\sin\theta=\frac{2\times 10+3\times 2+6(-11)}{\sqrt{4+9+36}\sqrt{100+4+121}}$$

$$=\frac{20+6-66}{7\times 15}=\frac{-40}{105}$$

$$\sin\theta=\frac{-8}{21} \text{ or } \theta=\sin^{-1}\left(\frac{-8}{21}\right)$$

17. $(x^3+y^3)dy-x^2ydx=0$ is rearranged as

$$\frac{dy}{dx}=\frac{x^2y}{x^3+y^3}$$

It is a homogeneous differential equation.

Let $\frac{y}{x} = v \Rightarrow y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore \quad v + x\frac{dv}{dx} = \frac{v}{1+v^3}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v}{1+v^3} - v = \frac{-v^4}{1+v^3}$$

$$\Rightarrow \quad \frac{1+v^3}{v^4}\,dv = -\frac{dx}{x}$$

Integrating both sides, we get

$$\int\left(\frac{1}{v^4} + \frac{1}{v}\right)dv = -\int\frac{dx}{x}$$

$$\Rightarrow \quad -\frac{1}{3v^3} + \log|v| = -\log|x| + C$$

$$\Rightarrow \quad -\frac{x^3}{3y^3} + \log\left|\left(\frac{y}{x}\right)\right| = -\log|x| + C$$

$\Rightarrow \quad \frac{x^3}{3y^3} + \log|y| = C$ is the solution of the given differential equation.

18. Given differential equation is $\frac{dy}{dx} + y\cot x = 4x\,\mathrm{cosec}\,x$ and is of the type $\frac{dy}{dx} + Py = Q$

$$\therefore \quad I.F. = e^{\int \cot x\,dx} = e^{\log|\sin x|} = \sin x$$

Its solution is given by

$$\Rightarrow \quad \sin x \,.\, y = \int 4x\,\mathrm{cosec}\,x.\sin x\,dx$$

$$\Rightarrow \quad y\sin x = \int 4x\,dx = \frac{4x^2}{2} + C$$

$$\Rightarrow \quad y\sin x = 2x^2 + C$$

Now $y = 0$ when $x = \frac{\pi}{2}$

$$\therefore \quad 0 = 2\times\frac{\pi^2}{4} + C \Rightarrow C = -\frac{\pi^2}{2}$$

Hence the particular solution of given differential equation is

$$y\sin x = 2x^2 - \frac{\pi^2}{2}$$

19. Let $|A| = \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$

Apply $C_1 \to aC_1, C_2 \to bC_2, C_3 \to cC_3$

$$\Rightarrow \quad |A| = \frac{1}{abc}\begin{vmatrix} a(a^2+1) & ab^2 & c^2a \\ a^2b & b(b^2+1) & c^2b \\ a^2c & b^2c & c(c^2+1) \end{vmatrix}$$

Take a, b, c common respectively from R_1, R_2 and R_3

$$|A| = \frac{abc}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

Apply $C_1 \to C_1 + C_2 + C_3$

$$|A| = \begin{vmatrix} a^2+b^2+c^2+1 & b^2 & c^2 \\ a^2+b^2+c^2+1 & b^2+1 & c^2 \\ a^2+b^2+c^2+1 & b^2 & c^2+1 \end{vmatrix}$$

$$= (a^2+b^2+c^2+1)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

Apply $R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\therefore \quad |A| = (a^2+b^2+c^2+1)\begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along C_1

$$|A| = a^2+b^2+c^2+1$$

20. Let $P(A)$ = Probability that A hits the target $= \frac{1}{3}$

$P(B)$ = Probability that B hits the target $= 2/5$

(i) P(target is hit) $= P$(at least one of A, B hits)

$= 1 - P$ (none hits)

$$= 1 - \frac{2}{3} \times \frac{3}{5} = \frac{9}{15} = \frac{3}{5}$$

(*ii*) P(exactly one of them hits) $= P(A \& \overline{B} \text{ or } \overline{A} \& B)$

$$= P(A) \times P(\overline{B}) + P(\overline{A}).P(B)$$

$$= \frac{1}{3} \times \frac{3}{5} + \frac{2}{3} \times \frac{2}{5} = \frac{7}{15}$$

21. $y^x + x^y = a^b$ ...(i)

Let $v = y^x$

$u = x^y$

Taking log on either side of the two equation, we get

$\log v = x \log y, \qquad \log u = y \log x$

Differentiating w.r.t.x, we get

$$\frac{1}{v}\frac{dv}{dx} = x.\frac{1}{y}\frac{dy}{dx} + \log y, \quad \frac{1}{u}\frac{du}{dx} = \frac{y}{x} + \log x.\frac{dy}{dx}$$

$$\Rightarrow \frac{dv}{dx} = y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right], \frac{du}{dx} = x^y\left[\frac{y}{x} + \log x.\frac{dy}{dx}\right]$$

From (*i*), we have

$$u + v = a^b$$

$$\Rightarrow \frac{du}{dx} + \frac{dv}{dx} = 0$$

$$\Rightarrow y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] + x^y\left[\frac{y}{x} + \log x.\frac{dy}{dx}\right] = 0$$

$$\Rightarrow y^x.\frac{x}{y}\frac{dy}{dx} + x^y.\log x\frac{dy}{dx} = -y^x \log y - x^y.\frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y^x \log y - x^{y-1}y}{y^{x-1}x + x^y.\log x}$$

22. Given $\vec{a}.\vec{b} = \vec{a}.\vec{c}$

$\Rightarrow \vec{a}.\vec{b} - \vec{a}.\vec{c} = 0$

$\Rightarrow \vec{a}.(\vec{b} - \vec{c}) = 0$

$\Rightarrow$ either $\vec{b} = \vec{c}$ or $\vec{a} \perp \vec{b} - \vec{c}$

Also given $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$

$\Rightarrow \vec{a} \times \vec{b} - \vec{a} \times \vec{c} = 0 \qquad \Rightarrow \qquad \vec{a} \times (\vec{b} - \vec{c}) = 0$

$\Rightarrow \vec{a} \parallel \vec{b} - \vec{c}$ or $\vec{b} = \vec{c}$

But $\vec{a}$ cannot be both parallel and perpendicular to $(\vec{b} - \vec{c})$.

Hence $\vec{b} = \vec{c}$.

SECTION–C

23. Let x = Number of cakes of Ist type while

y = Number of cakes of IInd type

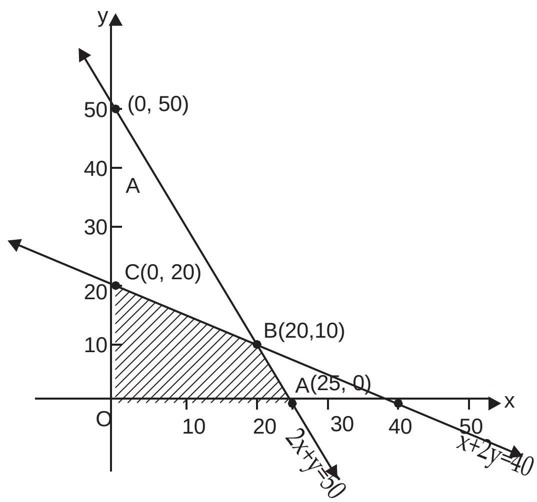
The linear programming problem is to maximise $Z = x + y$ subject to.

$200x + 100y \le 5000 \Rightarrow 2x + y \le 50$

$25x + 50y \le 1000 \Rightarrow x + 2y \le 40$

and $x \ge 0, y \ge 0$

To solve the LPP we draw the graph of the in equations and get the feasible solution shown (shaded) in the graph.



Corner points of the common shaded region are A (25, 0), B (20, 10) and C (0, 20).

Value of Z at each corner points:

$Z\big]_{(0,20)} = 0 + 20 = 20$

$Z\big]_{(20,10)} = 20 + 10 = 30$

$Z\big]_{(25,\,0)} = 25 + 0 = 25$

Hence 20 cakes of Ist kind and 10 cakes of IInd kind should be made to get maximum number of cakes.

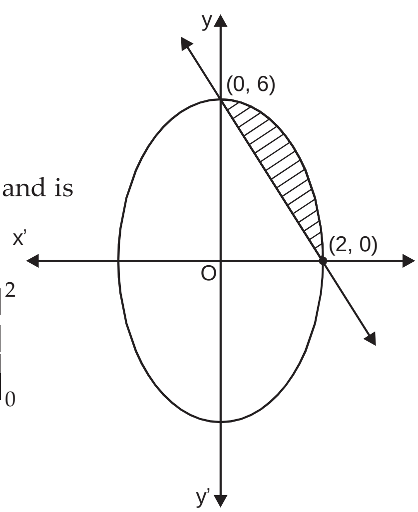
24. Given region is $\{(x, y): 9x^2 + y^2 \le 36 \text{ and } 3x + y \ge 6\}$

We draw the curves corresponding to equations

$$9x^2 + y^2 = 36 \quad \text{or} \quad \frac{x^2}{4} + \frac{y^2}{9} = 1 \text{ and } 3x + y = 6$$

The curves intersect at (2, 0) and (0, 6)

$\therefore$ Shaded area is the area enclosed by the two curves and is



$$= \int_0^2 \sqrt{9\left(1 - \frac{x^2}{4}\right)}dx - \int_0^2 (6 - 3x)dx$$

$$= 3\left[\frac{x}{4}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2} - 2x + \frac{x^2}{2}\right]_0^2$$

$$= 3\left[\frac{2}{4}\sqrt{4 - 4} + \frac{4}{2}\sin^{-1}\frac{2}{2} - 4 + \frac{4}{2}\right]$$

$$- 0 = 3\left[2\cdot\frac{\pi}{2} - 2\right] = 3(\pi - 2) \text{ square}$$

units

25. Given lines are

$$\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5} \qquad \ldots (i)$$

and $$\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5} \qquad \ldots (ii)$$

These lines will be coplanar if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

$\therefore$ $$\begin{vmatrix} -1+3 & 2-1 & 5-5 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix} = 2(5-10) - 1(-15+5) = 0$$

Hence lines are co-planar.

The equation of the plane containing two lines is

$$\begin{vmatrix} x+3 & y-1 & z-5 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix} = 0$$

$\Rightarrow \quad -5(x+3) + 10(y-1) - 5(z-5) = 0$

$\Rightarrow \quad -5x - 15 + 10y - 10 - 5z + 25 = 0$

$\Rightarrow \quad -5x + 10y - 5z + 0 = 0$

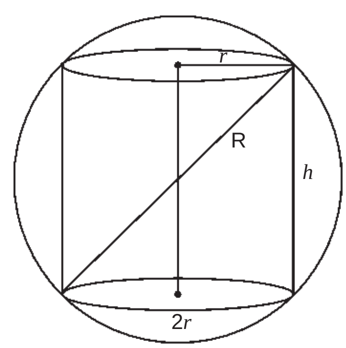
$\Rightarrow \quad -x + 2y - z = 0 \qquad$ or $\qquad x - 2y + z = 0$

26. Let r, h be the radius and height of the cylinder inscribed in the sphere of radius R.

$\therefore$ Using Pythagoras theorem

$$4r^2 + h^2 = 4R^2$$

$$\Rightarrow \quad r^2 = \frac{4R^2 - h^2}{4} \qquad \ldots (i)$$



Volume of cylinder $= V = \pi r^2 h$

$$\Rightarrow \quad V = \pi . h\left(\frac{4R^2 - h^2}{4}\right) = \pi R^2 h - \frac{\pi}{4}h^3$$

$$\Rightarrow \quad \frac{dV}{dh} = \pi R^2 - \frac{3\pi}{4}h^2 \qquad \ldots (ii)$$

For finding maximum volume

$$\frac{dV}{dh} = 0 \qquad \Rightarrow \qquad \pi R^2 = \frac{3\pi}{4}h^2$$

$$\Rightarrow \qquad h = \frac{2}{\sqrt{3}}R$$

Differentiating (*ii*) again

$$\frac{d^2V}{dh^2} = -\frac{6\pi}{4}h$$

$$\left.\frac{d^2V}{dh^2}\right|_{\left(h=\frac{2}{\sqrt{3}}R\right)} = -\frac{3\pi}{2}\left(\frac{2}{\sqrt{3}}R\right) = -\sqrt{3}R\pi < 0$$

Hence volume is maximum when $h = \frac{2}{\sqrt{3}}R$.

Maximum volume $= V\Big]_{h=\frac{2R}{\sqrt{3}}} = \pi h\left(\frac{4R^2 - h^2}{4}\right)$

$$V_{max} = \pi \times \frac{2R}{\sqrt{3}}\left(\frac{4R^2 - \frac{4R^2}{3}}{4}\right)$$

$$= \frac{2\pi R}{\sqrt{3}} \cdot \frac{2R^2}{3} = \frac{4\pi R^3}{3\sqrt{3}} \text{ cubic units.}$$

OR

The sides of the cuboid in the square base can be x, x and y

Let total surface area $= S = 2x^2 + 4xy$ …(*i*)

As volume of cuboid is $V = x^2y$ = constant

$\therefore \quad y = \frac{V}{x^2}$ … (*ii*)

$\therefore \quad S = 2x^2 + 4x.\frac{V}{x^2} = 2x^2 + \frac{4V}{x}$ [Substituting (*ii*) in (*i*)]

To find condition for minimum S we find $\frac{dS}{dx}$

$\Rightarrow \quad \frac{dS}{dx} = 4x - \frac{4V}{x^2}$ … (*iii*)

If $\frac{dS}{dx} = 0 \quad \Rightarrow \quad 4x^3 = 4V$

$\Rightarrow \quad x^3 = V \Rightarrow x = V^{\frac{1}{3}}$

Differentiating (*iii*) again w.r.t. x

$$\frac{d^2S}{dx^2} = 4 + \frac{8V}{x^3}$$

$$\left.\frac{d^2S}{dx^2}\right|_{\left(x=V^{1/3}\right)} = 4 + \frac{8V}{V} = 12 > 0$$

$\therefore$ Surface area is minimum when $x = V^{\frac{1}{3}}$

Put value of x in (ii) $\qquad y = \dfrac{V}{V^{\frac{2}{3}}} = V^{\frac{1}{3}}$

$\therefore \qquad x = y = V^{\frac{1}{3}}$

Hence cuboid of minimum surface area is a cube.

27. Given linear in equations are

$$3x - 2y + 3z = 8$$
$$2x + y - z = 1$$
$$4x - 3y + 2z = 4$$

The given equations can be expressed as $AX = B$

$$\begin{pmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ 4 \end{pmatrix} \Rightarrow$$

$\therefore \quad X = A^{-1}B$

To find A^{-1} we first find Adj. A

Co-factors of elements of A are

$c_{11} = -1, \quad c_{12} = -8, \quad c_{13} = -10$
$c_{21} = -5, \quad c_{22} = -6, \quad c_{23} = 1$
$c_{31} = -1, \quad c_{32} = 9, \quad c_{33} = 7$

$$\text{Matrix of co-factors} = \begin{pmatrix} -1 & -8 & -10 \\ -5 & -6 & 1 \\ -1 & 9 & 7 \end{pmatrix}$$

$$\text{Adj } A = \begin{pmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{pmatrix}$$

$$|A| = 3(2-3) + 2(4+4) + 3(-6-4)$$
$$= -3 + 16 - 30 = -17 \neq 0$$

$$\therefore \quad A^{-1} = -\frac{1}{17}\begin{pmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{17}\begin{pmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{pmatrix}\begin{pmatrix} 8 \\ 1 \\ 4 \end{pmatrix}$$

$$= -\frac{1}{17}\begin{pmatrix} -8-5-4 \\ -64-6+36 \\ -80+1+28 \end{pmatrix}$$

$$= -\frac{1}{17}\begin{pmatrix} -17 \\ -34 \\ -51 \end{pmatrix}$$

$\Rightarrow$ $x = 1, y = 2, z = 3$ is the required solution of the equations.

28. Let $I = \int \frac{x^4}{(x-1)(x^2+1)}\,dx$

Suppose $\frac{x^4}{(x-1)(x^2+1)} = \frac{x^4 - 1 + 1}{(x-1)(x^2+1)}$

$$= x + 1 + \frac{1}{(x-1)(x^2+1)}$$

Also let $\frac{1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx + c}{x^2+1}$

$\Rightarrow$ $1 = A(x^2+1) + (Bx + C)(x-1)$

Equating coefficients of similar terms

$$A + B = 0$$

$$-B + C = 0 \quad \Rightarrow \quad B = C$$

$$A - C = 1$$

$\therefore$ $A - B = 1$

$$\underline{A + B = 0}$$

$\Rightarrow$ $2A = 1 \quad \Rightarrow \quad A = \frac{1}{2} \quad \Rightarrow \quad B = -\frac{1}{2} = C$

$$\therefore I = \int \left(x + 1 + \frac{1}{2}\cdot\frac{1}{x-1} - \frac{1}{2}\cdot\frac{x+1}{x^2+1} \right) dx$$

$$= \frac{x^2}{2} + x + \frac{1}{2}\log|x-1| - \frac{1}{4}\int \frac{2x}{x^2+1}\,dx - \frac{1}{2}\int \frac{dx}{x^2+1}\,dx$$

$$= \frac{x^2}{2} + x + \frac{1}{2}\log|x-1| - \frac{1}{4}\log|x^2+1| - \frac{1}{2}\tan^{-1} x + C$$

OR

Given $I = \int_1^4 [\,|x-1| + |x-2| + |x-4|\,]\,dx$

$$= \int_1^4 (x-1)dx + \int_1^2 -(x-2)dx + \int_2^4 (x-2)dx + \int_1^4 -(x-4)dx$$

$$= \left.\frac{x^2}{2} - x\right]_1^4 + \left[-\frac{x^2}{2} + 2x\right]_1^2 + \left.\frac{x^2}{2} - 2x\right]_2^4 + \left.\left(-\frac{x^2}{2} + 4x\right)\right]_1^4$$

$$= \left(\frac{16}{2} - 4 - \frac{1}{2} + 1\right) + \left(-2 + 4 + \frac{1}{2} - 2\right) + \left(\frac{16}{2} - 8 - 2 + 4\right) + \left(-\frac{16}{2} + 16 + \frac{1}{2} - 4\right)$$

$$= \left(5 - \frac{1}{2}\right) + \frac{1}{2} + 2 + 4 + \frac{1}{2}$$

$$= 11 + \frac{1}{2} = \frac{23}{2}$$

29. Total no. of cards in the deck = 52

Number of red cards = 26

No. of cards drawn = 2 simultaneously

$\therefore \quad x$ = value of random variable = 0, 1, 2

x_i	$P(x)$	$x_i P(x)$	$x_1^2 P(x)$
0	$\frac{{}^{26}C_0 \times {}^{26}C_2}{{}^{52}C_2} = \frac{25}{102}$	0	0
1	$\frac{{}^{26}C_1 \times {}^{26}C_1}{{}^{52}C_2} = \frac{52}{102}$	$\frac{52}{102}$	$\frac{52}{102}$
2	$\frac{{}^{26}C_0 \times {}^{26}C_2}{{}^{52}C_2} = \frac{25}{102}$	$\frac{50}{102}$	$\frac{100}{102}$
		$\Sigma x_i P(x) = 1$	$\Sigma x_i^2 P(x) = \frac{152}{102}$

Mean $= \mu = \Sigma x_i P(x) = 1$

Variance $= \sigma^2 = \Sigma x_i^2 P(x) - \mu^2$

$$= \frac{152}{102} - 1 = \frac{50}{102} = \frac{25}{51}$$

$$= 0.49$$

Set–II

7. Let $I = \int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx$

Let $e^{2x} + e^{-2x} = t$

$\Rightarrow \quad 2(e^{2x} - e^{-2x})dx = dt$

$\therefore \quad I = \frac{1}{2}\int \frac{dt}{t}$

$= \frac{1}{2} \log |t| + c$

$= \frac{1}{2} \log |e^{2x} + e^{-2x}| + c$

10. Using equality of two matrices, we have

$3x - 2y = 3$

$x = -3$

$\therefore \quad 3(-3) - 2y = 3$

$\Rightarrow \quad -2y = 12$

$\Rightarrow \quad y = -6$

$\therefore \quad x = -3, y = -6$

13. The given line is

$$\frac{x-2}{3} = \frac{2y-5}{4} = \frac{3-z}{-6}$$

It is rearranged as

$$\frac{x-2}{3} = \frac{y-5/2}{2} = \frac{z-3}{6}$$

DR's of the line are = 3, 2, 6

The given equation of plane is $x + 2y + 2z - 5 = 0$

The DR's of its normal are = 1, 2, 2

To find angle between line and plane

$$\sin\theta = \frac{3(1) + 2(2) + 6(2)}{\sqrt{9+4+36}\sqrt{1+4+4}} = \frac{19}{21}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{19}{21}\right)$$

15. The differential equation given is

$$(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{2}{x^2 - 1}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2x}{x^2 - 1} y = \frac{2}{(x^2 - 1)^2}$$

It is an equation of the type $\frac{dy}{dx} + Py = Q$

$$\therefore \quad I.F. = e^{\int \frac{2x}{x^2-1}dx} = e^{\log(x^2-1)} = x^2 - 1$$

Its solution is given by

$$(x^2 - 1)y = \int (x^2 - 1)\frac{2}{(x^2-1)^2}\,dx$$

$$\Rightarrow \quad (x^2 - 1)y = 2.\frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C$$

$$\Rightarrow \quad y = \frac{1}{x^2-1}\log\left|\frac{x-1}{x+1}\right| + \frac{C}{x^2-1} \text{ is required solution.}$$

16. Let $|A| = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$

Apply $C_1 \to C_1 + C_2 + C_3$

$$|A| = \begin{vmatrix} 1+x+x^2 & x & x^2 \\ 1+x+x^2 & 1 & x \\ 1+x+x^2 & x^2 & 1 \end{vmatrix}$$

$$\Rightarrow \quad |A| = (1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & x \\ 1 & x^2 & 1 \end{vmatrix}$$

Apply $R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$

$$\Rightarrow \quad |A| = (1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1-x & x-x^2 \\ 0 & x^2-x & 1-x^2 \end{vmatrix}$$

Take $(1 - x)$ common from R_2 and R_3

$$|A| = (1+x+x^2)(1-x)^2\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & x \\ 0 & -x & 1+x \end{vmatrix}$$

Expanding along C_1

$$|A| = (1+x+x^2)(1-x)^2(1+x+x^2)$$

$$= (1-x^3)^2 \qquad [\because 1-x^3 = (1-x)(1+x+x^2)]$$

18. Given $y = a\cos(\log x) + b\sin(\log x)$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-a\sin(\log x)}{x} + \frac{b\cos(\log x)}{x}$$

$$\Rightarrow \quad x\frac{dy}{dx} = -a\sin(\log x) + b\cos(\log x)$$

Differentiating again w.r.t. x

$$\Rightarrow \quad x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{-a\cos(\log x)}{x} - \frac{b\sin(\log x)}{x}$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + \frac{xdy}{dx} = -y$$

$$\therefore \quad x^2\frac{d^2y}{dx^2} + \frac{xdy}{dx} + y = 0$$

26. Here number of throws = 4

$P(\text{doublet}) = p = \frac{6}{36} = \frac{1}{6}$

$P(\text{not doublet}) = q = \frac{30}{36} = \frac{5}{6}$

Let X denotes number of successes, then

$$P(X=0) = {}^4C_0 p^0 q^4 = 1 \times 1 \times \left(\frac{5}{6}\right)^4 = \frac{625}{1296}$$

$$P(X=1) = {}^4C_1 \frac{1}{6} \times \left(\frac{5}{6}\right)^3 = 4 \times \frac{125}{1296} = \frac{500}{1296}$$

$$P(X=2) = {}^4C_2 \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^2 = 6 \times \frac{25}{1296} = \frac{150}{1296}$$

$$P(X=3) + {}^4C_3 \left(\frac{1}{6}\right)^3 \times \frac{5}{6} = \frac{20}{1296}$$

$$P(X=4) = {}^4C_4 \left(\frac{1}{6}\right)^4 = \frac{1}{1296}$$

Being a binomial distribution with

$n = 4,\ p = \frac{1}{6}$ and $q = \frac{5}{6}$

$\mu = \text{mean} = np = 4 \times \frac{1}{6} = \frac{2}{3}$

$\mu^2 = \text{variance} = npq = 4 \times \frac{1}{6} \times \frac{5}{6} = \frac{5}{9}.$

28. The region given is

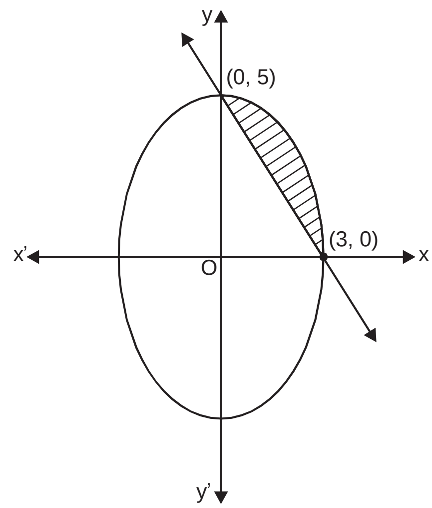
$\{(x, y): 25x^2 + 9y^2 \le 225 \text{ and } 5x + 3y \ge 15\}$

Consider the equations

$25x^2 + 9y^2 = 225$ and $5x + 3y = 15$

$\Rightarrow \quad \frac{x^2}{9} + \frac{y^2}{25} = 1$ which is an ellipse.

The two curve intersect at points (0, 5) and (3, 0) obtained by equating values of y from two equations. Hence the sketch of the curves is as shown in the figure and required area is the shaded region.



The required included area is

$$= \int_0^3 5\sqrt{1-\frac{x^2}{9}}\,dx - \int_0^3 \frac{15-5x}{3}\,dx$$

$$= \frac{5}{3}\left[\left(\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\frac{x}{3}-3x+\frac{x^2}{2}\right)\right]_0^3$$

$$= \frac{5}{3}\left(\frac{3}{2}\sqrt{9-9}+\frac{9}{2}\sin^{-1}\frac{3}{3}-9+\frac{9}{2}-0\right)$$

$$= \frac{5}{3}\left(\frac{9}{2}\times\frac{\pi}{2}-\frac{9}{2}\right) = \frac{15}{2}\left(\frac{\pi}{2}-1\right) \text{ square units.}$$

Set–III

1. Using equality of two matrices

$7y = -21 \quad \Rightarrow \quad y = -3$

$2x - 3y = 11$

$\Rightarrow \; 2x + 9 = 11$

$\Rightarrow \; \boldsymbol{x = 1}$

$\therefore \; x = 1, y = -3$

4. Let $I = \int \frac{e^{ax}-e^{-ax}}{e^{ax}+e^{-ax}}\,dx$

$$= \frac{1}{a}\int \frac{a(e^{ax}-e^{-ax})}{e^{ax}+e^{-ax}}\,dx$$

$$= \frac{1}{a}\log|e^{ax}+e^{-ax}|+C$$

$\left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|+C\right]$

15. Given

$$\sqrt{1-x^2}+\sqrt{1-y^2} = a(x-y)$$

Let $x = \sin A \quad \Rightarrow \quad A = \sin^{-1} x$

$y = \sin B \quad \Rightarrow \quad B = \sin^{-1} y$

$\therefore \; \sqrt{1-\sin^2 A}+\sqrt{1-\sin^2 B} = a(\sin A - \sin B)$

$\Rightarrow \; \cos A + \cos B = a.2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$\Rightarrow \; 2\cos\frac{A+B}{2}.\cos\frac{A-B}{2} = 2a\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$$\Rightarrow \frac{1}{a} = \tan\frac{A-B}{2}$$

$$\Rightarrow \tan^{-1}\frac{1}{a} = \frac{A-B}{2}$$

$$\Rightarrow 2\tan^{-1}\frac{1}{a} = \sin^{-1}x - \sin^{-1}y$$

Differentiating w.r.t. x, we get

$$0 = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} = \sqrt{\frac{1-y^2}{1-x^2}}$$

17. Let $|A| = \begin{vmatrix} a+bx & c+dx & p+qx \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix}$

Apply $R_2 \to R_2 - xR_1$

$$|A| = \begin{vmatrix} a+bx & c+dx & p+qx \\ b-bx^2 & d-dx^2 & q-qx^2 \\ u & v & w \end{vmatrix}$$

Taking $1-x^2$ common from R_2

$$|A| = (1-x^2)\begin{vmatrix} a+bx & c+dx & p+qx \\ b & d & q \\ u & v & w \end{vmatrix}$$

Apply $R_1 \to R_1 - xR_2$, we get

$$|A| = (1-x^2)\begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix} = \text{RHS}$$

18. Given differential equation is

$$xy\frac{dy}{dx} = (x+2)(y+2)$$

$$\Rightarrow \frac{y}{y+2}dy = \frac{x+2}{x}dx$$

Integrating both sides

$$\int\frac{y}{y+2}dy = \int\left(1+\frac{2}{x}\right)dx$$

$$\Rightarrow \int\left(1-\frac{2}{y+2}\right)dy = \int\left(1+\frac{2}{x}\right)dx$$

$\Rightarrow \quad y - 2\log|y+2| = x + 2\log|x| + c \qquad \ldots (i)$

The curve represented by (i) passes through $(1, -1)$. Hence

$$-1 - 2\log 1 = 1 + 2\log|1| + C$$

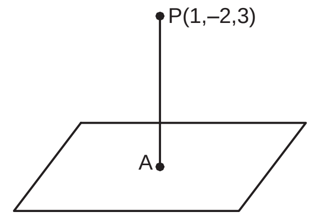
$\Rightarrow \quad C = -2$

$\therefore$ The required curve will be

$$y - 2\log|y+2| = x + 2\log|x| - 2$$

20. Let the foot of the perpendicular on the plane be A.

$PA \perp$ to the plane



$$2x - 3y + 4z + 9 = 0$$

$\therefore \quad DR$'s of $PA = 2, -3, 4$

Equation of PA can be written as

$$\frac{x-1}{2} = \frac{y+2}{-3} = \frac{z-3}{4} = \lambda$$

General points line $PA = (2\lambda + 1, -3\lambda - 2, 4\lambda + 3)$

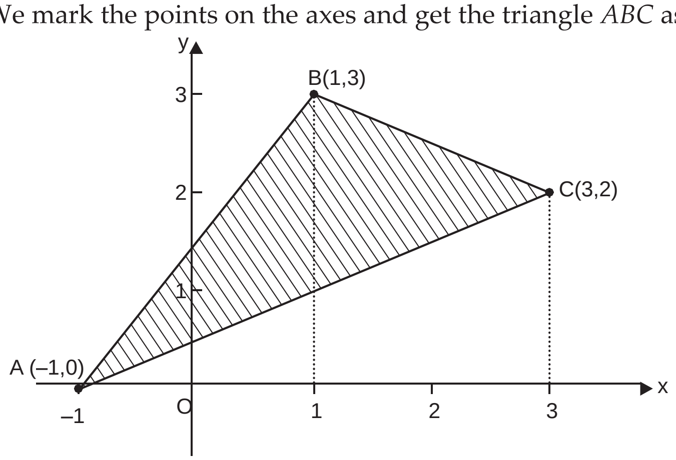
The point is on the plane hence

$$2(2\lambda + 1) - 3(-3\lambda - 2) + 4(4\lambda + 3) + 9 = 0$$

$\Rightarrow \quad 29\lambda + 29 = 0 \quad \text{or} \quad \lambda = 1$

$\therefore$ Co-ordinates of foot of perpendicular are $(-1, 1, -1)$.

24. We mark the points on the axes and get the triangle ABC as shown in the figure



Required area of triangle $= \int_{-1}^{1} AB + \int_{1}^{3} BC - \int_{-1}^{3} AC$

Equation of line $AB \Rightarrow y = \frac{3}{2}(x+1)$

Equation of line BC $y = -\frac{x}{2} + \frac{7}{2}$

Equation of line $AC \Rightarrow y = \frac{x}{2} + \frac{1}{2}$

$\therefore$ Area of $\Delta ABC = \int_{-1}^{1}\left(\frac{3}{2}x+\frac{3}{2}\right)dx + \int_{1}^{3}\left(-\frac{x}{2}+\frac{7}{2}\right)dx - \int_{-1}^{3}\left(\frac{x}{2}+\frac{1}{2}\right)dx$

$$= \left[\frac{3x^2}{4}+\frac{3}{2}x\right]_{-1}^{1} + \left[\left(\frac{-x^2}{4}+\frac{7}{2}x\right)\right]_{1}^{3} - \left[\left(\frac{x^2}{4}+\frac{x}{2}\right)\right]_{-1}^{3}$$

$$= \left(\frac{3}{4}+\frac{3}{2}-\frac{3}{4}+\frac{3}{2}\right)+\left(\frac{-9}{4}+\frac{21}{2}+\frac{1}{4}-\frac{7}{2}\right)-\left(\frac{9}{4}+\frac{3}{2}-\frac{1}{4}+\frac{1}{2}\right)$$

$$= 3+\frac{-9+42+1-14}{4}-\left(\frac{9+6-1+2}{4}\right)$$

$= 3+5-4 = 4$ square units.

27. Total no. of bulbs = 30

Number of defective bulbs = 6

Number of good bulbs = 24

Number of bulbs drawn = 4 = n

p = probability of drawing defective bulb $= \frac{6}{30} = \frac{1}{5}$

q = probability of drawing good bulb $= \frac{4}{5}$

The given probability distribution is a binomial distribution with

$n = 4,\ p = \frac{1}{5},\ q = \frac{4}{5}$

Where $P(r = 0, 1, 2, 3, 4 \text{ success}) = {}^4C_r \left(\frac{1}{5}\right)^r \left(\frac{4}{5}\right)^{4-r}$

Hence mean $= \mu = np$

$\therefore \quad \mu = 4 \times \frac{1}{5} = \frac{4}{5}$

Variance $= \sigma^2 = npq$

$\therefore \quad \sigma^2 = 4 \times \frac{1}{5} \times \frac{4}{5} = \frac{16}{25}$

EXAMINATION PAPERS – 2010

MATHEMATICS CBSE (Delhi)

CLASS – XII

Time allowed: 3 hours **Maximum marks: 100**

General Instructions:

1. ***All*** *questions are compulsory.*
2. *The question paper consists of* ***29*** *questions divided into three Sections* ***A, B*** *and* ***C.*** *Section A comprises of* ***10*** *questions of* ***one*** *mark each, Section B comprises of* ***12*** *questions of* ***four*** *marks each and Section C comprises of* ***7*** *questions of* ***six*** *marks each.*
3. ***All*** *questions in Section A are to be answered in* ***one*** *word,* ***one*** *sentence or as per the exact requirement of the question.*
4. *There is no overall choice. However, internal choice has been provided in* ***4*** *questions of* ***four*** *marks each and* ***2*** *questions of* ***six*** *marks each. You have to attempt only* ***one*** *of the alternatives in all such questions.*
5. *Use of calculator is* ***not*** *permitted.*

Set–I

SECTION–A

Question numbers 1 to 10 carry 1 mark each.

1. What is the range of the function $f(x) = \dfrac{|x-1|}{(x-1)}$?

2. What is the principal value of $\sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$?

3. If $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$, then for what value of α is A an identity matrix?

4. What is the value of the determinant $\begin{vmatrix} 0 & 2 & 0 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix}$?

5. Evaluate : $\int \dfrac{\log x}{x}\, dx.$

6. What is the degree of the following differential equation?

$$5x\left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} - 6y = \log x$$

7. Write a vector of magnitude 15 units in the direction of vector $\hat{i} - 2\hat{j} + 2\hat{k}$.

8. Write the vector equation of the following line:

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{6-z}{2}$$

9. If $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 7 & 11 \\ k & 23 \end{pmatrix}$, then write the value of k.

10. What is the cosine of the angle which the vector $\sqrt{2}\hat{i} + \hat{j} + \hat{k}$ makes with y-axis?

SECTION–B

11. On a multiple choice examination with three possible answers (out of which only one is correct) for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing?

12. Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $(2\vec{a} + \vec{b})$ and $(\vec{a} - 3\vec{b})$ respectively, externally in the ratio 1 : 2. Also, show that P is the mid-point of the line segment RQ.

13. Find the Cartesian equation of the plane passing through the points $A(0, 0, 0)$ and $B(3, -1, 2)$ and parallel to the line $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$.

14. Using elementary row operations, find the inverse of the following matrix :

$$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$$

15. Let Z be the set of all integers and R be the relation on Z defined as $R = \{(a, b) ; a, b \in Z,$ and $(a-b)$ is divisible by 5.$\}$ Prove that R is an equivalence relation.

16. Prove the following:

$$\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), \quad x \in (0, 1)$$

OR

Prove the following :

$$\cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right)$$

17. Show that the function f defined as follows, is continuous at $x = 2$, but not differentiable:

$$f(x) = \begin{cases} 3x-2, & 0 < x \le 1 \\ 2x^2 - x, & 1 < x \le 2 \\ 5x-4, & x > 2 \end{cases}$$

OR

Find $\frac{dy}{dx}$, if $y = \sin^{-1}[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}]$.

18. Evaluate : $\int e^x \left(\dfrac{\sin 4x - 4}{1 - \cos 4x}\right) dx.$

OR

Evaluate : $\int \dfrac{1 - x^2}{x(1 - 2x)}\, dx.$

19. Evaluate : $\int_{\pi/6}^{\pi/3} \dfrac{\sin x + \cos x}{\sqrt{\sin 2x}}\, dx.$

20. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.

21. Find the general solution of the differential equation

$$x \log x \cdot \frac{dy}{dx} + y = \frac{2}{x} \cdot \log x$$

OR

Find the particular solution of the differential equation satisfying the given conditions:

$\dfrac{dy}{dx} = y \tan x$, given that $y = 1$ when $x = 0$.

22. Find the particular solution of the differential equation satisfying the given conditions:

$x^2 dy + (xy + y^2)\, dx = 0$; $y = 1$ when $x = 1$.

SECTION–C

Question numbers 23 to 29 carry 6 marks each.

23. A small firm manufactures gold rings and chains. The total number of rings and chains manufactured per day is atmost 24. It takes 1 hour to make a ring and 30 minutes to make a chain. The maximum number of hours available per day is 16. If the profit on a ring is Rs. 300 and that on a chain is Rs 190, find the number of rings and chains that should be manufactured per day, so as to earn the maximum profit. Make it as an L.P.P. and solve it graphically.

24. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn at random and are found to be both clubs. Find the probability of the lost card being of clubs.

OR

From a lot of 10 bulbs, which includes 3 defectives, a sample of 2 bulbs is drawn at random. Find the probability distribution of the number of defective bulbs.

25. The points $A(4, 5, 10)$, $B(2, 3, 4)$ and $C(1, 2, -1)$ are three vertices of a parallelogram $ABCD$. Find the vector equations of the sides AB and BC and also find the coordinates of point D.

26. Using integration, find the area of the region bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

OR

Evaluate: $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\, dx.$

27. Show that the right circular cylinder, open at the top, and of given surface area and maximum volume is such that its height is equal to the radius of the base.

28. Find the values of x for which $f(x) = [x(x-2)]^2$ is an increasing function. Also, find the points on the curve, where the tangent is parallel to x-axis.

29. Using properties of determinants, show the following:

$$\begin{vmatrix} (b+c)^2 & ab & ca \\ ab & (a+c)^2 & bc \\ ac & bc & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3$$

Set-II

Only those questions, not included in Set I, are given.

3. What is the principal value of $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$?

7. Find the minor of the element of second row and third column (a_{23}) in the following determinant:

$$\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$$

11. Find all points of discontinuity of f, where f is defined as follows :

$$f(x) = \begin{cases} |x| + 3, & x \le -3 \\ -2x, & -3 < x < 3 \\ 6x + 2, & x \ge 3 \end{cases}$$

OR

Find $\frac{dy}{dx}$, if $y = (\cos x)^x + (\sin x)^{1/x}$.

12. Prove the following:

$$\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x \in (0, 1)$$

OR

Prove the following:

$$\cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right)$$

14. Let * be a binary operation on Q defined by

$$a * b = \frac{3ab}{5}$$

Show that * is commutative as well as associative. Also find its identity element, if it exists.

18. Evaluate: $\int_0^{\pi} \frac{x}{1 + \sin x}\, dx.$

20. Find the equations of the normals to the curve $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0.$

23. Evaluate $\int_1^3 (3x^2 + 2x)\, dx$ as limit of sums.

OR

Using integration, find the area of the following region:

$$\left\{(x, y) \,;\, \frac{x^2}{9} + \frac{y^2}{4} \le 1 \le \frac{x}{3} + \frac{y}{2}\right\}$$

29. Write the vector equations of the following lines and hence determine the distance between them:

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}\,;\, \frac{x-3}{4} = \frac{y-3}{6} = \frac{z+5}{12}$$

Set-III

Only those questions, not included in Set I and Set II, are given.

1. Find the principal value of $\sin^{-1}\left(-\frac{1}{2}\right) + \cos^{-1}\left(-\frac{1}{2}\right).$

9. If A is a square matrix of order 3 and $|3A| = K|A|$, then write the value of K.

11. There are two Bags, Bag I and Bag II. Bag I contains 4 white and 3 red balls while another Bag II contains 3 white and 7 red balls. One ball is drawn at random from one of the bags and it is found to be white. Find the probability that it was drawn from Bag I.

14. Prove that : $\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3) = \pi.$

OR

If $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$, find the value of x.

17. Show that the relation S in the set R of real numbers, defined as $S = \{(a, b) : a, b \in R \text{ and } a \le b^3\}$ is neither reflexive, nor symmetric nor transitive.

19. Find the equation of tangent to the curve $y = \frac{x-7}{(x-2)(x-3)}$, at the point, where it cuts the x-axis.

23. Find the intervals in which the function f given by

$$f(x) = \sin x - \cos x, \;\; 0 \le x \le 2\pi$$

is strictly increasing or strictly decreasing.

24. Evaluate $\int_1^4 (x^2 - x)\, dx$ as limit of sums.

OR

Using integration find the area of the following region :

$$\{(x, y) : |x - 1| \le y \le \sqrt{5 - x^2}\}$$

SOLUTIONS

Set–I

SECTION–A

1. We have given

$$f(x) = \frac{|x-1|}{(x-1)}$$

$$|x-1| = \begin{cases} (x-1), & \text{if } x-1>0 \text{ or } x>1 \\ -(x-1), & \text{if } x-1<0 \text{ or } x<1 \end{cases}$$

(*i*) For $x > 1$, $\quad f(x) = \frac{(x-1)}{(x-1)} = 1$

(*ii*) For $x < 1$, $\quad f(x) = \frac{-(x-1)}{(x-1)} = -1$

$\therefore$ Range of $f(x) = \frac{|x-1|}{(x-1)}$ is $\{-1, 1\}$.

2. Let $x = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$\Rightarrow \quad \sin x = -\frac{\sqrt{3}}{2} \quad \Rightarrow \quad \sin x = \sin\left(-\frac{\pi}{3}\right) \qquad \left[\because \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3}\right]$

$\Rightarrow \quad x = -\frac{\pi}{3}$

The principal value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is $-\frac{\pi}{3}$.

3. We have given

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

For the identity matrix, the value of A_{11} and A_{12} should be 1 and value of A_{12} and A_{21} should be 0.

i.e., $\quad \cos\alpha = 1 \quad$ and $\quad \sin\alpha = 0$

As we know $\cos 0° = 1$ and $\sin 0° = 0$

$\Rightarrow \quad \alpha = 0°$

4. $\begin{vmatrix} 0 & 2 & 0 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix} = 0\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ 4 & 6 \end{vmatrix} + 0\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$ (expanding the given determinant by R_1)

$$= -2\begin{vmatrix} 2 & 4 \\ 4 & 6 \end{vmatrix}$$

$$= -2(12 - 16) = 8$$

The value of determinant is 8.

5. We have given

$$\int \frac{\log x}{x}\, dx$$

Let $\log x = t \Rightarrow \frac{1}{x}\, dx = dt$

Given integral $= \int t\, dt$

$$= \frac{t^2}{2} + c = \frac{(\log x)^2}{2} + c$$

6. $5x\left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} - 6y = \log x$

Degree of differential equation is the highest power of the highest derivative. In above $\frac{d^2y}{dx^2}$ is the highest order of derivative.

$\therefore$ Its degree = 1.

7. Let $\vec{A} = \hat{i} - 2\hat{j} + 2\hat{k}$

Unit vector in the direction of $\vec{A}$ is $\hat{A} = \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{(1)^2 + (-2)^2 + (2)^2}} = \frac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k})$

$\therefore$ Vector of magnitude 15 units in the direction of $\vec{A} = 15\hat{A} = 15\,\frac{(\hat{i} - 2\hat{j} + 2\hat{k})}{3}$

$$= \frac{15}{3}\hat{i} - \frac{30}{3}\hat{j} + \frac{30}{3}\hat{k}$$

$$= 5\hat{i} - 10\hat{j} + 10\hat{k}$$

8. We have given line as

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{-2}$$

By comparing with equation

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c},$$

We get given line passes through the point (x_1, x_2, x_3) *i.e.*, $(5, -4, 6)$ and direction ratios are (a, b, c) *i.e.*, $(3, 7, -2)$.

Now, we can write vector equation of line as

$$\vec{A} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda\,(3\hat{i} + 7\hat{j} - 2\hat{k})$$

9. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ k & 23 \end{bmatrix}$

$$\text{LHS} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(3) + (2)(2) & (1)(1) + (2)(5) \\ (3)(3) + (4)(2) & (3)(1) + (4)(5) \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ 17 & 23 \end{bmatrix}$$

Now comparing LHS to RHS, we get

$\therefore \quad k = 17$

10. We will consider

$$\vec{a} = \sqrt{2}\hat{i} + \hat{j} + \hat{k}$$

Unit vector in the direction of $\vec{a}$ is $\hat{a} = \dfrac{\sqrt{2}\hat{i} + \hat{j} + \hat{k}}{\sqrt{(\sqrt{2})^2 + (1)^2 + (1)^2}}$

$$= \frac{\sqrt{2}\hat{i} + \hat{j} + \hat{k}}{\sqrt{4}} = \frac{\sqrt{2}\hat{i} + \hat{j} + \hat{k}}{2}$$

$$= \frac{\sqrt{2}}{2}\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}$$

The cosine of the angle which the vector $\sqrt{2}\hat{i} + \hat{j} + \hat{k}$ makes with y-axis is $\left(\dfrac{1}{2}\right)$.

SECTION–B

11. No. of questions $= n = 5$

Option given in each question $= 3$

p = probability of answering correct by guessing $= \dfrac{1}{3}$

q = probability of answering wrong by guessing $= 1 - p = 1 - \dfrac{1}{3} = \dfrac{2}{3}$

This problem can be solved by binomial distribution.

$$P(r) = {}^{n}C_r \left(\frac{2}{3}\right)^{n-r}\left(\frac{1}{3}\right)^{r}$$

where r = four or more correct answers = 4 or 5

(*i*) $P(4) = {}^{5}C_4 \left(\dfrac{2}{3}\right)\left(\dfrac{1}{3}\right)^4$ (*ii*) $P(5) = {}^{5}C_5 \left(\dfrac{1}{3}\right)^5$

$\therefore \quad P = P(4) + P(5)$

$$= {}^5C_4 \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^4 + {}^5C_5 \left(\frac{1}{3}\right)^5$$

$$= \left(\frac{1}{3}\right)^4 \left[\frac{10}{3} + \frac{1}{3}\right] = \frac{1}{3 \times 3 \times 3 \times 3}\left[\frac{11}{3}\right] = \frac{11}{243} = 0 \cdot 045$$

12. The position vector of the point R dividing the join of P and Q externally in the ratio 1 : 2 is

$$\vec{OR} = \frac{1(\vec{a} - 3\vec{b}) - 2(2\vec{a} + \vec{b})}{1-2}$$

$$= \frac{\vec{a} - 3\vec{b} - 4\vec{a} - 2\vec{b}}{-1} = \frac{-3\vec{a} - 5\vec{b}}{-1} = 3\vec{a} + 5\vec{b}$$

Mid-point of the line segment RQ is

$$\frac{(3\vec{a} + 5\vec{b}) + (\vec{a} - 3\vec{b})}{2} = 2\vec{a} + \vec{b}$$

As it is same as position vector of point P, *so* P is the mid-point of the line segment RQ.

13. Equation of plane is given by

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

Given plane passes through (0, 0, 0)

$\therefore \quad a(x - 0) + b(y - 0) + c(z - 0) = 0 \qquad \ldots(i)$

Plane (i) passes through (3, –1, 2)

$\therefore \quad 3a - b + 2c = 0 \qquad \ldots(ii)$

Also plane (i) is parallel to the line

$$\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$$

$$a - 4b + 7c = 0 \qquad \ldots(iii)$$

Eliminating a, b, c from equations (i), (ii) and (iii), we get

$$\begin{vmatrix} x & y & z \\ 3 & -1 & 2 \\ 1 & -4 & 7 \end{vmatrix} = 0$$

$$\Rightarrow \quad x\begin{vmatrix} -1 & 2 \\ -4 & 7 \end{vmatrix} - y\begin{vmatrix} 3 & 2 \\ 1 & 7 \end{vmatrix} + z\begin{vmatrix} 3 & -1 \\ 1 & -4 \end{vmatrix} = 0$$

$$\Rightarrow \quad x(-7 + 8) - y(21 - 2) + z(-12 + 1) = 0$$

$\Rightarrow \quad x - 19y - 11z = 0$, which is the required equation

14. Given, $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

We can write, $\quad A = IA$

i.e., $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} A$ $[R_1 \to R_1 - R_2]$

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} A$ $[R_2 \to R_2 - R_1]$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} A$ $[R_1 \to R_1 - 2R_2]$

$A^{-1} = \begin{bmatrix} & -5 \\ -1 & 2 \end{bmatrix}$

15. We have provided

$R = \{(a, b) : a, b \in Z, \text{ and } (a - b) \text{ is divisible by } 5\}$

(*i*) As $(a - a) = 0$ is divisible by 5.

$\therefore$ $(a, a) \in R \ \forall \ a \in R$

Hence, R is reflexive.

(*ii*) Let $(a, b) \in R$

$\Rightarrow$ $(a - b)$ is divisible by 5.

$\Rightarrow$ $-(b - a)$ is divisible by 5. $\Rightarrow$ $(b - a)$ is divisible by 5.

$\therefore$ $(b, a) \in R$

Hence, R is symmetric.

(*iii*) Let $(a, b) \in R$ and $(b, c) \in Z$

Then, $(a - b)$ is divisible by 5 and $(b - c)$ is divisible by 5.

$(a - b) + (b - c)$ is divisible by 5.

$(a - c)$ is divisible by 5.

$\therefore$ $(a, c) \in R$

$\Rightarrow$ R is transitive.

Hence, R is an equivalence relation.

16. We have to prove

$$\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), \ x \in (0, 1)$$

$$\text{L.H.S.} = \tan^{-1}\sqrt{x} = \frac{1}{2}[2\tan^{-1}\sqrt{x}]$$

$$= \frac{1}{2}\left[\cos^{-1}\frac{(1)^2 - (\sqrt{x})^2}{(1)^2 + (\sqrt{x})^2}\right]$$

$$= \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right) = \text{R.H.S.}$$ **Hence Proved.**

OR

$$\cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\sin^{-1}\left(\frac{56}{65}\right)$$

$$\text{LHS}=\cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)$$

$$=\sin^{-1}\left(\frac{5}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right) \qquad \left[\because \cos^{-1}\left(\frac{12}{13}\right)=\sin^{-1}\left(\frac{5}{13}\right)\right]$$

$$=\sin^{-1}\left[\frac{5}{13}\times\sqrt{1-\left(\frac{3}{5}\right)^2}+\frac{3}{5}\times\sqrt{1-\left(\frac{5}{13}\right)^2}\right]$$

$$=\sin^{-1}\left[\frac{5}{13}\times\frac{4}{5}+\frac{3}{5}\times\frac{12}{13}\right]=\sin^{-1}\frac{56}{65}=\text{RHS}$$

LHS = RHS **Hence Proved**

17. We have given, $f(x)=\begin{cases}3x-2, & 0<x\le 1\\ 2x^2-x, & 1<x\le 2\\ 5x-4, & x>2\end{cases}$

At $x=2$,

(*i*) RHL

$$=\lim_{x\to 2^+} f(x)$$
$$=\lim_{h\to 0} f(2+h)$$
$$=\lim_{h\to 0}\{5(2+h)-4\}$$
$$=10-4=6$$

LHL

$$=\lim_{x\to 2^-} f(x)$$
$$=\lim_{x\to 2^-} f(x)$$
$$=\lim_{h\to 0}\{2(2-h)^2-(2-h)\}$$
$$=\lim_{h\to 0}\{(2-h)(4-2h-1)\}=2\times 3=6$$

Also, $f(2)=2(2)^2-2=8-2=6$

$\because$ LHL = RHL = $f(2)$

$\therefore$ $f(x)$ is continuous at $x=2$

(*ii*) LHD

$$=\lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$$
$$=\lim_{h\to 0}\frac{[2(2-h)^2-(2-h)]-(8-2)}{-h}$$
$$=\lim_{h\to 0}\frac{[8+2h^2-8h-2+h)-6}{-h}$$
$$=\lim_{h\to 0}\frac{2h^2-7h}{-h}$$
$$=\lim_{h\to 0}(-2h+7)=7$$

RHD

$$=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$
$$=\lim_{h\to 0}\frac{[5(2+h)-4]-(8-2)}{h}$$
$$=\lim_{h\to 0}\frac{5h}{h}$$
$$=\lim_{h\to 0}(5)$$
$$=5$$

Q $\quad$ LHD $\neq$ RHD

$\therefore$ $f(x)$ is not differentiable at $x = 2$

OR

We have given

$$y = \sin^{-1}[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}].$$

$$= \sin^{-1}[x\sqrt{1-(\sqrt{x})^2} - \sqrt{x}\sqrt{1-x^2}]$$

$$\Rightarrow \quad y = \sin^{-1}x - \sin^{-1}\sqrt{x}$$

[using $\sin^{-1}x - \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}]$]

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x(1-x)}}$$

18. $\displaystyle\int e^x\left(\frac{\sin 4x - 4}{1-\cos 4x}\right)dx$

$$= \int e^x\left(\frac{2\sin 2x\cos 2x - 4}{2\sin^2 2x}\right)dx \qquad [\sin 4x = 2\sin 2x\cos 2x \text{ and } 1-\cos 4x = 2\sin^2 2x]$$

$$= \int e^x(\cot 2x - 2\,\text{cosec}^2\,2x)\,dx$$

$$= \int \cot 2x\,.\,e^x\,dx - 2\int e^x\,\text{cosec}^2\,2x\,dx$$

$$= [\cot 2x\,.\,e^x - \int(-2\,\text{cosec}^2\,2x)\,.\,e^x dx] - 2\int e^x\,\text{cosec}^2\,2x\,dx$$

$$= \cot 2x\,.\,e^x + 2\int \text{cosec}^2\,2x\,.\,e^x\,dx - 2\int \text{cosec}^2\,2x\,.\,e^x\,dx = e^x\cot 2x + c$$

OR

We have given

$$\int\frac{1-x^2}{x(1-2x)}dx = \int\frac{1-x^2}{x-2x^2}dx$$

$$= \int\frac{x^2-1}{2x^2-x}dx = \int\frac{1}{2}\left(\frac{2x^2-2}{2x^2-x}\right)dx$$

$$= \frac{1}{2}\int\frac{(2x^2-x)+(x-2)}{2x^2-x}dx$$

$$= \frac{1}{2} \int \left(1 + \frac{x - 2}{2x^2 - x} \right) dx \qquad \ldots(i)$$

By partial fraction

$$\frac{x-2}{2x^2-x}=\frac{x-2}{x(2x-1)}=\frac{A}{x}+\frac{B}{2x-1}$$

$$x-2=A(2x-1)+Bx \qquad \ldots(ii)$$

Equating co-efficient of x and constant term, we get

$$2A+B=1 \quad \text{and} \quad -A=-2$$

$\Rightarrow$ $\qquad A=2,\ B=-3$

$$\therefore \quad \frac{x-2}{2x^2-x}=\frac{2}{x}+\frac{3}{1-2x}$$

From equation (i)

$$\int \frac{1-x^2}{x(1-2x)}\,dx=\frac{1}{2}\int 1\,dx+\frac{1}{2}\int\left(\frac{2}{x}+\frac{3}{1-2x}\right)dx$$

$$=\frac{1}{2}x+\log|x|-\frac{3}{4}\log|1-2x|+c$$

19. Given integral can be written as

$$\mathrm{I}=\int_{\pi/6}^{\pi/3}\frac{\sin x+\cos x}{\sqrt{1-(1-\sin 2x)}}\,dx=\int_{\pi/6}^{\pi/3}\frac{\sin x+\cos x}{\sqrt{1-(\sin x-\cos x)^2}}\,dx$$

Put $\sin x-\cos x=t$

so that, $\qquad (\cos x+\sin x)=\frac{dt}{dx}$

when $x=\frac{\pi}{6}$, $t=\sin\frac{\pi}{6}-\cos\frac{\pi}{6}=\frac{1}{2}-\frac{\sqrt{3}}{2}$

when $x=\frac{\pi}{3}$, $t=\sin\frac{\pi}{3}-\cos\frac{\pi}{3}=\frac{\sqrt{3}}{2}-\frac{1}{2}$

$$\Rightarrow \quad \mathrm{I}=\int_{\frac{1}{2}-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}-\frac{1}{2}}\frac{dt}{\sqrt{1-t^2}}=\left[\sin^{-1}t\right]_{\frac{1}{2}-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}-\frac{1}{2}}$$

$$=\sin^{-1}\left[\frac{\sqrt{3}}{2}-\frac{1}{2}\right]-\sin^{-1}\left[\frac{1}{2}-\frac{\sqrt{3}}{2}\right]$$

$$=\sin^{-1}\left[\frac{\sqrt{3}}{2}-\frac{1}{2}\right]+\sin^{-1}\left[\frac{\sqrt{3}}{2}-\frac{1}{2}\right]=2\sin^{-1}\frac{1}{2}(\sqrt{3}-1)$$

20. Let $P(x_1, y_1)$ be the required point. The given curve is

$$y=x^3 \qquad \ldots(i)$$

$$\frac{dy}{dx}=3x^2$$

$$\left(\frac{dy}{dx}\right)_{x_1,y_1}=3x_1^2$$

$\because$ the slope of the tangent at $(x_1, y_1) = y_1$

$$3x_1^2 = y_1 \qquad \ldots(ii)$$

Also, (x_1, y_1) lies on (i) so $\quad y_1 = x_1^3 \qquad \ldots(iii)$

From (ii) and (iii), we have

$$3x_1^2 = x_1^3 \quad \Rightarrow \quad x_1^2\,(3 - x_1) = 0$$

$\Rightarrow \qquad x_1 = 0 \quad$ or $\quad x_1 = 3$

When $x_1 = 0$, $y_1 = (0)^3 = 0$

When $x_1 = 3$, $y_1 = (3)^3 = 27$

$\therefore$ the required points are (0, 0) and (3, 27).

21. $x \log x \dfrac{dy}{dx} + y = \dfrac{2}{x} \log x$

$\Rightarrow \qquad \dfrac{dy}{dx} + \dfrac{1}{x \log x}\, y = \dfrac{2}{x^2} \qquad \ldots(i)$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

where $P = \dfrac{1}{x \log x}$ and $Q = \dfrac{2}{x^2}$

$\therefore \qquad$ I.F. $= e^{\int P dx} = e^{\int \frac{1}{x \log x} dx}$

[Let $\log x = t \;\therefore\; \dfrac{1}{x}\, dx = dt$]

$$= e^{\int \frac{1}{t} dt} = e^{\log t} = t = \log x$$

$\therefore \qquad y \log x = \displaystyle\int \frac{2}{x^2} \log x \, dx + C \qquad$ [$\therefore$ solution is $y\,(\text{I. F.}) = \displaystyle\int Q\,(\text{I. F.})\; dx + C$]

$\Rightarrow \qquad y \log x = 2 \displaystyle\int \underset{\text{I}}{\log x} \,.\, \underset{\text{II}}{x^{-2}} \; dx + c$

$\Rightarrow \qquad y \log x = 2\left[\log x \left[\dfrac{x^{-1}}{-1}\right] - \displaystyle\int \frac{1}{x}\left[\frac{x^{-1}}{-1}\right] dx\right] + C$

$\Rightarrow \qquad y \log x = 2\left[-\dfrac{\log x}{x} + \displaystyle\int x^{-2} \; dx\right] + C$

$\Rightarrow \qquad y \log x = 2\left[-\dfrac{\log x}{x} - \dfrac{1}{x}\right] + C$

$\Rightarrow \qquad y \log x = -\dfrac{2}{x}(1 + \log x + C)$, which is the required solution

OR

$$\frac{dy}{dx} = y \tan x \qquad \Rightarrow \frac{dy}{y} = \tan x \, dx$$

By integrating both sides, we get

$$\int \frac{dy}{y} = \int \tan x \, . \, dx$$

$$\log y = \log |\sec x| + C \qquad \text{...}(i)$$

By putting $x = 0$ and $y = 1$ (as given), we get

$$\log 1 = \log(\sec 0) + C$$

$$C = 0$$

$\therefore$ $(i) \Rightarrow$ $\log y = \log|\sec x|$

$\Rightarrow$ $y = \sec x$

22. $x^2 dy + y(x + y)\, dx = 0$

$x^2 dy = -y(x + y)\, dx$

$$\frac{dy}{dx} = -y\frac{(x + y)}{x^2}$$

$$\frac{dy}{dx} = -\left(\frac{xy + y^2}{x^2}\right) \qquad \text{...}(i)$$

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in equation (i)

$$v + x\frac{dv}{dx} = -\left(\frac{vx^2 + v^2x^2}{x^2}\right) \quad \Rightarrow \quad v + x\frac{dv}{dx} = -(v + v^2)$$

$\Rightarrow$ $x\frac{dv}{dx} = -2v - v^2$

$\Rightarrow$ $\frac{dv}{v^2 + 2v} = -\frac{dx}{x}$ (by separating variable)

$\Rightarrow$ $\int \frac{1}{v^2 + 2v} dv = -\int \frac{1}{x} dx$ (Integrating both sides)

$\Rightarrow$ $\int \frac{1}{v^2 + 2v + 1 - 1} dv = -\int \frac{1}{x} dx$

$\Rightarrow$ $\int \frac{1}{(v + 1)^2 - 1^2} dv = -\int \frac{1}{x} dx$

$\Rightarrow$ $\frac{1}{2}\log\left|\frac{v + 1 - 1}{v + 1 + 1}\right| = -\log x + \log C$

$\Rightarrow$ $\frac{1}{2}\log\left|\frac{v}{v + 2}\right| = -\log x + \log C$

$$\Rightarrow \quad \log\left|\frac{v}{v+2}\right| + 2\log x = 2\log C$$

$$\Rightarrow \quad \log\left|\frac{v}{v+2}\right| + \log x^2 = \log k, \qquad \text{where } k = C^2$$

$$\Rightarrow \quad \log\left|\frac{vx^2}{v+2}\right| = \log k \quad \Rightarrow \quad \frac{vx^2}{v+2} = k$$

$$\Rightarrow \quad \frac{\frac{y}{x}\cdot x^2}{\frac{y}{x}+2} = k \qquad \left[\because \frac{y}{x} = v\right]$$

$$\Rightarrow \quad x^2 y = k(y+2x) \qquad \ldots(ii)$$

It is given that $y = 1$ and $x = 1$, putting in (ii), we get

$$1 = 3k \quad \Rightarrow \quad k = \frac{1}{3}$$

Putting $k = \frac{1}{3}$ in (ii), we get

$$x^2 y = \left(\frac{1}{3}\right)(y+2x)$$

$$\Rightarrow \quad 3x^2 y = (y+2x)$$

SECTION–C

23. Total no. of rings & chain manufactured per day = 24.

Time taken in manufacturing ring = 1 hour

Time taken in manufacturing chain = 30 minutes

One time available per day = 16

Maximum profit on ring = Rs 300

Maximum profit on chain = Rs 190

Let gold rings manufactured per day = x

Chains manufactured per day = y

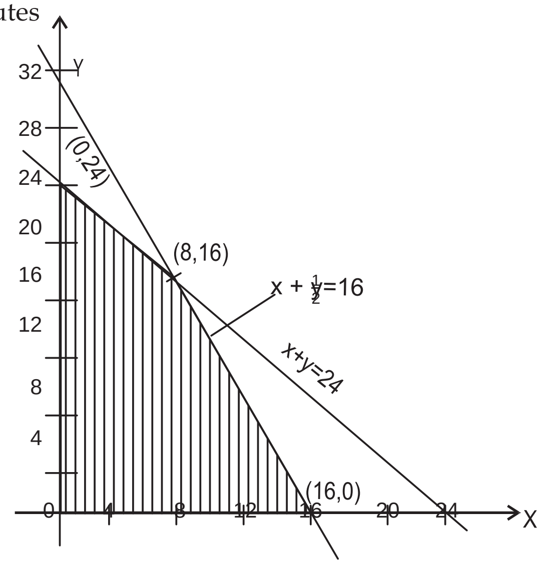
L.P.P. is

maximize $Z = 300x + 190y$

Subject to $x \geq 0$, $y \geq 0$

$$x + y \leq 24$$

$$x + \frac{1}{2}y \leq 16$$



Possible points for maximum Z are (16, 0), (8, 16) and (0, 24).

At (16, 0), $\quad Z = 4800 + 0 = 4800$

At (8, 16), $Z = 2400 + 3040 = 5440 \quad \leftarrow$ Maximum

At (0, 24), $Z = 0 + 4560 = 4560$

Z is maximum at (8, 16).

$\therefore$ 8 gold rings & 16 chains must be manufactured per day.

24. Let A_1, E_1 and E_2 be the events defined as follows:

A : cards drawn are both club

E_1: lost card is club

E_2: lost card is not a club

Then, $P(E_1) = \frac{13}{52} = \frac{1}{4}, \quad P(E_2) = \frac{39}{52} = \frac{3}{4}$

$P(A/E_1) =$ Probability of drawing both club cards when lost card is club $= \frac{12}{51} \times \frac{11}{50}$

$P(A/E_2) =$ Probability of drawing both club cards when lost card is not club $= \frac{13}{51} \times \frac{12}{50}$

To find : $P(E_1/A)$

By Baye's Theorem,

$$P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2)}$$

$$= \frac{\frac{1}{4} \times \frac{12}{51} \times \frac{11}{50}}{\frac{1}{4} \times \frac{12}{51} \times \frac{11}{50} + \frac{3}{4} \times \frac{13}{51} \times \frac{12}{50}} = \frac{12 \times 11}{12 \times 11 + 3 \times 13 \times 12} = \frac{11}{11 + 39} = \frac{11}{50}$$

OR

There are 3 defective bulbs & 7 non-defective bulbs.

Let X denote the random variable of "the no. of defective bulb."

Then X can take values 0, 1, 2 since bulbs are replaced

$p = P(D) = \frac{3}{10} \quad$ and $\quad q = P(\overline{D}) = 1 - \frac{3}{10} = \frac{7}{10}$

We have

$$P(X = 0) = \frac{{}^7C_2 \times {}^3C_0}{{}^{10}C_2} = \frac{7 \times 6}{10 \times 9} = \frac{7}{15}$$

$$P(X = 1) = \frac{{}^7C_1 \times {}^3C_1}{{}^{10}C_2} = \frac{7 \times 3 \times 2}{10 \times 9} = \frac{7}{15}$$

$$P(X = 2) = \frac{{}^7C_0 \times {}^3C_2}{{}^{10}C_2} = \frac{1 \times 3 \times 2}{10 \times 9} = \frac{1}{15}$$

$\therefore$ Required probability distribution is

X	0	1	2
$P(x)$	7/15	7/15	1/15

25. The points A (4, 5, 10), B(2, 3, 4) and C (1, 2, – 1) are three vertices of parallelogram $ABCD$.

Let coordinates of D be (x, y, z)

Direction vector along AB is

$$\vec{a} = (2-4)\,\hat{i} + (3-5)\,\hat{j} + (4-10)\,\hat{k} = -2\hat{i} - 2\hat{j} - 6\hat{k}$$

$\therefore$ Equation of line AB, is given by

$$\vec{b} = (4\hat{i} + 5\hat{j} + 10\hat{k}) + \lambda\,(2\hat{i} + 2\hat{j} + 6\hat{k})$$

Direction vector along BC is

$$\vec{c} = (1-2)\,\hat{i} + (2-3)\,\hat{j} + (-1-4)\,\hat{k} = -\hat{i} - \hat{j} - 5\hat{k}$$

$\therefore$ Equation of a line BC, is given by .

$$\vec{d} = (2\hat{i} + 3\hat{j} + 4\hat{k}) + \mu\,(\hat{i} + \hat{j} + 5\hat{k})$$

Since $ABCD$ is a parallelogram AC and BD bisect each other

$$\therefore \qquad \left[\frac{4+1}{2}, \frac{5+2}{2}, \frac{10-1}{2}\right] = \left[\frac{2+x}{2}, \frac{3+y}{2}, \frac{4+z}{2}\right]$$

$\Rightarrow \qquad 2 + x = 5, \qquad 3 + y = 7, \qquad 4 + z = 9$

$\Rightarrow \qquad x = 3, \quad y = 4, \quad z = 5$

Co-ordinates of D are (3, 4, 5).

26. Given curve

$$x^2 = 4y \qquad \ldots(i)$$

Line equation

$$x = 4y - 2 \qquad \ldots(ii)$$

Equation (i) represents a parabola with vertex at the origin and axis along (+)ve direction of y-axis.

Equation (ii) represents a straight line which meets the coordinates axes at (–2, 0) and $\left(0, \frac{1}{2}\right)$ respectively.

By solving two equations, we obtain

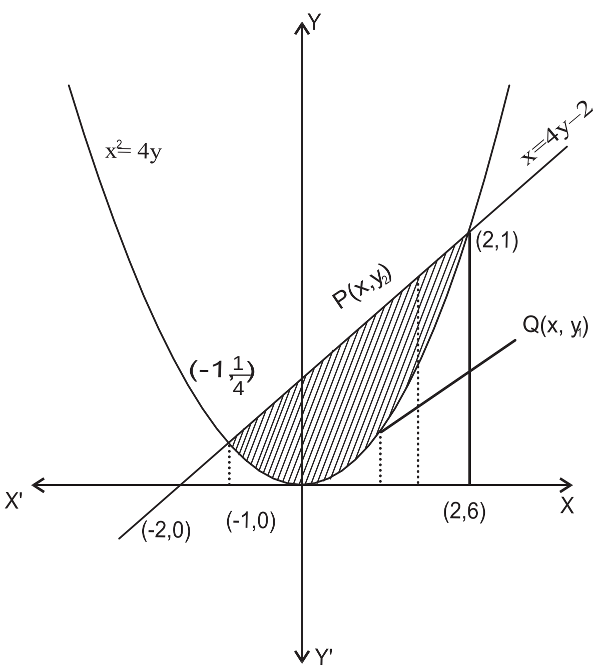
$$x = x^2 - 2$$

$\Rightarrow \quad x^2 - x - 2 = 0 \quad$ (by eliminating y)

$\Rightarrow \quad (x-2)(x+1) = 0$

$\Rightarrow \quad x = -1, 2$

The point of intersection of given parabola & line are (2, 1) and $\left(-1, \frac{1}{4}\right)$.



$\therefore$ required area $= \int_{-1}^{2} (y_2 - y_1)\, dx.$ ...(*iii*)

$\because P(x, y_2)$ and $Q(x, y_1)$ lies on (*ii*) and (*i*) respectively

$\therefore \quad y_2 = \dfrac{x+2}{4}$ and $y_1 = \dfrac{x^2}{4}$

$\therefore$ (*iii*) $\Rightarrow$ Area $= \displaystyle\int_{-1}^{2} \left(\frac{x+2}{4} - \frac{x^2}{4} \right) dx$

$$= \int_{-1}^{2} \frac{x}{4}\, dx + \frac{1}{2}\int_{-1}^{2} dx - \frac{1}{4}\int_{-1}^{2} x^2\, dx = \left[\frac{x^2}{8} + \frac{1}{2}x - \frac{x^3}{12} \right]_{-1}^{2}$$

$$= \left[\frac{4}{8} + \frac{2}{2} - \frac{8}{12} \right] - \left[\frac{1}{8} - \frac{1}{2} + \frac{1}{12} \right] = \frac{9}{8} \text{ sq. units.}$$

OR

$$I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\, dx$$

$$I = \int_0^{\pi} \frac{x \dfrac{\sin x}{\cos x}}{\dfrac{1}{\cos x} + \dfrac{\sin x}{\cos x}}\, dx = \int_0^{\pi} \frac{x \sin x}{1 + \sin x}\, dx \qquad \text{...}(i)$$

$$I = \int_0^{\pi} \frac{(\pi - x) \sin (\pi - x)}{1 + \sin (\pi - x)}\, dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx \right]$$

$$I = \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + \sin x}\, dx \qquad \text{...}(ii)$$

$$\therefore \quad 2I = \int_0^{\pi} \frac{\pi \sin x}{1 + \sin x}\, dx \qquad \text{[Using (i) and (ii)]}$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{\sin x (1 - \sin x)}{(1 + \sin x)(1 - \sin x)}\, dx$$

$$= \pi \int_0^{\pi} \frac{\sin x - \sin^2 x}{1 - \sin^2 x}\, dx = \pi \int_0^{\pi} \left[\frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} \right] dx$$

$$= \pi \int_0^{\pi} \tan x \sec x\, dx - \pi \int_0^{\pi} \tan^2 x\, dx$$

$$= \pi \int_0^{\pi} \tan x \sec x\, dx - \pi \int_0^{\pi} (\sec^2 x - 1)\, dx$$

$$= \pi \int_0^{\pi} \sec x \tan x\, dx - \pi \int_0^{\pi} \sec^2 x\, dx + \pi \int_0^{\pi} dx$$

$$= \pi \left[\sec x \right]_0^{\pi} - \pi \left[\tan x \right]_0^{\pi} + \pi \left[x \right]_0^{\pi} + C = \pi[-1 - 1] - 0 + \pi[\pi - 0] = \pi(\pi - 2)$$

$$\Rightarrow \quad I = \frac{\pi}{2}(\pi - 2)$$

27. Let r be the radius and h be the height of the cylinder of given surface s. Then,

$$s = \pi r^2 + 2\pi hr$$

$$h = \frac{s - \pi r^2}{2\pi r} \qquad \ldots(i)$$

Then $$v = \pi r^2 h = \pi r^2 \left[\frac{s - \pi r^2}{2\pi r}\right] \qquad \text{[From eqn. } (i)]$$

$$v = \frac{sr - \pi r^3}{2}$$

$$\frac{dv}{dr} = \frac{s - 3\pi r^2}{2} \qquad \ldots(ii)$$

For maximum or minimum value, we have

$$\frac{dv}{dr} = 0$$

$$\Rightarrow \quad \frac{s - 3\pi r^2}{2} = 0 \quad \Rightarrow \quad s = 3\pi r^2$$

$$\Rightarrow \quad \pi r^2 + 2\pi rh = 3\pi r^2$$

$$\Rightarrow \quad r = h$$

Differentiating equation (ii) w.r.t. r, we get

$$\frac{d^2v}{dr^2} = -3\pi r < 0$$

Hence, when $r = h$, *i.e.*, when the height of the cylinder is equal to the radius of its base v is maximum.

28. We have given

$$y = [x(x-2)]^2 \qquad \ldots(i)$$

$$= x^2(x^2 - 4x + 4) = x^4 - 4x^3 + 4x^2$$

$$\frac{dy}{dx} = 4x^3 - 12x^2 + 8x$$

For the increasing function,

$$\frac{dy}{dx} > 0$$

$$\Rightarrow \quad 4x^3 - 12x^2 + 8x > 0 \quad \Rightarrow \quad 4x(x^2 - 3x + 2) > 0$$

$$\Rightarrow \quad 4x(x-1)(x-2) > 0$$

For $0 < x < 1$, $\frac{dy}{dx} = (+)(-)(-) = (+)$ ve

For $x > 2$, $\frac{dy}{dx} = (+)(+)(+) = (+)$ ve

The function is increasing for $0 < x < 1$ and $x > 2$

If tangent is parallel to x-axis, then $\frac{dy}{dx} = 0$

$\Rightarrow \quad 4x(x-1)(x-2) = 0$

$\Rightarrow \quad x = 0, 1, 2$

For $x = 0$, $f(0) = 0$

For $x = 1$, $f(1) = [1(1-2)]^2 = 1$

For $x = 2$, $f(2) = [2 \times 0]^2 = 0$

$\therefore$ Required points are (0, 0), (1, 1), (2, 0).

29. To prove: $\begin{vmatrix} (b+c)^2 & ab & ca \\ ab & (a+c)^2 & bc \\ ac & bc & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$

Let $\quad \Delta = \begin{vmatrix} (b+c)^2 & ab & ca \\ ab & (a+c)^2 & bc \\ ac & bc & (a+b)^2 \end{vmatrix}$

[Multiplying R_1, R_2 and R_3 by a, b, c respectively]

$$\Delta = \frac{1}{abc}\begin{vmatrix} a(b+c)^2 & ba^2 & a^2c \\ ab^2 & b(a+c)^2 & b^2c \\ ac^2 & bc^2 & (a+b)^2c \end{vmatrix}$$

$$= \frac{1}{abc}\, abc \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

$[C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3]$

$$= \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (c+a)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c+a)(b+c-a) & 0 & a^2 \\ 0 & (c+a+b)(b+c-a) & b^2 \\ (c+a+b)(c-a-b) & (c+a+b)(c-a-b) & (a+b)^2 \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix} \qquad (R_3 \to R_3 - (R_1 + R_2))$$

$$= \frac{(a+b+c)^2}{ab} \begin{vmatrix} ab+ac-a^2 & 0 & a^2 \\ 0 & bc+ba-b^2 & b^2 \\ -2ab & -2ab & 2ab \end{vmatrix}$$

$$= \frac{(a+b+c)^2}{ab} \begin{vmatrix} ab+ac & a^2 & a^2 \\ b^2 & bc+ba & b^2 \\ 0 & 0 & 2ab \end{vmatrix} \qquad [C_1 \to C_1 + C_3,\ C_2 \to C_2 + C_3]$$

$$= \frac{(a+b+c)^2}{ab} . ab . 2ab \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 2ab\,(a+b+c)^2 \begin{vmatrix} b+c & a \\ b & c+a \end{vmatrix}$$

$$= 2ab\,(a+b+c)^2 \,\{(b+c)(c+a) - ab\}$$

$$= 2abc\,(a+b+c)^3 = \text{RHS}$$

Set–II

3. Let $\quad x = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$\Rightarrow \quad \cos x = -\frac{\sqrt{3}}{2}$

$\Rightarrow \quad \cos x = \cos\left[\pi - \frac{\pi}{6}\right] = \cos\frac{5\pi}{6} \quad [\text{as } \cos \pi/6 = \sqrt{3}/2]$

$\Rightarrow \quad x = \frac{5\pi}{6}$

The principal value of $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is $\frac{5\pi}{6}$.

7. We have given

$$\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$$

Minor of an element

$$a_{23} = M_{23} = \begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix} = 10 + 3 = 13$$

11. We have given

$$f(x)=\begin{cases}|x|+3, & x\le -3\\ -2x, & -3<x<3\\ 6x+2, & x\ge 3\end{cases}$$

(i) For $x = -3$

$$\text{LHL} = \lim_{x\to -3^-} f(x) = \lim_{h\to 0} f(3-h) = \lim_{h\to 0} -2(3-h) = -6$$

$$\text{RHL} = \lim_{x\to 3^+} f(x) = \lim_{h\to 0} f(3+h) = \lim_{h\to 0} 6(3+h)+2 = 20$$

$$\text{LHL} \ne \text{RHL}$$

At $x = 3$, function is not continuous.

OR

Given, $y = (\cos x)^x + (\sin x)^{1/x}$

$$= e^{x\log(\cos x)} + e^{1/x \log(\sin x)}$$

By differentiating w.r.t. x

$$\frac{dy}{dx} = e^{x\log(\cos x)}\left\{\log(\cos x) + \frac{x}{\cos x} - (\sin x)\right\}$$

$$+ e^{\frac{1}{x}\log(\sin x)}\left[-\log(\sin x)\frac{1}{x^2} + \frac{\cos x}{x\sin x}\right]$$

$$= (\cos x)^x\{\log(\cos x) - x\tan x\} + (\sin x)^{1/x}\left\{-\frac{1}{x^2}\log\sin x + \frac{\cot x}{x}\right\}$$

14. For commutativity, condition that should be fulfilled is

$$a*b = b*a$$

Consider $a*b = \dfrac{3ab}{5} = \dfrac{3ba}{5} = b*a$

$\therefore$ $a*b = b*a$

Hence, * is commutative.

For associativity, condition is $(a*b)*c = a*(b*c)$

Consider $(a*b)*c = \left(\dfrac{3ab}{5}\right)*c = \dfrac{9abc}{25} = \dfrac{3}{5}a\left(\dfrac{3}{5}bc\right) = \dfrac{3}{5}a(b*c) = a*(b*c)$

Hence, $(a*b)*c = a*(b*c)$

$\therefore$ * is associative.

Let $e \in Q$ be the identity element,

Then $a*e = e*a = a$

$\Rightarrow$ $\dfrac{3ae}{5} = \dfrac{3ea}{5} = a \quad \Rightarrow \quad e = \dfrac{5}{3}\cdot$

18. $I = \int_0^{\pi} \frac{x}{1 + \sin x}\, dx$ ...(*i*)

$$I = \int_0^{\pi} \frac{\pi - x}{1 + \sin (\pi - x)}\, dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx\right]$$

$$= \int_0^{\pi} \frac{\pi - x}{1 + \sin x}\, dx \qquad \text{...}(ii)$$

Adding equations (*i*) and (*ii*), we get

$$2I = \int_0^{\pi} \frac{\pi}{1 + \sin x}\, dx$$

$$= \pi \int_0^{\pi} \frac{1 - \sin x}{(1 + \sin x)(1 - \sin x)}\, dx = \pi \int_0^{\pi} \frac{1 - \sin x}{\cos^2 x}\, dx$$

$$= \pi \int_0^{\pi} (\sec^2 x - \sec x \tan x)\, dx$$

$$= \pi \left[\tan x - \sec x\right]_0^{\pi} = \pi[(0 + 1) - (0 - 1)] = 2\pi$$

$\Rightarrow \quad 2I = 2\pi$ or $I = \pi$

20. Given equation of curve

$$y = x^3 + 2x + 6 \qquad \text{...}(i)$$

Equation of line

$$x + 14y + 4 = 0 \qquad \text{...}(ii)$$

Differentiating (*i*) w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 + 2 \quad \Rightarrow \quad \frac{dx}{dy} = \frac{1}{3x^2 + 2}$$

$\therefore$ Slope of normal $= \frac{-1}{3x^2 + 2}$.

and it is parallel to equation of line.

$$\therefore \qquad \frac{-1}{3x^2 + 2} = \frac{-1}{14}$$

$$\Rightarrow \qquad 3x^2 + 2 = 14 \;\Rightarrow\; 3x^2 = 12$$

$$x^2 = 4 \;\Rightarrow\; x = \pm 2$$

From equation of curve,

if $x = 2, y = 18$; if $x = -2, y = -6$

$\therefore$ Equation of normal at (2, 18) is

$$y - 18 = -\frac{1}{14}(x - 2) \quad \text{or} \quad x + 14y - 254 = 0$$

and for (–2, –6) it is

$$y + 6 = -\frac{1}{14}(x + 2) \quad \text{or} \quad x + 14y + 86 = 0$$

23. $\int_1^3 (3x^2 + 2x)\,dx$

We have to solve this by the help of limit of sum.

So, $a = 1, b = 3$

$$f(x) = 3x^2 + 2x, \qquad h = \frac{3-1}{n} \Rightarrow nh = 2$$

$$\because \quad \int_1^3 (3x^2 + 2x)\,dx = \lim_{h \to 0} h\,[f(1) + f(1+h) + f(1+2h) + \ldots f(1 + \overline{(n-1)}\;h)] \qquad \ldots(i)$$

$$f(1) = 3(1)^2 + 2(1)$$

$$f(1+h) = 3\,(1+h)^2 + 2\,(1+h) = 3h^2 + 8h + 5$$

$$f\,(1+2h) = 3\,(1+2h)^2 + 2\,(1+2h) = 12h^2 + 16h + 5$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$f(1 - \overline{(n-1)}\;h) = 3\,(1 + (n-1)\,h)^2 + 2\,(1 + (n-1)\,h)$$

$$= 3\,(n-1)^2\,h^2 + 8\,(n-1)\,h + 5$$

By putting all values in equation (i), we get

$$\int_1^3 (3x^2 + 2x)\,dx = \lim_{h \to 0} h\,[(5) + (3h^2 + 8h + 5) + (12h^2 + 16h + 5) + \ldots$$

$$+ [3\,(n-1)^2\,h^2 + 8\,(n-1)\,h + 5]]$$

$$= \lim_{h \to 0} h\,[3h^2\,\{1 + 4 + \ldots + (n-1)^2\} + 8h\,\{1 + 2 + \ldots + (n-1)\} + 5n]$$

$$= \lim_{h \to 0} h\left[3h^2 \cdot \frac{(n-1)\,(2n-1)\,n}{6} + \frac{8h\,(n-1)\,n}{2} + 5n\right]$$

$$\left[\because \{1 + 4 + \ldots + (n-1)^2 = \frac{(n-1)\,(2n-1)\,n}{6} \text{ and } \{1 + 2 + \ldots + (n-1) = \frac{(n-1)\,n}{2}\right]$$

$$= \lim_{h \to 0}\left[\frac{(nh-h)\,(nh)\,(2nh-h)}{2} + 4\,(nh-h)\,(nh) + 5nh\right] = \lim_{h \to 0}\left[\frac{(2-h)\,(2)\,(4-h)}{2} + 4\,(2-h)\,(2) + 10\right]$$

$$= \left[\frac{2 \times 2 \times 4}{2} + 4 \times 2 \times 2 + 10\right] \text{[by applying limit]} = 34$$

OR

We have given

$$\left\{(x, y)\,;\, \frac{x^2}{9} + \frac{y^2}{4} \le 1 \le \frac{x}{3} + \frac{y}{2}\right\}$$

There are two equations

(i) y_1 = equation of ellipse

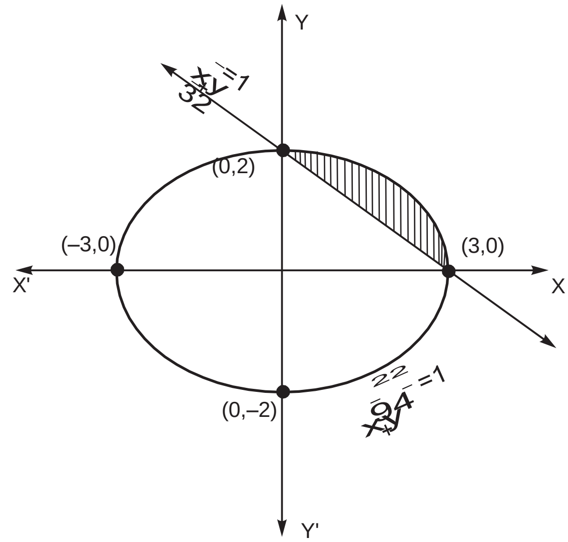
i.e., $\frac{x^2}{9}+\frac{y^2}{4}=1$

$\Rightarrow \quad y_1=\frac{2}{3}\sqrt{9-x^2}$

and y_2 = equation of straight line

i.e., $\frac{x}{3}+\frac{y}{2}=1$

$\Rightarrow \quad y_2=\frac{2}{3}(3-x)$



$\therefore$ We have required area

$$=\int_0^3 (y_1-y_2)\,dx$$

$$=\int_0^3 \left\{\frac{2}{3}\sqrt{9-x^2}-\frac{2}{3}(3-x)\right\}dx$$

$$=\frac{2}{3}\int_0^3 \left\{\sqrt{9-x^2}-(3-x)\right\}dx$$

$$=\frac{2}{3}\left[\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\frac{x}{3}-3x+\frac{x^2}{2}\right]_0^3$$

$$=\frac{2}{3}\left[\left(\frac{3}{2}\sqrt{0}+\frac{9}{2}\sin^{-1}(1)-9+\frac{9}{2}\right)-(0+0-0+0)\right]$$

$$=\frac{2}{3}\left[\frac{9}{2}\cdot\frac{\pi}{2}-\frac{9}{2}\right]=\frac{3}{2}(\pi-2) \text{ sq. units.}$$

29. Let

Line 1 : $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+4}{6}=\mu$ …(i)

From above, a point (x, y, z) on line 1 will be $(2\mu+1,\ 3\mu+2,\ 6\mu-4)$

Line 2 : $\frac{x-3}{4}=\frac{y-3}{6}=\frac{z+5}{12}=\lambda$ …(ii)

From above, a point (x, y, z) on line 2 will be $(4\lambda+3,\ 6\lambda+3,\ 12\lambda-5)$

Position vector from equation (i), we get

$$\vec{r}=(2\mu+1)\hat{i}+(3\mu+2)\hat{j}+(6\mu-4)\hat{k}$$

$$=(\hat{i}+2\hat{j}-4\hat{k})+\mu(2\hat{i}+3\hat{j}+6\hat{k})$$

$$\vec{a}_1=\hat{i}+2\hat{j}-4\hat{k},\ \vec{b}_1=2\hat{i}+3\hat{j}+6\hat{k}$$

Position vector from equation (ii), we get

$$\vec{r}=(4\lambda+3)\hat{i}+(6\lambda+3)\hat{j}+(12\lambda-5)\hat{k}=(3\hat{i}+3\hat{j}-5\hat{k})+\lambda(4\hat{i}+6\hat{j}+12\hat{k})$$

$$\vec{a}_2 = 3\hat{i} + 3\hat{j} - 5\hat{k}, \quad \vec{b}_2 = 4\hat{i} + 6\hat{j} + 12\hat{k}$$

From b_1 and b_2 we get $\vec{b}_2 = 2\vec{b}_1$

$$\text{Shortest distance} = \left| \frac{|(\vec{a}_2 - \vec{a}_1) \times \vec{b}\,|}{|\vec{b}\,|} \right|$$

$$(\vec{a}_2 - \vec{a}_1) = (3\hat{i} + 3\hat{j} - 5\hat{k}) - (\hat{i} + 2\hat{j} - 4\hat{k}) = 2\hat{i} + \hat{j} - \hat{k}$$

$$(\vec{a}_2 - \vec{a}_1) \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 2 & 3 & 6 \end{vmatrix} = 9\hat{i} - 14\hat{j} + 4\hat{k}$$

$$|(\vec{a}_2 - \vec{a}_1) \times \vec{b}\,| = \sqrt{(9)^2 + (-14)^2 + (4)^2} = \sqrt{81 + 196 + 16} = \sqrt{293}$$

$$|\vec{b}\,| = \sqrt{(2)^2 + (3)^2 + (6)^2} = \sqrt{4 + 9 + 36} = 7$$

$$\text{Shortest distance} = \frac{\sqrt{293}}{7} \text{ units}$$

Set–III

1. We have given

$$\sin^{-1}\left(-\frac{1}{2}\right) + \cos^{-1}\left(-\frac{1}{2}\right)$$

But, as we know $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}\cdot$

$\therefore$ principal value is $\dfrac{\pi}{2}\cdot$

9. Given $\quad |3A| = K|A|$, where A is a square matrix of order 3. ...(*i*)

We know that $\quad |3A| = (3)^3 |A| = 27|A|$ ...(*ii*)

By comparing equations (*i*) and (*ii*), we get

$$K = 27$$

11. Let A, E_1, E_2 be the events defined as follow:

A : Ball drawn is white

E_1 : Bag I is chosen, $\quad E_2$: Bag II is chosen

Then we have to find $P(E_1 / A)$

Using Baye's Theorem

$$P(E_1 / A) = \frac{P(E_1)P(A / E_1)}{P(E_1)P(A / E_1) + P(E_2)P(A / E_2)} = \frac{\frac{1}{2} \times \frac{4}{5}}{\frac{1}{2} \times \frac{4}{7} + \frac{1}{2} \times \frac{3}{10}} = \frac{\frac{4}{7}}{\frac{40 + 21}{70}} = \frac{40}{61}$$

14. $\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3) = \pi$

Consider L.H.S.

$$\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3) \qquad \ldots(i)$$

Let $Z = \tan^{-1}(1)$

$\tan Z = 1$

$$Z = \frac{\pi}{4} \qquad \ldots(ii)$$

And we know $\tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1} \frac{x+y}{1-xy} \qquad \ldots(iii)$

Putting value of (*ii*) and (*iii*) in equation (*i*), we get

$$\text{LHS} = \frac{\pi}{4} + \pi + \tan^{-1}\frac{x+y}{1-xy} = \frac{\pi}{4} + \pi + \tan^{-1}\frac{2+3}{1-2\times 3}$$

$$= \frac{\pi}{4} + \pi + \tan^{-1}(-1) = \frac{\pi}{4} + \pi - \frac{\pi}{4} = \pi = \text{RHS}$$

OR

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$$

Consider above equation

We know $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}$

$$\Rightarrow \quad \tan^{-1}\left\{\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right\} = \frac{\pi}{4}$$

$$\Rightarrow \quad \frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1} = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{2x^2-4}{-3} = 1 \Rightarrow 2x^2 - 4 = -3$$

$$\Rightarrow \quad 2x^2 = 1 \quad \text{or} \quad x = \pm\frac{1}{\sqrt{2}}$$

i.e.,

$$x = \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$$

17. We have given

$$S = \{(a, b) : a, b \in R \text{ and } a \le b^3\}$$

(*i*) Consider $a = \frac{1}{2}$

Then $\quad (a, a) = \left(\frac{1}{2}, \frac{1}{2}\right) \in R$

But $\quad \frac{1}{2} \leq \left(\frac{1}{2}\right)^3$ is not true

$\therefore$ $(a, a) \notin R$, for all $a \in R$

Hence, R is not reflexive.

(*ii*) Let $a = \frac{1}{2},\ b = 1$

Then, $\frac{1}{2} \leq (1)^3$ *i.e.,* $\frac{1}{2} \leq 1$

$\Rightarrow$ $(a, b) \in R$

But $1 \not\leq \left(\frac{1}{2}\right)^3$ $\quad \therefore$ $(b, a) \notin R$

Hence, $(a, b) \in R$ but $(b, a) \notin R$

(*iii*) Let $a = 3, b = \frac{3}{2}, c = \frac{4}{3}$

Then $\quad 3 \leq \left(\frac{3}{2}\right)^3$ $\quad$ *i.e.,* $3 \leq 27$

$\therefore$ $\quad (a, b) \in R$

Also, $\quad \frac{3}{2} \leq \left(\frac{4}{3}\right)^3$ $\quad$ *i.e.,* $\frac{3}{2} \leq \frac{64}{27}$

$\therefore$ $\quad (b, c) \in R$

But $\quad 3 \not\leq \left(\frac{4}{3}\right)^3$ $\quad$ *i.e.,* $3 \not\leq \frac{64}{27}$

$\therefore$ $\quad (a, c) \notin R$

Hence, $\quad (a, b) \in R$, $(b, c) \in R$ but $(a, c) \notin R$

$\Rightarrow$ R is not transitive.

19. We have given

$$y = \frac{x-7}{(x-2)(x-3)} \qquad \ldots(i)$$

Let (*i*) cuts the x-axis at $(x, 0)$

then $y = \frac{x-7}{(x-2)(x-3)} = 0 \quad \Rightarrow \quad x = 7$

$\therefore$ the required point is $(7, 0)$.

Differentiating equation (*i*) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(x-2)(x-3)\,1 - (x-7)\,[(x-2)+(x-3)]}{[(x-2)(x-3)]^2}$$

$$= \frac{x^2 - 5x + 6 - 2x^2 + 19x - 35}{(x^2 - 5x + 6)^2} = \frac{-x^2 + 14x - 29}{(x^2 + 6 - 5x)^2}$$

$$\left.\frac{dy}{dx}\right]_{(7,\,0)} = \frac{-49 + 98 - 29}{(49 - 35 + 6)^2} = \frac{20}{400} = \frac{1}{20}$$

$\therefore$ Equation of tangent is

$$y - y_1 = \frac{1}{20}(x - x_2)$$

$$\Rightarrow \quad y - 0 = \frac{1}{20}(x - 7) \quad \text{or} \quad x - 20y - 7 = 0$$

23. $f(x) = \sin x - \cos x,\ 0 \le x \le 2\pi$

Differentiating w.r.t. x, we get

$$f'(x) = \cos x + \sin x = \sqrt{2}\sin\left(\frac{\pi}{4} + x\right)$$

For critical points, $\dfrac{dy}{dx} = 0$

$$\Rightarrow \quad \cos x + \sin x = 0$$

$$\Rightarrow \quad \sin x = -\cos x \quad \Rightarrow \quad \tan x = -1$$

$$\Rightarrow \quad \tan x = \tan\left(-\frac{\pi}{4}\right)$$

$$\Rightarrow \quad x = n\pi - \frac{\pi}{4} = -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4} + \text{K}$$

(*i*) For $0 < x < \dfrac{3\pi}{4}$,

$\dfrac{\pi}{4} < x + \dfrac{\pi}{4} < \pi$ *i.e.*, It lies in quadrant I, II.

$\Rightarrow \sqrt{2}\sin\left(\dfrac{\pi}{2} + x\right) > 0,$ Hence, function is increasing.

(*ii*) For $\dfrac{3\pi}{4} < x < \dfrac{7\pi}{4}$

$\pi < x + \dfrac{\pi}{4} < 2\pi$ *i.e.*, It lies in quadrant III, IV.

$\Rightarrow \sqrt{2}\sin\left(\dfrac{\pi}{4} + x\right) < 0,$ Hence, function is decreasing.

(*iii*) For $\dfrac{7\pi}{4} < x < 2\pi$

$2\pi < x + \dfrac{\pi}{4} < \dfrac{9\pi}{4}$ *i.e.*, It lies in quadrant I.

$\Rightarrow \sqrt{2}\sin\left(\dfrac{\pi}{4} + x\right) > 0,$ Hence, function is increasing.

Internal where function is strictly increasing is

$$\left(0, \frac{3\pi}{4}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right)$$

Interval where function is strictly decreasing is

$$\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$$

24. $\int_1^4 (x^2 - x)\, dx$

We have to solve it by using limit of sums.

Here, $a = 1, b = 4, h = \frac{b-a}{n} = \frac{4-1}{n}$ *i.e.,* $nh = 3$

Limit of sum for $\int_1^4 (x^2 - x)\, dx$ is

$$= \lim_{h \to 0} h\,[f(1) + f(1+h) + f(1+2h) + \ldots + f\{1 + (n-1)\,h\}]$$

Now, $f(1) = 1 - 1 = 0$

$$f(1+h) = (1+h)^2 - (1+h) = h^2 + h$$

$$f(1+2h) = (1+2h)^2 - (1+2h) = 4h^2 + 2h$$

....

....

....

$$f[1 + (n-1)\,h] = \{1 + (n-1)\,h\}^2 - \{1 + (n-1)\,h\}$$

$$= (n-1)^2\, h^2 + (n-1)\,h$$

$$\therefore \quad \int_1^2 (x^2 - x)\, dx = \lim_{h \to 0} h\,[0 + h^2 + h + 4h^2 + 2h + \ldots : (n-1)^2\, h^2 + (n-1)\,h]$$

$$= \lim_{h \to 0} h\,[h^2\,\{1 + 4 + \ldots + (n-1)^2\} + h\,\{1 + 2 + \ldots + (n-1)\}]$$

$$= \lim_{h \to 0} h\left[h^2 \cdot \frac{(n)(n-1)(2n-1)}{6} + h\,\frac{n(n-1)}{2}\right]$$

$$\left[\because\ 1 + 4 + \ldots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6}\quad 1 + 2 + \ldots + (n-1) = \frac{n(n-1)}{2}\right]$$

$$= \lim_{h \to 0}\left[\frac{nh\,(nh-h)\,(2nh-h)}{6} + \frac{nh\,(nh-h)}{2}\right]$$

$$= \lim_{h \to 0}\left[\frac{(3-h)\,(3)\,(6-h)}{6} + \frac{(3-h)\,(3)}{2}\right]$$

$$= \left(\frac{3 \times 3 \times 6}{6}\right) + \left(\frac{3 \times 3}{2}\right) = 9 + \frac{9}{2} = \frac{27}{2}$$

OR

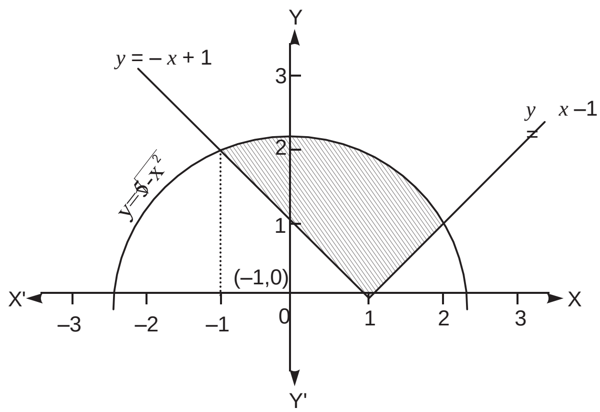
We have provided

$$(x, y) : |x-1| \le y \le \sqrt{5-x^2}$$

Equation of curve is $y = \sqrt{5-x^2}$ or $y^2 + x^2 = 5$, which is a circle with centre at (0, 0) and radius $\frac{5}{2}$.

Equation of line is $y = |x-1|$

Consider, $\quad y = x-1 \quad$ and $\quad y = \sqrt{5-x^2}$



Eliminating y, we get

$$x - 1 = \sqrt{5-x^2}$$

$\Rightarrow \qquad x^2 + 1 - 2x = 5 - x^2$

$\Rightarrow \qquad 2x^2 - 2x - 4 = 0$

$\Rightarrow \qquad x^2 - x - 2 = 0$

$\Rightarrow \qquad (x-2)(x+1) = 0$

$\Rightarrow \qquad x = 2, -1$

The required area is

$$= \int_{-1}^{2} \sqrt{5-x^2}\, dx - \int_{-1}^{1} (-x+1)\, dx - \int_{1}^{2} (x-1)\, dx$$

$$= \left[\frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}}\right]_{-1}^{2} - \left[-\frac{x^2}{2} + x\right]_{-1}^{1} - \left[\frac{x^2}{2} - x\right]_{1}^{2}$$

$$= \left(1 + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}}\right) + 1 - \frac{5}{2}\sin^{-1}\left(-\frac{1}{\sqrt{5}}\right) - \left(\frac{-1}{2} + 1 + \frac{1}{2} + 1\right) - \left(2 - 2 - \frac{1}{2} + 1\right)$$

$$= \frac{5}{2}\left(\sin^{-1}\frac{2}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{5}}\right) + 2 - 2 - \frac{1}{2}$$

$$= \frac{5}{2}\sin^{-1}\left[\frac{2}{\sqrt{5}}\sqrt{1-\frac{1}{5}} + \frac{1}{\sqrt{5}}\sqrt{1-\frac{4}{5}}\right] - \frac{1}{2}$$

$$= \frac{5}{2}\left[\sin^{-1}\left(\frac{4}{5} + \frac{1}{5}\right)\right] - \frac{1}{2}$$

$$= \frac{5}{2}\sin^{-1}(1) - \frac{1}{2}$$

$$= \left(\frac{5\pi}{4} - \frac{1}{2}\right) \text{ sq. units}$$

EXAMINATION PAPERS – 2010

MATHEMATICS CBSE (All India)
CLASS – XII

Time allowed: 3 hours **Maximum marks: 100**

General Instructions: ***As given in CBSE Examination paper (Delhi)*** **– 2010.**

Set–I

SECTION–A

Question numbers 1 to 10 carry 1 mark each.

1. If $f : R \to R$ be defined by $f(x) = (3 - x^3)^{1/3}$, then find $fof(x)$.

2. Write the principal value of $\sec^{-1}(-2)$.

3. What positive value of x makes the following pair of determinants equal?

$$\begin{vmatrix} 2x & 3 \\ 5 & x \end{vmatrix}, \quad \begin{vmatrix} 16 & 3 \\ 5 & 2 \end{vmatrix}$$

4. Evaluate : $\int \sec^2 (7 - 4x)\, dx$

5. Write the adjoint of the following matrix :

$$\begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}$$

6. Write the value of the following integral :

$$\int_{-\pi/2}^{\pi/2} \sin^5 x\, dx$$

7. A is a square matrix of order 3 and $|A| = 7$. Write the value of $|adj.\, A|$.

8. Write the distance of the following plane from the origin :

$$2x - y + 2z + 1 = 0$$

9. Write a vector of magnitude 9 units in the direction of vector $-2\hat{i} + \hat{j} + 2\hat{k}$.

10. Find λ if $(2\hat{i} + 6\hat{j} + 14\hat{k}) \times (\hat{i} - \lambda\hat{j} + 7\hat{k}) = \vec{0}$.

SECTION–B

Question number 11 to 22 carry 4 marks each.

11. A family has 2 children. Find the probability that both are boys, if it is known that

(*i*) at least one of the children is a boy

(*ii*) the elder child is a boy.

12. Show that the relation S in the set $A = \{x \in Z : 0 \le x \le 12\}$ given by $S = \{(a, b) : a, b \in Z, |a - b|$ is divisible by 4$\}$ is an equivalence relation. Find the set of all elements related to 1.

13. Prove the following :

$$\tan^{-1} x + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

OR

Prove the following:

$$\cos[\tan^{-1}\{\sin(\cot^{-1} x)\}] = \sqrt{\frac{1+x^2}{2+x^2}}$$

14. Express the following matrix as the sum of a symmetric and skew symmetric matrix, and verify you result:

$$\begin{pmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{pmatrix}$$

15. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 4\hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$, find a vector of magnitude 6 units which is parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$.

OR

Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c} \cdot \vec{d} = 18$.

16. Find the points on the line $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2}$ at a distance of 5 units from the point $P(1, 3, 3)$.

OR

Find the distance of the point $P(6, 5, 9)$ from the plane determined by the points $A(3, -1, 2)$, $B(5, 2, 4)$ and $C(-1, -1, 6)$.

17. Solve the following differential equation :

$$(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{1}{x^2 - 1}; \quad |x| \ne 1$$

OR

Solve the following differential equation :

$$\sqrt{1 + x^2 + y^2 + x^2y^2} + xy\frac{dy}{dx} = 0$$

18. Show that the differential equation $(x - y)\frac{dy}{dx} = x + 2y$, is homogeneous and solve it.

19. Evaluate the following :

$$\int \frac{x+2}{\sqrt{(x-2)(x-3)}}\,dx$$

20. Evaluate the following :

$$\int_1^2 \frac{5x^2}{x^2+4x+3}\,dx$$

21. If $y = e^{a\sin^{-1}x}$, $-1 \le x \le 1$, then show that

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2 y = 0$$

22. If $y = \cos^{-1}\left(\dfrac{3x + 4\sqrt{1-x^2}}{5}\right)$, find $\dfrac{dy}{dx}$.

SECTION–C

Question number 2 to 29 carry 6 marks each.

23. Using properties of determinants, prove the following:

$$\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x)$$

OR

Find the inverse of the following matrix using elementary operations :

$$A = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

24. A bag contains 4 balls. Two balls are drawn at random, and are found to be white. What is the probability that all balls are white?

25. One kind of cake requires 300 g of flour and 15 g of fat, another kind of cake requires 150 g of flour and 30 g of fat. Find the maximum number of cakes which can be made from $7 \cdot 5$ kg of flour and 600 g of fat, assuming that there is no shortage of the other ingredients used in making the cakes. Make it as an L.P.P. and solve it graphically.

26. Find the coordinates of the foot of the perpendicular and the perpendicular distance of the point $P(3, 2, 1)$ from the plane $2x - y + z + 1 = 0$. Find also, the image of the point in the plane.

27. Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$.

OR

Using integration, find the area of the triangle ABC, coordinates of whose vertices are $A(4, 1)$, $B(6, 6)$ and $C(8, 4)$.

28. If the length of three sides of a trapezium other than the base is 10 cm each, find the area of the trapezium, when it is maximum.

29. Find the intervals in which the following function is :

(*a*) strictly increasing

(*b*) strictly decreasing

Set-II

Only those questions, not included in Set I, are given

6. Write the principal value of $\cot^{-1}(-\sqrt{3})$.

10. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}\cdot\vec{b}| = |\vec{a}\times\vec{b}|$, then what is the angle between $\vec{a}$ and $\vec{b}$?

11. Prove the following :

$$\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$$

OR

Solve for x :

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$$

14. If $A = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix}$, then find the value of $A^2 - 3A + 2I$.

18. Evaluate:

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}\,dx$$

20. Show that the following differential equation is homogeneous, and then solve it :

$$y\,dx + x\log\left(\frac{y}{x}\right)dy - 2x\,dy = 0$$

23. Find the equations of the tangent and the normal to the curve

$$x = 1 - \cos\theta, \quad y = \theta - \sin\theta\,; \quad \text{at } \theta = \frac{\pi}{4}\cdot$$

24. Find the equation of the plane passing through the point $P(1, 1, 1)$ and containing the line $\vec{r} = (-3\hat{i} + \hat{j} + 5\hat{k}) + \lambda(3\hat{i} - \hat{j} - 5\hat{k})$. Also, show that the plane contains the line

$$\vec{r} = (-\hat{i} + 2\hat{j} + 5\hat{k}) + \mu(\hat{i} - 2\hat{j} - 5\hat{k})$$

Set-III

Only those questions, not included in Set I and Set II are given

6. Find the value of $\sin^{-1}\left(\frac{4\pi}{5}\right)$.

7. Vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = \frac{2}{3}$ and $(\vec{a} \times \vec{b})$ is a unit vector. Write the angle between $\vec{a}$ and $\vec{b}$.

11. Show that the relation S defined on the set $N \times N$ by

$$(a, b)\ S(c, d) \quad \Rightarrow \quad a + d = b + c$$

15. For the following matrices$A$ and B, verify that $(AB)' = B'A'$.

$$A = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}, \quad B = (-1, 2, 1)$$

17. Solve the following differential equation :

$$(x^2 + 1)\frac{dy}{dx} + 2xy = \sqrt{x^2 + 4}$$

OR

Solve the following differential equation :

$$(x^3 + x^2 + x + 1)\frac{dy}{dx} = 2x^2 + x$$

20. If $y = \operatorname{cosec}^{-1} x, x > 1.$ then show that

$$x(x^2 - 1)\frac{d^2y}{dx^2} + (2x^2 - 1)\frac{dy}{dx} = 0$$

23. Using matrices, solve the following system of equations :

$$x + 2y - 3z = -4$$
$$2x + 3y + 2z = 2$$
$$3x - 3y - 4z = 11$$

OR

If a, b, c are positive and unequal, show that the following determinant is negative :

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

25. Show that the volume of the greatest cylinder that can be inscribed in a cone of height 'h' and semi-vertical angle 'α' is $\frac{4}{27}\pi h^3 \tan^2 \alpha$.

SOLUTIONS

Set–I

SECTION–A

1. If $f : R \to R$ be defined by

$$f(x) = (3 - x^3)^{1/3}$$

then $\quad (fof)\, x = f(f(x))$

$$= f\,[(3 - x^3)^{1/3}]$$

$$= [3 - \{(3 - x^3)^{1/3}\}^3]^{1/3} = [3 - (3 - x^3)]^{1/3} = (x^3)^{1/3} = x$$

2. Let $\quad x = \sec^{-1}(-2)$

$\Rightarrow \quad \sec x = -2$

$\Rightarrow \quad \sec x = -\sec\dfrac{\pi}{3} = \sec\left(\pi - \dfrac{\pi}{3}\right) = \sec\dfrac{2\pi}{3}$

$\Rightarrow \quad x = \dfrac{2\pi}{3}$.

3. We have given

$$\begin{vmatrix} 2x & 3 \\ 5 & x \end{vmatrix} = \begin{vmatrix} 16 & 3 \\ 5 & 2 \end{vmatrix}$$

$\Rightarrow \quad 2x^2 - 15 = 32 - 15 \qquad$ (solving the determinant)

$\Rightarrow \quad 2x^2 = 32 \quad \Rightarrow \quad x^2 = 16 \quad \Rightarrow \quad x = \pm 4$

But we need only positive value

$\therefore \quad x = 4$

4. Let $\mathrm{I} = \int \sec^2 (7 - 4x)\, dx$

Let $\quad 7 - 4x = m, \quad -4dx = dm$

$\Rightarrow \quad \mathrm{I} = \dfrac{-1}{4}\int \sec^2 m\, dm$

$$= -\frac{1}{4}\tan m + c = -\frac{1}{4}\tan(7 - 4x) + c$$

5. We have given matrix :

$$\begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$$

$C_{11} = 3 \qquad C_{12} = -4$

$C_{21} = 1 \qquad C_{22} = 2$

$\therefore \quad Adj.\ A = \begin{bmatrix} 3 & 1 \\ -4 & 2 \end{bmatrix}$

6. $\int_{-\pi/2}^{\pi/2} \sin^5 x \, dx$

Let $f(x) = \sin^5 x$

$$f(-x) = [\sin(-x)]^5$$
$$= (-\sin x)^5 = -\sin^5 x$$
$$= -f(x)$$

Thus, $f(x)$ is an odd function.

$$\therefore \quad \int_{-\pi/2}^{\pi/2} \sin^5 x \, dx = 0$$

7. A is a square matrix of order 3 and $|A| = 7$

then $\quad |adj. A| = |A|^2 = (7)^2 = 49$

8. We have given plane

$$2x - y + 2z + 1 = 0$$

$$\text{Distance from origin} = \left| \frac{(2 \times 0) - (1 \times 0) + (2 \times 0) + 1}{\sqrt{(2)^2 + (-1)^2 + (2)^2}} \right| = \left| \frac{1}{\sqrt{4+1+4}} \right| = \frac{1}{3}$$

9. Let $\quad \vec{r} = -2\hat{i} + \hat{j} + 2\hat{k}$

Unit vector in the direction of $\vec{r} = \hat{r} = \dfrac{\vec{r}}{|\vec{r}|}$

$\therefore$ Vector of magnitude $9 = 9\hat{r}$

$$\text{Units in the direction of } \vec{r} = 9\left[\frac{-2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{(2)^2 + (1)^2 + (2)^2}}\right]$$

$$= 9\left[\frac{-2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{4+1+4}}\right] = -6\hat{i} + 3\hat{j} + 6\hat{k}$$

10. We have given

$$(2\hat{i} + 6\hat{j} + 14\hat{k}) \times (\hat{i} - \lambda\hat{j} + 7\hat{k}) = \vec{0}$$

$$\Rightarrow \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 14 \\ 1 & -\lambda & 7 \end{vmatrix} = \vec{0}$$

$$\Rightarrow \quad \hat{i}\begin{vmatrix} 6 & 14 \\ -\lambda & 7 \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & 14 \\ 1 & 7 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & 6 \\ 1 & -\lambda \end{vmatrix} = \vec{0}$$

$$\Rightarrow \quad \hat{i}(42 + 14\lambda) - 0\hat{j} + \hat{k}(-2\lambda - 6) = \vec{0}$$

$$\Rightarrow \quad 42 + 14\lambda = 0 \quad \Rightarrow \quad 14\lambda = -42 \quad \Rightarrow \quad \lambda = -3$$

Also, $-2\lambda - 6 = 0 \quad \Rightarrow \quad \lambda = -3$

$\therefore \quad \lambda = -3$

SECTION–B

11. A family has 2 children, then

Sample space = $S = \{BB, BG, GB, GG\}$

where B = Boy, G = Girl

(i) Let us define the following events:

A : at least one of the children is boy : $\{BB, BG, GB\}$

B : both are boys: $\{BB\}$

$\therefore \quad A \cap B : \{BB\}$

$$\Rightarrow \quad P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/4}{3/4} = \frac{1}{3}$$

(ii) Let A : elder boy child : $\{BB, BG\}$

B : both are boys: $\{BB\}$

$\therefore \quad A \cap B : \{BB\}$

$$\Rightarrow \quad P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/4}{2/4} = \frac{1}{2}$$

12. We have given,

$A = \{x \in Z : 0 \le x \le 12\}$ and

$S = \{(a, b) : a, b \in A, |a - b| \text{ is divisible by } 4\}$

(i) for $(a, a) \in S$, $|a - a| = 0$ is divisible by 4.

$\therefore$ It is reflexive.

(ii) Let $(a, b) \in S$

Then $|a - b|$ is divisible by 4

$\Rightarrow$ $|-(b - a)|$ is divisible by 4 $\Rightarrow |b - a|$ is divisible by 4

$\therefore \quad (a, b) \in S \Rightarrow (b, a) \in S$

$\therefore$ It is symmetric.

(iii) Let $(a, b) \in S$ and $(b, c) \in S$

$\Rightarrow$ $|a - b|$ is divisible by 4 and $|b - c|$ is divisible by 4

$\Rightarrow$ $(a - b)$ is divisible by 4 and $(b - c)$ is divisible by 4

$\Rightarrow$ $|a - c| = |(a - b) + (b - c)|$ is divisible by 4

$\therefore \quad (a, c) \in S$

$\therefore$ It is transitive.

From above we can say that the relation S is reflexive, symmetric and transitive.

$\therefore$ Relation S is an equivalence relation.

The set of elements related to 1 are {9, 5, 1}.

13. We have to prove: $\tan^{-1} x + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = \tan^{-1}\left(\dfrac{3x - x^3}{1-3x^2}\right)$

LHS $= \tan^{-1} x + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$

$$= \tan^{-1}\left[\frac{x + \dfrac{2x}{1-x^2}}{1 - x\left(\dfrac{2x}{1-x^2}\right)}\right] \qquad \left[\text{As we know } \tan^{-1} a + \tan^{-1} b = \tan^{-1}\left[\frac{a+b}{1-ab}\right]\right]$$

$$= \tan^{-1}\left[\frac{x - x^3 + 2x}{1 - x^2 - 2x^2}\right] = \tan^{-1}\left[\frac{3x - x^3}{1-3x^2}\right] = \text{RHS}$$

OR

$\cos[\tan^{-1}\{\sin(\cot^{-1} x)\}] = \sqrt{\dfrac{1+x^2}{2+x^2}}$

LHS $= \cos[\tan^{-1}\{\sin(\cot^{-1} x)\}]$

Let $x = \cot\theta$

LHS $= \cos[\tan^{-1}(\sin\theta)]$

$$= \cos\left[\tan^{-1}\left(\sqrt{\frac{1}{1+\cot^2\theta}}\right)\right] = \cos\left[\tan^{-1}\left(\sqrt{\frac{1}{1+x^2}}\right)\right] \qquad \ldots(i)$$

Let $\theta_1 = \tan^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right) \Rightarrow \tan\theta_1 = \dfrac{\sqrt{1}}{\sqrt{1+x^2}}$

$\cos\theta_1 = \dfrac{1+x^2}{\sqrt{2+x^2}} \Rightarrow \theta_1 = \cos^{-1}\dfrac{1+x^2}{\sqrt{2+x^2}}$

Now, put θ_1 in equation (*i*), we get

$$\cos\left[\cos^{-1}\sqrt{\frac{1+x^2}{2+x^2}}\right] = \sqrt{\frac{1+x^2}{2+x^2}}$$

14. Consider

$$A = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

We can write $A = \dfrac{1}{2}(A + A') + \dfrac{1}{2}(A - A')$ …(*i*)

where, $\dfrac{1}{2}(A + A')$ is a symmetric matrix

and $\dfrac{1}{2}(A - A')$ is a skew symmetric matrix.

Now, $A' = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$

$$\frac{1}{2}(A + A') = \frac{1}{2}\left[\begin{pmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 1/2 & -5/2 \\ 1/2 & -2 & -2 \\ -5/2 & -2 & 2 \end{bmatrix} \quad \ldots(ii)$$

$$\frac{1}{2}(A - A') = \frac{1}{2}\left[\begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}\right]$$

$$= \frac{1}{2}\begin{bmatrix} 0 & -5 & -3 \\ 5 & 0 & -6 \\ 3 & 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -5/2 & -3/2 \\ 5/2 & 0 & -3 \\ 3/2 & 3 & 0 \end{bmatrix} \quad \ldots(iii)$$

Putting value of equations (*ii*) and (*iii*) in equation (*i*),

$$A = \begin{bmatrix} 3 & 1/2 & -5/2 \\ 1/2 & -2 & -2 \\ -5/2 & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -5/2 & -3/2 \\ 5/2 & 0 & -3 \\ 3/2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

Hence Proved.

15. Given, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 4\hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$

Consider, $\vec{r} = 2\vec{a} - \vec{b} + 3\vec{c}$

$-3 \;/\; 2 \;-3 \Big|$ $\quad 0 \Big|$

$$= 2\hat{i} + 2\hat{j} + 2\hat{k} - 4\hat{i} + 2\hat{j} - 3\hat{k} + 3\hat{i} - 6\hat{j} + 3\hat{k} = \hat{i} - 2\hat{j} + 2\hat{k}$$

Since the required vector has magnitude 6 units and parallel to $\vec{r}$.

$\therefore$ Required vector $= 6\hat{r}$

$$= 6\left[\frac{\hat{i}-2\hat{j}+2\hat{k}}{\sqrt{(1)^2+(-2)^2+(2)^2}}\right] = 6\left[\frac{\hat{i}-2\hat{j}+2\hat{k}}{\sqrt{1+4+4}}\right] = 2\hat{i}-4\hat{j}+4\hat{k}$$

OR

Given,

$$\vec{a} = \hat{i}+4\hat{j}+2\hat{k},\quad \vec{b} = 3\hat{i}-2\hat{j}+7\hat{k},\quad \vec{c} = 2\hat{i}-\hat{j}+4\hat{k}$$

Vector $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$ *i.e.*, $\vec{d}$ is parallel to vector $\vec{a}\times\vec{b}$.

$$\therefore \qquad \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 2 \\ 3 & -2 & 7 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} 4 & 2 \\ -2 & 7 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 4 \\ 3 & -2 \end{vmatrix} = 32\hat{i}-\hat{j}-14\hat{k}$$

Now let $\vec{d} = \mu(32\hat{i}-\hat{j}-14\hat{k})$

Also, $\qquad \vec{c}\cdot\vec{d} = 18$

$\Rightarrow \qquad (2\hat{i}-\hat{j}+4\hat{k})\cdot\mu(32\hat{i}-\hat{j}-14\hat{k}) = 18$

$\Rightarrow \qquad \mu(64+1-56) = 18 \;\Rightarrow\; 9\mu = 18$ or $\mu = 2$

$\therefore \qquad \vec{d} = 2(32\hat{i}-\hat{j}-14\hat{k}) = 64\hat{i}-2\hat{j}-28\hat{k}$

16. Given cartesian form of line as:

$$\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2} = \mu$$

$\therefore$ General point on line is $(3\mu-2,\ 2\mu-1,\ 2\mu+3)$

Since distance of points on line from $P(1, 3, 3)$ is 5 units.

$\therefore \qquad \sqrt{(3\mu-2-1)^2+(2\mu-1-3)^2+(2\mu+3-3)^2} = 5$

$\Rightarrow \qquad (3\mu-3)^2+(2\mu-4)^2+(2\mu)^2 = 25$

$\Rightarrow \qquad 17\mu^2-34\mu = 0 \;\Rightarrow\; 17\mu(\mu-2) = 0 \;\Rightarrow\; \mu = 0, 2$

$\therefore$ Required point on line is $(-2, -1, 3)$ for $\mu = 0$, or $(4, 3, 7)$ for $\mu = 2$.

OR

Plane determined by the points $A(3, -1, 2)$, $B(5, 2, 4)$ and $C(-1, -1, 6)$ is

$$\begin{vmatrix} x-3 & y+1 & z-2 \\ 5-3 & 2+1 & 4-2 \\ -1-3 & -1+1 & 6-2 \end{vmatrix} = 0 \quad\Rightarrow\quad \begin{vmatrix} x-3 & y+1 & z-2 \\ 2 & 3 & 2 \\ -4 & 0 & 4 \end{vmatrix} = 0$$

$$\Rightarrow \quad (x-3)\begin{vmatrix} 3 & 2 \\ 0 & 4 \end{vmatrix} - (y+1)\begin{vmatrix} 2 & 2 \\ -4 & 4 \end{vmatrix} + (z-2)\begin{vmatrix} 2 & 3 \\ -4 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \quad 12x - 36 - 16y - 16 + 12z - 24 = 0$$

$$\Rightarrow \quad 3x - 4y + 3z - 19 = 0$$

Distance of this plane from point P (6, 5, 9) is

$$\left|\frac{(3\times 6)-(4\times 5)+(3\times 9)-19}{\sqrt{(3)^2+(4)^2+(3)^2}}\right| = \left|\frac{18-20+27-19}{\sqrt{9+16+9}}\right| = \frac{6}{\sqrt{34}} \text{ units}$$

17. Given, $(x^2-1)\dfrac{dy}{dx} + 2xy = \dfrac{1}{x^2-1}$; $|x| \neq 1$

By simplifying the equation, we get

$$\frac{dy}{dx} + \frac{2x}{x^2-1}\,y = \frac{1}{(x^2-1)^2}$$

This is a linear differential equation of the form $\dfrac{dy}{dx} + Py = Q$

Here $P = \dfrac{2x}{x^2-1}$, $Q = \dfrac{1}{(x^2-1)^2}$

$$\text{I.F.} = e^{\int \frac{2x}{x^2-1}dx} = e^{\log|x^2-1|} = x^2-1$$

$\therefore$ Solution is $(x^2-1)\,y = \displaystyle\int (x^2-1)\,;\frac{1}{(x^2-1)^2}\,dx = \int \frac{1}{x^2-1}\,dx$

$$\Rightarrow \quad (x^2-1)\,y = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C$$

OR

Given, $\quad \sqrt{1+x^2+y^2+x^2y^2} + xy\dfrac{dy}{dx} = 0$

By simplifying the equation, we get

$$xy\frac{dy}{dx} = -\sqrt{1+x^2+y^2+x^2y^2}$$

$$\Rightarrow \quad xy\frac{dy}{dx} = -\sqrt{(1+x^2)(1+y^2)} = -\sqrt{(1+x^2)}\,\sqrt{(1+y^2)}$$

$$\Rightarrow \quad \frac{y}{\sqrt{1+y^2}}\,dy = -\frac{\sqrt{1+x^2}}{x}\,dx$$

Integrating both sides, we get

$$\int \frac{y}{\sqrt{1+y^2}}\,dy = -\int \frac{\sqrt{1+x^2}}{x}\,dx \qquad \text{...}(i)$$

Let $1+y^2=t \quad \Rightarrow \quad 2y\,dy=dt$ (For LHS)

Let $1+x^2=m^2 \quad \Rightarrow \quad 2x\,dx=2m\,dm \quad \Rightarrow \quad x\,dx=m\,dm$ (For RHS)

$\therefore$ (i) $\Rightarrow \quad \frac{1}{2}\int \frac{1}{\sqrt{t}}\,dt=-\int \frac{m}{m^2-1}\,.\,m\,dm$

$\Rightarrow \quad \frac{1}{2}\frac{t^{1/2}}{1/2}+\int \frac{m^2}{m^2-1}\,dm=0 \quad \Rightarrow \quad \sqrt{t}+\int \frac{m^2+1-1}{m^2-1}\,dm=0$

$\Rightarrow \quad \sqrt{t}+\int\left(1+\frac{1}{m^2-1}\right)dm=0 \quad \Rightarrow \quad \sqrt{t}+m+\frac{1}{2}\log\left|\frac{m-1}{m+1}\right|=0$

Now substituting these value of t and m, we get

$$\sqrt{1+y^2}+\sqrt{1+x^2}+\frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right|+C=0$$

18. Given, $\quad (x-y)\frac{dy}{dx}=x+2y$

By simplifying the above equation we get

$$\frac{dy}{dx}=\frac{x+2y}{x-y} \qquad \ldots(i)$$

Let $\quad F(x,y)=\frac{x+2y}{x-y}$

then $\quad F(Ax,Ay)=\frac{Ax+2Ay}{Ax-Ay}=\frac{A(x+2y)}{A(x-y)}=F(x,y)$

$\therefore$ $F(x,y)$ and hence the equation is homogeneous

Now let $y=vx$

$\Rightarrow \quad \frac{dy}{dx}=v+x\frac{dv}{dx}$

Substituting these values in equation (i), we get

$$v+x\frac{dv}{dx}=\frac{x+2vx}{x-vx}$$

$\Rightarrow \quad x\frac{dv}{dx}=\frac{1+2v}{1-v}-v=\frac{1+2v-v+v^2}{1-v}=\frac{1+v+v^2}{1-v}$

$\Rightarrow \quad \frac{1-v}{1+v+v^2}\,dv=\frac{dx}{x}$

By integrating both sides, we get

$$\int \frac{1-v}{v^2+v+1}\,dv=\int \frac{dx}{x} \qquad \ldots(ii)$$

LHS $\int \frac{1-v}{v^2+v+1}\,dv$

Let $1-v = A(2v+1)+B$

$= 2Av + (A+B)$

Comparing both sides, we get

$2A = -1, \quad A+B = 1$

or $A = -\frac{1}{2}, \quad B = \frac{3}{2}$

$$\therefore \int \frac{1-v}{v^2+v+1}\,dv = \int \frac{-\frac{1}{2}(2v+1)+\frac{3}{2}}{v^2+v+1}\,dv$$

$$= -\frac{1}{2}\int \frac{2v+1}{v^2+v+1}\,dv + \frac{3}{2}\int \frac{dv}{v^2+v+1}$$

$$= -\frac{1}{2}\int \frac{(2v+1)}{v^2+v+1}\,dv + \frac{3}{2}\int \frac{dv}{\left(v+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= -\frac{1}{2}\log|v^2+v+1| + \frac{3}{2}\times\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{v+\frac{1}{2}}{\sqrt{3}/2}\right)$$

Now substituting it in equation (*ii*), we get

$$-\frac{1}{2}\log|v^2+v+1| + \sqrt{3}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) = \log x + C$$

$$\Rightarrow \quad -\frac{1}{2}\log\left|\frac{y^2}{x^2}+\frac{y}{x}+1\right| + \sqrt{3}\tan^{-1}\left(\frac{\frac{2y}{x}+1}{\sqrt{3}}\right) = \log x + C$$

$$\Rightarrow \quad -\frac{1}{2}\log|x^2+xy+y^2| + \frac{1}{2}\log x^2 + \sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) = \log x + C$$

$$\Rightarrow \quad -\frac{1}{2}\log|x^2+xy+y^2| + \sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) = C$$

19. Given, $\int \frac{(x+2)\,dx}{\sqrt{(x-2)(x-3)}}\,dx$

$$= \int \frac{(x+2)\,dx}{\sqrt{x^2-5x+6}}\,dx$$

$$= \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2-5x+6}}\,dx$$

$$= \frac{1}{2}\int \frac{(2x-5)+9}{\sqrt{x^2-5x+6}}\,dx$$

$$= \underbrace{\frac{1}{2}\int \frac{2x-5}{\sqrt{x^2-5x+6}}\,dx}_{\mathrm{I}_1} + \underbrace{\frac{9}{2}\int \frac{dx}{\sqrt{x^2-5x+6}}}_{\mathrm{I}_2}$$

For I_1

Let $x^2 - 5x + 6 = m$

$\Rightarrow \quad (2x-5)\,dx = dm = \frac{1}{2}\int \frac{1}{\sqrt{m}}\,dm$

$\therefore \quad \mathrm{I}_1 = \frac{1}{2} \times 2\sqrt{m} = \sqrt{m} = \sqrt{x^2-5x+6}$ …(*i*)

$$\mathrm{I}_2 = \frac{9}{2}\int \frac{1}{\sqrt{x^2-5x+6}}\,dx = \frac{9}{2}\int \frac{dx}{\sqrt{\left(x-\frac{5}{2}\right)^2 - \frac{25}{4} + 6}}$$

$$= \frac{9}{2}\int \frac{dx}{\sqrt{\left(x-\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$$

$$= \frac{9}{2}\log\left|\left(x-\frac{5}{2}\right) + \sqrt{x^2-5x+6}\right| \qquad \ldots(ii)$$

Thus, $\displaystyle\int \frac{(x+2)}{\sqrt{(x-2)\,(x-3)}}\,dx = I_1 + I_2 = \sqrt{x^2-5x+6} + \frac{9}{2}\log\left|\left(x-\frac{5}{2}\right) + \sqrt{x^2-5x+6}\right| + C$

20. Given, $\displaystyle\int_1^2 \frac{5x^2}{x^2+4x+3}\,dx$

$$= 5\int_1^2 \frac{(x^2+4x+3)-(4x+3)}{x^2+4x+3}\,dx = 5\int_1^2 dx - 5\int_1^2 \frac{4x+3}{x^2+4x+3}\,dx$$

$$= 5[x]_1^2 - 5\int_1^2 \frac{4x+8-5}{x^2+4x+3}\,dx = 5 - 5\left[\int_1^2 \frac{2\,(2x+4)}{x^2+4x+3}\,dx - 5\int_1^2 \frac{dx}{x^2+4x+3}\right]$$

$$= 5 - 10\int_1^2 \frac{2x+4}{x^2+4x+3}\,dx + 25\int_1^2 \frac{dx}{(x+2)^2-1}$$

$$= 5 - \left[10\log|x^2+4x+3| - \frac{25}{2}\log\left|\frac{x+1}{x+3}\right|\right]_1^2$$

$$= 5 - \left[10\log 15 - \frac{25}{2}\log\frac{3}{5} - 10\log 8 + \frac{25}{2}\log\frac{1}{2}\right] = 5 + 10\log\frac{8}{15} + \frac{25}{2}\log\frac{6}{5}$$

21. We have given,

$$y = e^{a \sin^{-1} x}, \quad -1 \le x \le 1 \qquad \ldots(i)$$

and we have to prove

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0 \qquad \ldots(ii)$$

Now differentiating equation (i) w.r.t. x, we get

$$\frac{dy}{dx} = e^{a \sin^{-1} x} \cdot \frac{a}{\sqrt{1 - x^2}}$$

$$\Rightarrow \quad \sqrt{1 - x^2}\,\frac{dy}{dx} = ay \quad \Rightarrow (1 - x^2)\left(\frac{dy}{dx}\right)^2 = a^2y^2 \quad \text{(Squaring both sides)}$$

Now again differentiating w.r.t. x, we get

$$2(1 - x^2)\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = a^2\left(2y\frac{dy}{dx}\right)$$

Dividing both sides by $2\dfrac{dy}{dx}$, we get

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = a^2y$$

$$\Rightarrow \quad (1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0 \qquad \textbf{Hence Proved.}$$

22. Given, $y = \cos^{-1}\left(\dfrac{3x + 4\sqrt{1 - x^2}}{5}\right)$

Let $x = \cos\alpha$ so that $\alpha = \cos^{-1} x$

$$\Rightarrow \quad y = \cos^{-1}\left[\frac{3\cos\alpha + 4\sqrt{1 - \cos^2\alpha}}{5}\right] = \cos^{-1}\left[\frac{3}{5}\cos\alpha + \frac{4}{5}\sin\alpha\right] \qquad \ldots(i)$$

Let $\dfrac{3}{5} = \cos\theta$, then $\dfrac{4}{5} = \sin\theta$

$$\therefore \quad y = \cos^{-1}[\cos\alpha\cos\theta + \sin\alpha\sin\theta] = \cos^{-1}[\cos(\alpha - \theta)] = \alpha - \theta$$

$$\Rightarrow \quad y = \cos^{-1} x - \cos^{-1}\frac{3}{5}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1 - x^2}} - 0 = \frac{-1}{\sqrt{1 - x^2}}$$

SECTION–C

23. $$\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x)$$

$$\text{LHS} = \begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$$

By splitting into two parts, we get

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + pxyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + (-1)^2 \ pxyz \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$$

[In second determinant, replacing c_1 and c_3 and then c_1 with c_2]

$$= (1+pxyz) \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$$

By applying $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$, we get

$$= (1+pxyz) \begin{vmatrix} x-y & (x-y)(x+y) & 0 \\ y-z & (y-z)(y+z) & 0 \\ z & z^2 & 1 \end{vmatrix}$$

$$= (1+pxyz)(x-y)(y-z) \begin{vmatrix} 1 & x+y & 0 \\ 1 & y+z & 0 \\ z & z^2 & 1 \end{vmatrix}$$

By expanding the determinant, we get

$\Rightarrow \quad (1+pxyz)(x-y)(y-z)[y+z-x-y]$

$\Rightarrow \quad (1+pxyz)(x-y)(y-z)(z-x)$

OR

$$A=\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}$$

Let $A = IA$

$$\Rightarrow \quad \begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A$$

Applying $R_2 \to R_2 + R_1$

$$\begin{bmatrix}1 & 2 & -2\\ 0 & 5 & -2\\ 0 & -2 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A$$

Applying $R_1 \to R_1 + R_3, R_2 \to R_2 + 2R_3$

$$\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 0\\ 0 & -2 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 1\\ 1 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}A$$

Applying $R_3 \to R_3 + 2R_2$

$$\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 1\\ 1 & 1 & 2\\ 2 & 2 & 5\end{bmatrix}A$$

Applying $R_1 \to R_1 + R_3$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}3 & 2 & 6\\ 1 & 1 & 2\\ 2 & 2 & 5\end{bmatrix}A$$

$$\therefore \qquad A^{-1}=\begin{bmatrix}3 & 2 & 6\\ 1 & 1 & 2\\ 2 & 2 & 5\end{bmatrix}$$

24. Let us define the following events.

E: drawn balls are white

A : 2 white balls in bag.

B: 3 white balls in bag.

C: 4 white balls in bag.

Then $P(A) = P(B) = P(C) = \frac{1}{3}$

$$P\left(\frac{E}{A}\right) = \frac{{}^2C_2}{{}^4C_2} = \frac{1}{6},$$

$$P\left(\frac{E}{B}\right) = \frac{{}^3C_2}{{}^4C_2} = \frac{3}{6}, \quad P\left(\frac{E}{C}\right) = \frac{{}^4C_2}{{}^4C_2} = 1$$

By applying Baye's Theorem

$$P\left(\frac{C}{E}\right) = \frac{P(C) \cdot P\left(\frac{E}{C}\right)}{P(A) \cdot P\left(\frac{E}{A}\right) + P(B) \cdot P\left(\frac{E}{B}\right) + P(C)\, P\left(\frac{E}{C}\right)}$$

$$= \frac{\frac{1}{3} \times 1}{\left(\frac{1}{3} \times \frac{1}{6}\right) + \left(\frac{1}{3} \times \frac{3}{6}\right) + \left(\frac{1}{3} \times 1\right)} = \frac{1}{\frac{1}{6} + \frac{3}{6} + 1} = \frac{3}{5}$$

25. Let number of first kind and second kind of cakes that can be made be x and y respectively

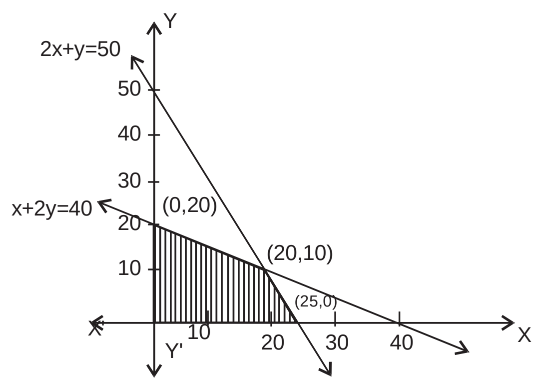
Then, the given problem is

Maximize, $z = x + y$

Subjected to $x \geq 0,\ y \geq 0$

$$300x + 150y \leq 7500 \Rightarrow 2x + y \leq 50$$

$$15x + 30y \leq 600 \Rightarrow x + 2y \leq 40$$



From graph, three possible points are

(25, 0), (20, 10), (0, 20)

At (25, 0), $z = x + y = 25 + 0 = 25$

At (20, 10), $z = x + y = 20 + 10$
$= 30 \leftarrow$ Maximum

At (0, 20), $z = 0 + 20 = 20$

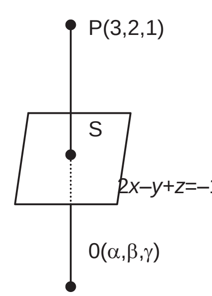
As Z is maximum at (20, 10), *i.e.*, $x = 20$, $y = 10$.

$\therefore$ 20 cakes of type I and 10 cakes of type II can be made.

26. Let $O(\alpha, \beta, \gamma)$ be the image of the point $P(3, 2, 1)$ in the plane

$$2x - y + z + 1 = 0$$

$\therefore$ PO is perpendicular to the plane and S is the mid point of PO and the foot of the perpendicular.



DR's of PS are 2, –1, 1

$\therefore$ Equation of PS are $\dfrac{x-3}{2} = \dfrac{y-2}{-1} = \dfrac{z-1}{1} = \mu$

$\therefore$ General point on line is $S(2\mu + 3, -\mu + 2, \mu + 1)$

If this point lies on plane, then

$$2(2\mu + 3) - (-\mu + 2) + 1(\mu + 1) + 1 = 0$$

$$\Rightarrow \quad 6\mu + 6 = 0 \Rightarrow \mu = -1$$

$\therefore$ Coordinates of S are (1, 3, 0).

As S is the mid point of PO,

$$\therefore \qquad \left(\frac{3+\alpha}{2}, \frac{2+\beta}{2}, \frac{1+\gamma}{2}\right) = (1, 3, 0)$$

By comparing both sides, we get

$$\frac{3+\alpha}{2} = 1 \quad \Rightarrow \quad \alpha = -1$$

$$\frac{2+\beta}{2} = 3 \quad \Rightarrow \quad \beta = 4$$

$$\frac{1+\gamma}{2} = 0 \quad \Rightarrow \quad \gamma = -1$$

$\therefore$ Image of point P is (–1, 4, –1).

27. Equation of circle is

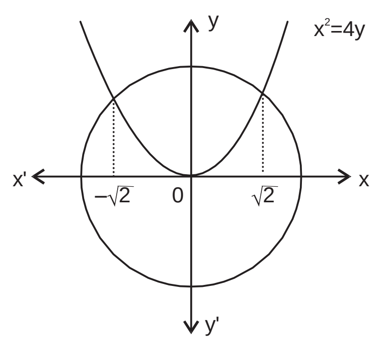
$$4x^2 + 4y^2 = 9 \qquad \ldots(i)$$

and equation of parabola is

$$x^2 = 4y \qquad \ldots(ii)$$

$$y = x^2 / 4 \qquad \ldots(iii)$$

By putting value of equation (iii) in equation (i), we get



$$4x^2 + 4\left(\frac{x^2}{4}\right)^2 = 9$$

$$\Rightarrow \quad x^4 + 16x^2 - 36 = 0$$

$$\Rightarrow \quad (x^2 + 18)(x^2 - 2) = 0$$

$$\Rightarrow \quad x^2 + 18 = 0, \; x^2 - 2 = 0$$

$$\Rightarrow \quad x = -\sqrt{18}, \; x = \pm\sqrt{2}$$

$$\Rightarrow \quad x = \pm\sqrt{2} \qquad (\because \; x = -\sqrt{18} \text{ is not possible})$$

$\therefore$ Required area $= 2\int_0^{\sqrt{2}} (y_1 - y_2)\, dx$

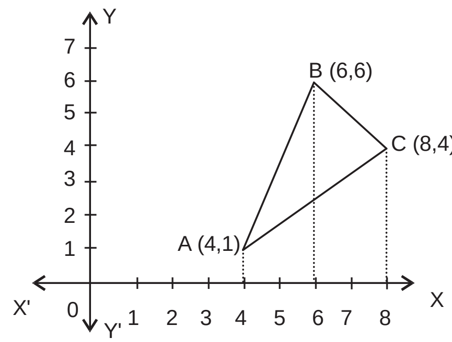
$$= 2\int_0^{\sqrt{2}} \left[\left\{\sqrt{\frac{9}{4} - x^2} - \frac{x^2}{4}\right\}\right] dx \qquad \left[\text{As } y_1 : x^2 + y^2 = \frac{9}{4}, \; y_2 : x^2 = 4y\right]$$

$$= 2\left[\frac{x}{2}\sqrt{\frac{9}{4} - x^2} + \frac{9}{8}\sin^{-1}\frac{x}{3/2} - \frac{x^3}{12}\right]_0^{\sqrt{2}}$$

$$= 2\left[\frac{\sqrt{2}}{4} + \frac{9}{8}\sin^{-1}\frac{2\sqrt{2}}{3} - \frac{\sqrt{2}}{6}\right] = \left(\frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}\right) \text{ sq. units.}$$

OR

Given triangle ABC, coordinates of whose vertices are $A\,(4, 1)$, $B(6, 6)$ and $C\,(8, 4)$.



Equation of AB is given by

$$y-6=\frac{6-1}{6-4}(x-6) \text{ or } y=\frac{5}{2}x-9$$

Equation of BC is given by

$$y-4=\frac{4-6}{8-6}(x-8) \quad \text{or} \quad y=-x+12$$

Equation of AC is given by

$$y-4=\frac{4-1}{8-4}(x-8) \quad \text{or} \quad y=\frac{3}{4}x-2$$

$\therefore$ Area of ΔABC

$$=\int_4^6 (y_{AB}-y_{AC})\,dx+\int_6^8 (y_{BC}-y_{AC})\,dx$$

$$=\int_4^6 \left(\frac{5}{2}x-9-\frac{3}{4}x+2\right)dx+\int_6^8 \left(-x+12-\frac{3}{4}x+2\right)dx$$

$$=\int_4^6 \left(\frac{7}{4}x-7\right)dx+\int_6^8 \left(-\frac{7x}{4}+14\right)dx$$

$$=\left[\frac{7x^2}{8}-7x\right]_4^6+\left[-\frac{7x^2}{8}+14x\right]_6^8=\left[\left(\frac{63}{2}-42\right)-(14-28)\right]+\left[(-56+112)-\left(\frac{-63}{2}+84\right)\right]$$

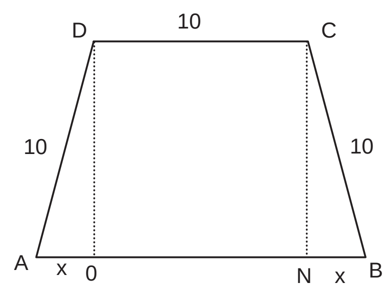
$$=\left[\frac{63}{2}-42-14+28-56+112+\frac{63}{2}-84\right]=63-56=7 \text{ sq. units.}$$

28. Given, the length of three sides of a trapezium other than the base is 10 cm, each

i.e., $AD=DC=BC=10$ cm.

Let $AO=NB=x$ cm.

$$DO=\sqrt{100-x^2} \text{ cm}$$



$$\text{Area }(A)=\frac{1}{2}(AB+DC)\,.\,DO$$

$$=\frac{1}{2}(10+2x+10)\sqrt{100-x^2}$$

$$\therefore \quad A=(x+10)\sqrt{100-x^2} \quad \ldots(i)$$

Differentiating w.r.t. x, we get

$$\frac{dA}{dx}=(x+10)\,.\,\frac{1}{2\sqrt{100-x^2}}(-2x)+\sqrt{100-x^2}\,.1$$

$$=\frac{-x(x+10)+(100-x^2)}{\sqrt{100-x^2}}=\frac{-2x^2-10x+100}{\sqrt{100-x^2}}$$

For maximum area, $\frac{dA}{dx} = 0$

$\Rightarrow \quad 2x^2 + 10x - 100 = 0 \quad \text{or} \quad x^2 + 5x - 50 = 0$

$\Rightarrow \quad (x+10)(x-5) = 0 \quad \Rightarrow \quad x = 5, -10$

$\Rightarrow \quad x = 5$

Now again differentiating w.r.t. x, we get

$$\frac{d^2A}{dx^2} = \frac{\sqrt{100-x^2}\,(-4x-10) - (-2x^2 - 10x + 100)\cdot\frac{(-2x)}{2\sqrt{100-x^2}}}{(100-x^2)}$$

For $x = 5$

$$\frac{d^2A}{dx^2} = \frac{\sqrt{100-25}\,(-20-10) - 0}{(100-25)} = \frac{\sqrt{75}\,(-30)}{75} < 0$$

$\therefore$ For $x = 5$, area is maximum

$$A_{max} = (5+10)\sqrt{100-25}\ \text{cm}^2 \qquad \text{[Using equation } (i)]$$

$$= 15\sqrt{75}\ \text{cm}^2 = 75\sqrt{3}\ \text{cm}^2$$

29. Question is incomplete.

Set–II

6. Let $\quad x = \cot^{-1}(-\sqrt{3})$

$\Rightarrow \quad \cot x = -\sqrt{3} = -\cot\frac{\pi}{6} = \cot\left(\pi - \frac{\pi}{6}\right) \Rightarrow \cot x = \cot\frac{5\pi}{6} \quad \Rightarrow \quad x = \frac{5\pi}{6}$

10. Given, $\vec{a}$ and $\vec{b}$ are two vectors such that

$$|\vec{a}\cdot\vec{b}| = |\vec{a}\times\vec{b}|$$

$\Rightarrow \quad |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|\sin\theta$

$\Rightarrow \quad \cos\theta = \sin\theta \quad \Rightarrow \quad \frac{\sin\theta}{\cos\theta} = 1$

$\Rightarrow \quad \tan\theta = 1 \quad \Rightarrow \quad \theta = \frac{\pi}{4}$

11. We have to prove

$$\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$$

LHS $\quad = \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{8}\right)$

$$= \tan^{-1}\left[\frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{3}\times\frac{1}{5}}\right] + \tan^{-1}\left[\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\times\frac{1}{8}}\right] \quad \left[\because\ \tan^{-1}a + \tan^{-1}b = \tan^{-1}\left(\frac{a+b}{1-ab}\right)\right]$$

$$= \tan^{-1}\left(\frac{4}{7}\right) + \tan^{-1}\left(\frac{3}{11}\right)$$

$$= \tan^{-1}\left(\frac{\frac{4}{7}+\frac{3}{11}}{1-\frac{4}{7}\times\frac{3}{11}}\right) = \tan^{-1}\left(\frac{65}{65}\right) = \tan^{-1}(1) = \frac{\pi}{4} = \text{RHS}$$

OR

Given, $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right] = \frac{\pi}{4}$$

$$\Rightarrow \quad \frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1} = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{2x^2-4}{-3} = 1$$

$$\Rightarrow \quad 2x^2-4 = -3 \quad \Rightarrow \quad 2x^2 = 1$$

$$\Rightarrow \quad x^2 = \frac{1}{2} \quad \Rightarrow \quad x = \pm\frac{1}{\sqrt{2}}$$

14. We have given

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

For $A^2 - 3A + 2I$

$$A^2 = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

$$3A = 3\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 3 \\ 6 & 3 & 9 \\ 3 & -3 & 0 \end{bmatrix}$$

$$2I = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^2 - 3A + 2I = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 3 \\ 6 & 3 & 9 \\ 3 & -3 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 3 & -3 & -4 \\ -3 & 2 & 0 \end{bmatrix}$$

18. $\displaystyle\int \frac{5x+3}{\sqrt{x^2+4x+10}}\,dx$

Let $5x + 3 = A(2x+4) + B = 2Ax + (4A + B)$

Comparing both sides, we get

$$2A = 5 \Rightarrow A = \frac{5}{2}$$

$$4A + B = 3 \Rightarrow B = -7$$

$$\therefore \quad \int \frac{5x+3}{\sqrt{x^2+4x+10}}\,dx = \int \frac{\frac{5}{2}(2x+4) - 7}{\sqrt{x^2+4x+10}}\,dx$$

$$= \frac{5}{2}\underbrace{\int \frac{2x+4}{\sqrt{x^2+4x+10}}\,dx}_{I_1} - 7\underbrace{\int \frac{dx}{\sqrt{x^2+4x+10}}}_{I_2}$$

For I_1

Let $x^2 + 4x + 10 = m \Rightarrow (2x+4)\,dx = dm$

$$\Rightarrow \quad I_1 = \frac{5}{2}\int \frac{1}{\sqrt{m}}\,dm = \frac{5}{2} \times 2\sqrt{m} = 5\sqrt{m} = 5\sqrt{x^2+4x+10} + C_1$$

$$I_2 = 7\int \frac{1}{\sqrt{x^2+4x+10}}\,dx = 7\int \frac{dx}{\sqrt{(x+2)^2 - 4 + 10}} = 7\int \frac{dx}{\sqrt{(x+2)^2 + 6}}$$

$$= 7 \log\left|(x+2) + \sqrt{x^2+4x+10}\right| + C_2$$

Thus, $\displaystyle\int \frac{5x+3}{\sqrt{x^2+4x+10}}\,dx = I_1 + I_2$

$$= 5\sqrt{x^2+4x+10} - 7\log\left|(x+2) + \sqrt{x^2+4x+10}\right| + C, \; C = C_1 + C_2$$

20. $y\,dx + x\log\left(\frac{y}{x}\right)dy - 2x\,dy = 0$

Simplifying the above equation, we get

$$\left[x\log\left(\frac{y}{x}\right) - 2x\right]dy = -y\,dx$$

$$\frac{dy}{dx} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)} \qquad \ldots(i)$$

Let $\quad F(x, y) = \dfrac{y}{2x - x\log\left(\frac{y}{x}\right)}$

$$F(\mu x, \mu y) = \frac{\mu y}{2\mu x + \mu x\log\left(\frac{\mu y}{\mu x}\right)} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)} = F(x, y)$$

$\therefore$ Function and hence the equation is homogeneous,

Let $\quad y = vx$

$\Rightarrow \quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Substituting in equation (i), we get

$$v + x\frac{dv}{dx} = \frac{vx}{2x - x\log v}$$

$\Rightarrow \quad x\dfrac{dv}{dx} = \dfrac{v}{2 - \log v} - v \qquad x\dfrac{dv}{dx} = \dfrac{v\log v - v}{2 - \log v}$

$\Rightarrow \quad \dfrac{2 - \log v}{v\log v - v}\, dv = \dfrac{dx}{x}$

Integrating both sides, we get

$$\int \frac{2 - \log v}{v\log v - v}\, dv = \int \frac{dx}{x}$$

$\Rightarrow \quad \displaystyle\int \frac{1 + (1 - \log v)}{v(\log v - 1)}\, dv = \int \frac{dx}{x}$

$\Rightarrow \quad \displaystyle\int \frac{dv}{v(\log v - 1)} - \int \frac{1}{v}\, dv = \int \frac{dx}{x}$

Let $\quad \log v - 1 = m \quad \Rightarrow \quad \dfrac{1}{v}\, dv = dm$

$\Rightarrow \quad \displaystyle\int \frac{1}{m}\, dm - \int \frac{1}{v}\, dv = \int \frac{dx}{x}$

$\Rightarrow \quad \log|m| - \log|v| = \log|x| + \log|c|$

$\Rightarrow \quad \log\left|\dfrac{m}{v}\right| = \log|cx|$

$\Rightarrow \quad \dfrac{m}{v} = cx \quad \Rightarrow \quad (\log v - 1) = vcx$

$\Rightarrow \quad \left[\log\left(\dfrac{y}{x}\right) - 1\right] = cy$

which is the required solution.

23. We have given

$$\left.\begin{aligned} x &= 1 - \cos\theta \\ y &= \theta - \sin\theta \end{aligned}\right\} \quad \theta = \frac{\pi}{4} \qquad \ldots(i)$$

At $\theta = \frac{\pi}{4}$

$$x = 1 - \cos\frac{\pi}{4} = 1 - \frac{1}{\sqrt{2}}, \qquad y = \frac{\pi}{4} - \sin\frac{\pi}{4} = \frac{\pi}{4} - \frac{1}{\sqrt{2}}$$

$\therefore$ point is $\left(1 - \frac{1}{\sqrt{2}}, \frac{\pi}{4} - \frac{1}{\sqrt{2}}\right)$

Now differentiating equation (i). w.r.t. θ, we get

$$\frac{dx}{d\theta} = \sin\theta \quad \text{and} \quad \frac{dy}{d\theta} = 1 - \cos\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{1 - \cos\theta}{\sin\theta} = \operatorname{cosec}\theta - \cot\theta$$

At $\theta = \frac{\pi}{4}$ $\quad \frac{dy}{dx} = \operatorname{cosec}\frac{\pi}{4} - \cot\frac{\pi}{4} = \sqrt{2} - 1$

which is slope of the tangent.

$\therefore$ Equation of the tangent is

$$y - \left(\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right) = (\sqrt{2} - 1)\left\{x - \left(1 - \frac{1}{\sqrt{2}}\right)\right\}$$

$$= (\sqrt{2} - 1)\, x - (\sqrt{2} - 1)\, \frac{(\sqrt{2} - 1)}{\sqrt{2}}$$

$$\Rightarrow \quad y - \left(\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right) = (\sqrt{2} - 1)x - \frac{2 + 1 - 2\sqrt{2}}{\sqrt{2}}$$

$$\Rightarrow \quad (\sqrt{2} - 1)\, x - y - \frac{3 - 2\sqrt{2}}{\sqrt{2}} + \frac{\pi}{4} - \frac{1}{\sqrt{2}} = 0$$

$$\Rightarrow \quad (\sqrt{2} - 1)\, x - y + \frac{\pi}{4} - \frac{4 - 2\sqrt{2}}{\sqrt{2}} = 0$$

$$\Rightarrow \quad (\sqrt{2} - 1)\, x - y + \frac{\pi}{4} - 2\sqrt{2} + 2 = 0$$

which is the equation of the tangent.

$$\text{Slope of the normal} = \frac{-1}{dy/dx} = \frac{-1}{\sqrt{2} - 1} = \frac{-(\sqrt{2} + 1)}{(\sqrt{2} - 1)(\sqrt{2} + 1)} = -(\sqrt{2} + 1)$$

Equation of the normal is

$$y - \left(\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right) = -(\sqrt{2} + 1)\left\{x - \left(1 - \frac{1}{\sqrt{2}}\right)\right\}$$

$$\Rightarrow \quad y - \frac{\pi}{4} + \frac{1}{\sqrt{2}} = -(\sqrt{2}+1)\, x + (\sqrt{2}+1)\frac{(\sqrt{2}-1)}{\sqrt{2}}$$

$$\Rightarrow \quad y - \frac{\pi}{4} + \frac{1}{\sqrt{2}} = -(\sqrt{2}+1)x + \frac{2-1}{\sqrt{2}}$$

$$\Rightarrow \quad (\sqrt{2}+1)\, x + y - \frac{\pi}{4} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$$

$$\Rightarrow \quad (\sqrt{2}+1)\, x + y - \frac{\pi}{4} = 0$$

which is the equation of the normal.

24. Plane through the point $P\,(1, 1, 1)$ is

$$[\vec{r} - (\hat{i} + \hat{j} + \hat{k})] \,.\, \vec{n} = 0 \qquad ...(i)$$

As plane contains the line $\vec{r} = (-3\hat{i} + \hat{j} + 5\hat{k}) + \lambda\,(3\hat{i} - \hat{j} - 5\hat{k})$

$$\therefore \quad [-3\hat{i} + \hat{j} + 5\hat{k} - \hat{i} - \hat{j} - \hat{k}].\,\vec{n} = 0$$

$$\Rightarrow \quad (-4\hat{i} + 4\hat{k})\,.\,\vec{n} = 0 \qquad ...(ii)$$

Also, $\quad (3\hat{i} - \hat{j} - 5\hat{k})\,.\,\vec{n} = 0 \qquad ...(iii)$

From (*ii*) and (*iii*), we get

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & 0 & 4 \\ 3 & -1 & -5 \end{vmatrix} = 4\hat{i} - 8\hat{j} + 4\hat{k}$$

Substituting $\vec{n}$ in (*i*), we get

$$[\vec{r} - (\hat{i} + \hat{j} + \hat{k})]\,.\,(4\hat{i} - 8\hat{j} + 4\hat{k}) = 0$$

$$\Rightarrow \quad \vec{r}\,.\,(4\hat{i} - 8\hat{j} + 4\hat{k}) - (4 - 8 + 4) = 0$$

$$\Rightarrow \quad \vec{r}\,.\,(\hat{i} - 2\hat{j} + \hat{k}) = 0$$

Which is the required equation of plane.

$\vec{r}\,.\,(\hat{i} - 2\hat{j} + \hat{k}) = 0$ contain the line

$$\vec{r} = (\hat{i} + 2\hat{j} + 5\hat{k}) + \mu\,(\hat{i} - 2\hat{j} - 5\hat{k})$$

if $\quad (-\hat{i} + 2\hat{j} + 5\hat{k})\,.\,(\hat{i} - 2\hat{j} + \hat{k}) = 0$

i.e., $\quad -1 - 4 + 5 = 0$, which is correct

and $\quad (\hat{i} - 2\hat{j} + \hat{k})\,.\,(\hat{i} - 2\hat{j} - 5\hat{k}) = 0$

i.e., $\quad 1 + 4 - 5 = 0$, which is correct.

Set–III

6. We are given $\sin^{-1}\left(\sin\frac{4\pi}{5}\right) = \sin^{-1}\left(\sin\left(\pi - \frac{\pi}{5}\right)\right)$

$$= \sin^{-1}\left(\sin\frac{\pi}{5}\right) = \frac{\pi}{5}$$

7. Angle b/w $\vec{a}$ and $\vec{b} = \sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} = \dfrac{1\times 3}{\sqrt{3}\times 2} = \dfrac{\sqrt{3}}{2}$

$\Rightarrow \qquad \theta = \sin^{-1}\dfrac{\sqrt{3}}{2} = \dfrac{\pi}{3}$

11. $(a, b)\ S(c, d) \quad \Rightarrow \quad a + d = b + c$

(i) For $(a, b) \in N \times N$

$a + b = b + a \Rightarrow (a, b)\ S(a, b)$

$\therefore$ S is reflexive.

(ii) Let $(a, b)\ S(c, d) \quad \Rightarrow \quad a + d = b + c$

$\Rightarrow \quad d + a = c + b \quad \Rightarrow \quad c + b = d + a$

$\therefore \quad (a, b)\ S(c, d) \quad \Rightarrow \quad (c, d)\ S(a, b)$

i.e., S is symmetric.

(iii) For $(a, b), (c, d), (e, f) \in N \times N$

Let $(a, b)\ S(c, d)$ and $(c, d)\ S(e, f)$

$\Rightarrow \quad a + d = b + c$ and $c + f = d + e$

$\Rightarrow \quad a + d + c + f = b + c + d + e$

$\Rightarrow \quad a + f = b + e$

$\Rightarrow \quad (a, b)\ S(e, f)$

$\therefore \quad (a, b)\ S(c, d)$ and $(c, d)\ S(e, f) \quad \Rightarrow \quad (a, b)\ S(e, f)$

$\therefore$ S is transitive.

$\therefore$ Relation S is an equivalence relation.

15. Given, $\quad A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, \qquad B = (-1 \quad 2 \quad 1)$

$$AB = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} [-1 \quad 2 \quad 1] = \begin{bmatrix} -1 & 2 & 1 \\ 4 & -8 & -4 \\ -3 & 6 & 3 \end{bmatrix},$$

$$(AB)' = \begin{bmatrix} -1 & 2 & 1 \\ 4 & -8 & -4 \\ -3 & 6 & 3 \end{bmatrix}' = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix}$$

$$B'A' = (-1 \quad 2 \quad 1)' \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} [1 \quad -4 \quad 3] = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix}$$

$\therefore \quad (AB)' = B'A'.$

17. $(x^2+1)\dfrac{dy}{dx} + 2xy = \sqrt{x^2+4}$

Simplifying the above equation,

$$\frac{dy}{dx} + \frac{2x}{x^2+1}\, y = \frac{\sqrt{x^2+4}}{(x^2+1)}$$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $P = \dfrac{2x}{x^2+1}$, $Q = \dfrac{\sqrt{x^2+4}}{(x^2+1)}$

$$\text{I.F.} = e^{\int \frac{2x}{x^2+1}\,dx} = e^{\log(x^2+1)} = (x^2+1)$$

$$\therefore \quad (x^2+1)\, y = \int (x^2+1) \cdot \frac{\sqrt{x^2+4}}{(x^2+1)}\, dx \quad = \int \sqrt{x^2+4}\, dx$$

$$\Rightarrow \quad (x^2+1)\cdot y = \frac{x}{2}\sqrt{x^2+4} + \frac{4}{2}\log|x + \sqrt{x^2+4}| + C$$

OR

$$(x^3+x^2+x+1)\frac{dy}{dx} = 2x^2 + x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x^2+x}{x^3+x^2+x+1} \quad \Rightarrow \quad dy = \frac{2x^2+x}{(x^2+1)(x+1)}\, dx$$

Integrating both sides, we get

$$\int dy = \int \frac{2x^2+x}{(x^2+1)(x+1)}\, dx \qquad \ldots(i)$$

By partial fraction,

$$\frac{2x^2+x}{(x^2+1)(x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} = A(x^2+1) + (Bx+C)(x+1)$$

$$2x^2 + x = x^2(A+B) + x(B+C) + (A+C)$$

Comparing both the sides, we get

$$A + B = 2, \quad B + C = 1 \quad \text{and} \quad A + C = 0$$

$= \frac{3}{2}, A = \frac{1}{2}, C = \frac{-1}{2}$

$$\therefore\ (i) \Rightarrow \quad y = \int \left[\frac{1/2}{x+1} + \frac{\frac{3}{2}x - \frac{1}{2}}{x^2+1} \right] dx$$

$$= \frac{1}{2} \int \frac{1}{x+1}\, dx + \frac{3}{2} \int \frac{x}{x^2+1}\, dx - \frac{1}{2} \int \frac{1}{x^2+1}\, dx$$

$$y = \frac{1}{2} \log |x+1| + \frac{3}{4} \log |x^2+1| - \frac{1}{2} \tan^{-1} x + C$$

20. Consider,

$$y = \text{cosec}^{-1} x$$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}} \quad \Rightarrow \quad x\sqrt{x^2-1}\,\frac{dy}{dx} = -1$$

Again differentiating w.r.t. x, we get

$$x\sqrt{x^2-1} \cdot \frac{d^2y}{dx^2} + \sqrt{x^2-1}\,\frac{dy}{dx} + x\,\frac{2x}{2\sqrt{x^2-1}}\,\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x(x^2-1)\frac{d^2y}{dx^2} + (2x^2-1)\frac{dy}{dx} = 0$$

23. We are given

$$x + 2y - 3z = -4$$
$$2x + 3y + 2z = 2$$
$$3x - 3y - 4z = 11$$

The matrix equation form of equations is

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix} \quad i.e., \quad AX = B \Rightarrow X = A^{-1} B$$

where $A^{-1} = \frac{1}{|A|} Adj. A.$

$$|A| = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{vmatrix} = 1 \begin{vmatrix} 3 & 2 \\ -3 & -4 \end{vmatrix} - 2 \begin{vmatrix} 2 & 2 \\ 3 & -4 \end{vmatrix} - 3 \begin{vmatrix} 2 & 3 \\ 3 & -3 \end{vmatrix}$$

$$= (-12 + 6) - 2(-8 - 6) - 3(-6 - 9) = -6 + 28 + 45 = 67 \neq 0$$

$$Adj.\ A = \begin{bmatrix} -6 & 14 & -15 \\ 17 & 5 & 9 \\ 13 & -8 & -1 \end{bmatrix}' = \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$$

$$X = \frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}\begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{67}\begin{bmatrix} 201 \\ -134 \\ 67 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore \quad x = 3,\ y = -2,\ z = 1$

OR

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix} \qquad \text{[by applying } R_1 \to R_1 + R_2 + R_3]$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$

$$\Delta = (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ b-c & c-a & a \\ c-a & a-b & b \end{vmatrix} = (a+b+c)\begin{vmatrix} b-c & c-a \\ c-a & a-b \end{vmatrix}$$

$$= (a+b+c)\,[(b-c)(a-b) - (c-a)^2]$$

$$= -(a+b+c)\,[a^2 + b^2 + c^2 - ab - bc - ca]$$

$$= -\frac{1}{2}(a+b+c)\,[(a^2 + b^2 - 2ab) + (b^2 + c^2 - 2bc) + (c^2 + a^2 - 2ac)]$$

$$\Rightarrow \qquad \Delta = -\frac{1}{2}(a+b+c)\,[(a-b)^2 + (b-c)^2 + (c-a)^2]$$

As $a \neq b \neq c$ and all are positive.

$$a+b+c > 0,\ \ (a-b)^2 > 0,\ (b-c)^2 > 0 \quad \text{and} \quad (c-a)^2 > 0$$

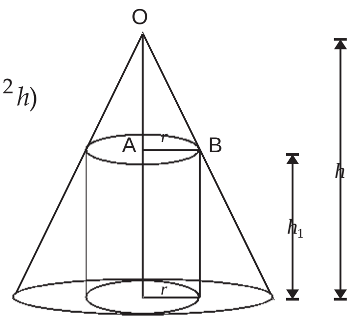
Hence, Δ is negative.

25. Let a cylinder of base radius r and height h_1 is included in a cone of height h and semi-vertical angle α.

Then $AB = r$, $OA = (h - h_1)$,

In right angled triangle OAB,

$$\frac{AB}{OA} = \tan\alpha \quad \Rightarrow \quad \frac{r}{h - h_1} = \tan\alpha$$



or $\quad r = (h - h_1)\tan\alpha$

$\therefore \quad V = \pi[(h - h_1)\tan\alpha]^2 \,.\, h_1 \quad$ (Q Volume of cylinder $= \pi r^2 h$)

$$V = \pi\tan^2\alpha \,.\, h_1 (h - h_1)^2 \qquad ...(i)$$

Differentiating w.r.t. h_1, we get

$$\frac{dV}{dh_1} = \pi\tan^2\alpha\,[h_1 \,.\, 2(h - h_1)(-1) + (h - h_1)^2 \times 1]$$

$$= \pi\tan^2\alpha\,(h - h_1)[-2h_1 + h - h_1]$$

$$= \pi\tan^2\alpha\,(h - h_1)(h - 3h_1)$$

For maximum volume V, $\dfrac{dV}{dh_1} = 0$

$\Rightarrow \quad h - h_1 = 0 \quad$ or $\quad h - 3h_1 = 0$

$\Rightarrow \quad h = h_1 \quad$ or $\quad h_1 = \dfrac{1}{3}h$

$\Rightarrow \quad h_1 = \dfrac{1}{3}h \qquad$ (Q $h = h_1$ is not possible)

Again differentiating w.r.t. h_1, we get

$$\frac{d^2V}{dh_1^2} = \pi\tan^2\alpha\,[(h - h_1)(-3) + (h - 3h_1)(-1)]$$

At $h_1 = \dfrac{1}{3}h$

$$\frac{d^2V}{dh_1^2} = \pi\tan^2\alpha\left[\left(h - \frac{1}{3}h\right)(-3) + 0\right] = -2\pi h\tan^2\alpha < 0$$

$\therefore \quad$ Volume is maximum for $h_1 = \dfrac{1}{3}h$

$$V_{max} = \pi\tan^2\alpha \,.\left(\frac{1}{3}h\right)\left(h - \frac{1}{3}h\right)^2 \qquad \text{[Using } (i)\text{]}$$

$$= \frac{4}{27}\pi h^3\tan^2\alpha$$

EXAMINATION PAPERS – 2010

MATHEMATICS CBSE (Foreign)

CLASS – XII

Time allowed: 3 hours **Maximum marks: 100**

General Instructions: ***As given in CBSE Examination paper (Delhi)*** **– 2010.**

Set–I

SECTION–A

Question number 1 to 10 carry 1 mark each.

1. Write a square matrix of order 2, which is both symmetric and skew symmetric.

2. If 'f' is an invertible function, defined as $f(x) = \dfrac{3x-4}{5}$, write $f^{-1}(x)$.

3. What is the domain of the function $\sin^{-1} x$?

4. What is the value of the following determinant?

$$\Delta = \begin{vmatrix} 4 & a & b+c \\ 4 & b & c+a \\ 4 & c & a+b \end{vmatrix}$$

5. Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 15$.

6. For what value of p, is $(\hat{i} + \hat{j} + \hat{k})\, p$ a unit vector?

7. If $\int (ax+b)^2\, dx = f(x) + c$, find $f(x)$.

8. Evaluate: $\int_0^1 \dfrac{1}{1+x^2}\, dx$.

9. Write the cartesian equation of the following line given in vector form :

$$\vec{r} = 2\hat{i} + \hat{j} - 4\hat{k} + \lambda(\hat{i} - \hat{j} - \hat{k})$$

10. From the following matrix equation, find the value of x :

$$\begin{pmatrix} x+y & 4 \\ -5 & 3y \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -5 & 6 \end{pmatrix}$$

SECTION–B

Question numbers 11 to 22 carry 3 marks each.

11. Consider $f : R \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(y) = \left(\frac{\sqrt{y+6} - 1}{3}\right)$.

OR

Let $A = N \times N$ and * be a binary operation on A defined by $(a, b) * (c, d) = (a + c, b + d)$. Show that * is commutative and associative. Also, find the identity element for * on A, if any.

12. Prove the following: $\tan\left[\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right] + \tan\left[\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right] = \frac{2b}{a}$.

13. Prove the following, using properties of determinants:

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$

OR

Find the inverse of $A = \begin{pmatrix} 3 & -1 \\ -4 & 1 \end{pmatrix}$ using elementary transformations.

14. If $y = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$, show that $\frac{dy}{dx} = \sec x$. Also find the value of $\frac{d^2y}{dx^2}$ at $x = \frac{\pi}{4}$.

15. If $y = \cos^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$, find $\frac{dy}{dx}$.

16. Evaluate: $\int \sin x \,.\, \sin 2x \,.\, \sin 3x \, dx$.

OR

Evaluate: $\int \frac{x^2 - 3x}{(x-1)(x-2)} dx$.

17. Evaluate: $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$.

18. Form the differential equation representing the family of ellipses foci on x-axis and centre at the origin.

19. Find the particular solution of the following differential equation satisfying the given condition :

$$(3x^2 + y)\frac{dx}{dy} = x, \; x > 0, \text{ when } x = 1, y = 1$$

OR

Solve the following differential equation: $y\,dx + x \log\left(\frac{y}{x}\right) dy = 2x\,dy$

20. Let $\vec{a} = \hat{i} - \hat{j}$, $\vec{b} = 3\hat{j} - \hat{k}$ and $\vec{c} = 7\hat{i} - \hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$, and $\vec{c} \cdot \vec{d} = 1$.

21. Find the shortest distance between the following pair of lines and hence write whether the lines are intersecting or not :

$$\frac{x-1}{2} = \frac{y+1}{3} = z; \frac{x+1}{5} = \frac{y-2}{1}; \; z = 2$$

22. An experiment succeeds twice as often as it fails. Find the probability that in the next six trails there will be at least 4 successes.

SECTION–C

Question numbers 2 to 29 carry 6 marks each.

23. A factory makes two types of items A and B, made of plywood. One piece of item A requires 5 minutes for cutting and 10 minutes for assembling. One piece of item B requires 8 minutes for cutting and 8 minutes for assembling. There are 3 hours and 20 minutes available for cutting and 4 hours for assembling. The profit on one piece of item A is Rs 5 and that on item B is Rs 6. How many pieces of each type should the factory make so as to maximise profit? Make it as an L.P.P. and solve it graphically.

24. An urn contains 4 white and 3 red balls. Let X be the number of red balls in a random draw of three balls. Find the mean and variance of X.

OR

In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{5}$ be the probability that he knows the answer and $\frac{2}{5}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $\frac{1}{3}$, what is the probability that the student knows the answer, given that he answered it correctly?

25. Find the coordinates of the point where the line through (3, –4, –5) and (2, –3, 1) crosses the plane determined by points A (1, 2, 3), B(2, 2, 1) and C (–1, 3, 6).

26. If $A = \begin{pmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{pmatrix}$, find A^{-1}. Using A^{-1} solve the following system of equations :

$$2x - 3y + 5z = 16; \quad 3x + 2y - 4z = -4; \quad x + y - 2z = -3$$

27. Using integration, find the area of the region bounded by the lines,

$$4x - y + 5 = 0; \; x + y - 5 = 0 \; \text{ and } \; x - 4y + 5 = 0$$

OR

Using integration, find the area of the following region : $\{(x, y) ; |x + 2| \le y \le \sqrt{20 - x^2}\}$.

28. The lengths of the sides of an isosceles triangle are $9 + x^2$, $9 + x^2$ and $18 - 2x^2$ units. Calculate the area of the triangle in terms of x and find the value of x which makes the area maximum.

29. Evaluate the following : $\int_0^{3/2} |x \cos \pi x| \, dx$.

Set–II

Only those questions, not included in Set I, are given

2. If $f : R \to R$ and $g : R \to R$ are given by $f(x) = \sin x$ and $g(x) = 5x^2$, find $gof(x)$.

3. From the following matrix equation, find the value of x :

$$\begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

11. Prove the following, using properties of determinants :

$$\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2\,(3abc - a^3 - b^3 - c^3)$$

OR

Find the inverse of the following matrix, using elementary transformations: $A = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$.

14. Differentiate the following function with respect to x : $f(x) = \tan^{-1}\left(\dfrac{1-x}{1+x}\right) - \tan^{-1}\left(\dfrac{x+2}{1-2x}\right)$.

17. Evaluate : $\int_{-5}^{5} |x + 2|\, dx$.

21. Find the cartesian and vector equations of the plane passing through the points (0, 0, 0) and (3, –1, 2) and parallel to the line $\dfrac{x-4}{1} = \dfrac{y+3}{-4} = \dfrac{z+1}{7}$.

23. Using matrices, solve the following system of equations :

$$3x - 2y + 3z = -1\,; \quad 2x + y - z = 6\,; \quad 4x - 3y + 2z = 5$$

24. Evaluate the following : $\int_{-1}^{3/2} |x \sin \pi x|\, dx$.

Set–III

Only those questions, not included in Set I and Set II are given

1. If $f(x) = 27x^3$ and $g(x) = x^{1/3}$, find $gof(x)$.

7. If $\begin{pmatrix} 3 & 4 \\ 2 & x \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} 19 \\ 15 \end{pmatrix}$, find the value of x.

13. Prove the following, using properties of determinants :

$$\begin{vmatrix} a+bx^2 & c+dx^2 & p+qx^2 \\ ax^2+b & cx^2+d & px^2+q \\ u & v & w \end{vmatrix} = (x^4 - 1) \begin{vmatrix} b & d & q \\ a & c & p \\ u & v & w \end{vmatrix}$$

OR

Using elementary transformations, find the inverse of the following matrix : $A = \begin{pmatrix} 6 & 5 \\ 5 & 4 \end{pmatrix}$.

17. If $x = a\left(\cos t + \log \tan \frac{t}{2}\right)$, $y = a(1 + \sin t)$, find $\frac{d^2y}{dx^2}$.

19. Evaluate the following : $\int_0^1 x^2 (1 - x)^n \, dx$.

21. The scalar product of the vector $\hat{i} + 2\hat{j} + 4\hat{k}$ with a unit vector along the sum of vectors $\hat{i} + 2\hat{j} + 3\hat{k}$ and $\lambda\hat{i} + 4\hat{j} - 5\hat{k}$ is equal to one. Find the value of λ.

23. If $A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & -1 \\ -2 & 1 & 1 \end{pmatrix}$, find A^{-1}. Using A^{-1}, solve the following system of equations :

$$2x + y + 3z = 9\,; \quad x + 3y - z = 2\,; \quad -2x + y + z = 7$$

27. The sum of the perimeter of a circle and a square is K, where K is some constant. Prove that the sum of their areas is least when the side of the square is double the radius of the circle.

SOLUTIONS

Set–I

SECTION–A

1. Square matrix of order 2, which is both symmetric and skew symmetric is

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2. We are given $f(x) = \frac{3x - 4}{5}$ which is invertible

Let $y = \frac{3x - 4}{5}$

$\Rightarrow \quad 5y = 3x - 4 \quad \Rightarrow \quad x = \frac{5y + 4}{3}$

$\therefore \quad f^{-1}(y) = \frac{5y + 3}{3}$ and $f^{-1}(x) = \frac{5x + 4}{3}$

3. $-1 \le x \le 1$ is the domain of the function $\sin^{-1} x$.

4. We are given

$$\Delta = \begin{vmatrix} 4 & a & b + c \\ 4 & b & c + a \\ 4 & c & a + b \end{vmatrix}$$

Applying $C_3 \to C_3 + C_2$

$$\Delta = \begin{vmatrix} 4 & a & b+c+a \\ 4 & b & c+a+b \\ 4 & c & a+b+c \end{vmatrix} = 4\,(a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix}$$

As we know if two columns are same in any determinant then its value is 0

$\therefore \quad \Delta = 0$

5. For a unit vector $\vec{a}$,

$$(\vec{x} - \vec{a}).(\vec{x} + \vec{a}) = 15$$

$$\vec{x}^2 - \vec{a}^2 = 15 \quad \Rightarrow \quad |\vec{x}|^2 - |\vec{a}|^2 = 15$$

$$\Rightarrow \quad |\vec{x}|^2 - 1 = 15 \qquad [|\vec{a}|^2 = 1]$$

$$\Rightarrow \quad |\vec{x}|^2 = 16 \quad \text{or} \quad |\vec{x}|^2 = (4)^2 \quad \text{or} \quad |\vec{x}| = 4$$

6. Let, $\vec{a} = p\,(\hat{i} + \hat{j} + \hat{k})$

Magnitude of $\vec{a}$ is $|\vec{a}|$

$$|\vec{a}| = \sqrt{(p)^2 + (p)^2 + (p)^2} = \pm\sqrt{3}p$$

As $\vec{a}$ is a unit vector,

$$\therefore \quad |\vec{a}| = 1 \quad \Rightarrow \quad \pm\sqrt{3}p = 1 \quad \Rightarrow \quad p = \pm\frac{1}{\sqrt{3}}$$

7. Given $\int (ax+b)^2\, dx = f(x) + C$

$$\Rightarrow \quad \frac{(ax+b)^3}{3a} + C = f(x) + C \quad \Rightarrow \quad f(x) = \frac{(ax+b)^3}{3a}$$

8. $\int_0^1 \frac{1}{1+x^2}\, dx$

$$\left[\tan^{-1} x\right]_0^1 = [\tan^{-1} 1 - \tan^{-1} 0] = \frac{\pi}{4}$$

9. Vector form of a line is given as :

$$\vec{r} = 2\hat{i} + \hat{j} - 4\hat{k} + \lambda\,(\hat{i} - \hat{j} - \hat{k})$$

Direction ratios of above equation are (1, –1, –1) and point through which the line passes is (2, 1, –4).

$\therefore$ Cartesian equation is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

i.e., $$= \frac{x-2}{1} = \frac{y-1}{-1} = \frac{z+4}{-1} \quad \text{or} \quad x - 2 = 1 - y = -z - 4$$

10. Given matrix equation

$$\begin{bmatrix} x+y & 4 \\ -5 & 3y \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -5 & 6 \end{bmatrix}$$

Comparing both sides we get,

$$x + y = 3 \quad \text{and} \quad 3y = 6 \qquad \ldots(i)$$

i.e., $y = 2$ and $x = 1$

$\therefore$ $x = 1,\ y = 2.$

SECTION–B

11. Given $f : R \to [-5, \infty)$, given by

$f(x) = 9x^2 + 6x - 5$

(*i*) Let $f(x_1) = f(x_2)$

$\Rightarrow \quad 9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$

$\Rightarrow \quad 9(x_1 - x_2)(x_1 + x_2) + 6(x_1 - x_2) = 0$

$\Rightarrow \quad (x_1 - x_2)[9(x_1 + x_2) + 6] = 0$

$\Rightarrow \quad x_1 - x_2 = 0$ or $9(x_1 + x_2) + 6 = 0 \qquad \ldots(i)$

$\Rightarrow \quad x_1 = x_2$ or $9(x_1 + x_2) = -6$ i.e., $(x_1 + x_2) = -\dfrac{6}{9}$ which is not possible.

$\therefore \quad x_1 = x_2$

So, we can say, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$\therefore$ f is one-one.

(*ii*) Let $y \in [-5, \infty]$

So that $y = f(x)$ for some

$x \in R_+ \Rightarrow \quad 9x^2 + 6x - 5 = y$

$\Rightarrow \quad 9x^2 + 6x - 5 - y = 0$

$\Rightarrow \quad 9x^2 + 6x - (5 + y) = 0 \qquad \Rightarrow \quad x = \dfrac{-6 \pm \sqrt{36 + 4(9)(5 + y)}}{2 \times 9}$

$\Rightarrow \quad x = \dfrac{-6 \pm 6\sqrt{1 + 5 + y}}{18} = \dfrac{-1 \pm \sqrt{y + 6}}{3}$

$\Rightarrow \quad x = \dfrac{-1 + \sqrt{y + 6}}{3}, \ \dfrac{-1 - \sqrt{y + 6}}{3}$

here $x = \dfrac{-1 + \sqrt{y + 6}}{3}$

$\in R_+ \quad \therefore$ f is onto.

Since function is one-one and onto, so it is invertible.

$$f^{-1}(y) = \frac{-1 + \sqrt{y + 6}}{3} \qquad i.e., \quad f^{-1}(x) = \frac{\sqrt{x + 6} - 1}{3}$$

OR

Given $A = N \times N$

* is a binary operation on A defined by

$$(a, b) * (c, d) = (a + c, b + d)$$

(*i*) Commutativity: Let $(a, b), (c, d) \in N \times N$

Then $\quad (a, b) * (c, d) = (a + c, b + d) = (c + a, \ d + b)$

$\qquad\qquad (\because a, b, c, d \in N, a + c = c + a$ and $b + d = d + c)$

$\qquad = (c, d) * b$

Hence, $\quad (a, b) * (c, d) = (c, d) * (a, b)$

$\therefore$ * is commutative.

(*ii*) Associativity: let $(a, b), (b, c), (c, d)$

Then $[(a, b) * (c, d)] * (e, f) = (a + c, b + d) * (e, f) = ((a + c) + e, (b + d) + f)$

$\qquad = \{a + (c + e), b + (d + f)]$ $\qquad (\because$ set N is associative)

$\qquad = (a, b) * (c + e, d + f) = (a, b) * \{(c, d) * (e, f)\}$

Hence, $\quad [(a, b) * (c, d)] * (e, f) = (a, b) * \{(c, d) * (e, f)\}$

$\therefore$ * is associative.

(*iii*) Let (x, y) be identity element for $\ast$ on A,

Then $(a, b) * (x, y) = (a, b)$

$\Rightarrow \quad (a + x, b + y) = (a, b)$

$\Rightarrow \quad a + x = a, \qquad b + y = b$

$\Rightarrow \quad x = 0, \qquad y = 0$

But $(0, 0) \notin A$

$\therefore$ For *, there is no identity element.

12. $\tan\left[\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right] + \tan\left[\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right] = \frac{2b}{a}$

L.H.S. $\quad \tan\left[\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right] + \tan\left[\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\left(\frac{a}{\ }\right)\right]$

$| \ b \ | \ |$ Let $\quad \cos^{-1}\frac{a}{b} = x \Rightarrow \frac{a}{b} = \cos x$

LHS $\quad = \tan\left[\frac{\pi}{4} + \frac{1}{2}x\right] + \tan\left[\frac{\pi}{4} - \frac{1}{2}x\right]$

$$= \frac{\tan\frac{\pi}{4} + \tan\frac{x}{2}}{1 - \tan\frac{\pi}{4}\tan\frac{x}{2}} + \frac{\tan\frac{\pi}{4} - \tan\frac{x}{2}}{1 + \tan\frac{\pi}{4}\tan\frac{x}{2}}$$

$$\left[\because \tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b} \text{ and } \tan(a - b) = \frac{\tan a - \tan b}{1 - \tan a \tan b}\right]$$

$$= \frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)} + \frac{1-\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)}$$

$$= \frac{\left[1+\tan\left(\frac{x}{2}\right)\right]^2 + \left[1-\tan\left(\frac{x}{2}\right)\right]^2}{1-\tan^2\frac{x}{2}} = \frac{2\left[1+\tan^2\frac{x}{2}\right]}{1-\tan^2\frac{x}{2}}$$

$$= \frac{2}{\cos x} \qquad \left(\because \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \frac{2}{a/b} = \frac{2b}{a}$$

LHS = RHS **Hence Proved.**

13. L.H.S. $= \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

$$= 2(a+b+c)\begin{vmatrix} 0 & -(a+b+c) & 0 \\ 0 & (a+b+c) & -(a+b+c) \\ 1 & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c+)^3\begin{vmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3[1(1-0)] = 2(a+b+c)^3 = \text{RHS}$$

OR

Given $A = \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix}$

We know that $A = IA$

$\Rightarrow \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Applying $R_1 \to R_1 + \frac{1}{2} R_2$

$$\begin{bmatrix} 1 & -1/2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 + 4R_1$

$$\begin{bmatrix} 1 & -1/2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ 4 & 3 \end{bmatrix} A$$

Applying $R_2 \to - R_2$

$$\begin{bmatrix} 1 & -1/2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ -4 & -3 \end{bmatrix} A$$

Applying $R_1 \to R_1 + \frac{1}{2} R_2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -4 & -3 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} {}_{-1} \begin{bmatrix} -1 & -1 \\ -4 & -3 \end{bmatrix}$$

$$\Rightarrow \qquad A^{-1} = \begin{bmatrix} -1 & -1 \\ -4 & -3 \end{bmatrix}$$

14. Given $y = \log \tan \left(\frac{\pi}{4} + \frac{x}{2}\right)$

By differentiating of w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)} \cdot \sec^2\left(\frac{\pi}{4} + \frac{x}{2}\right) \cdot \frac{1}{2}$$

$$= \frac{\cos\left(\frac{\pi}{4} + \frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4} + \frac{x}{2}\right)\cos^2\left(\frac{\pi}{4} + \frac{x}{2}\right)} = \frac{1}{2\sin\left(\frac{\pi}{4} + \frac{x}{2}\right)\cos\left(\frac{\pi}{4} + \frac{x}{2}\right)}$$

$$= \frac{1}{\sin 2\left(\frac{\pi}{4} + \frac{x}{2}\right)} = \frac{1}{\sin\left(\frac{\pi}{2} + x\right)} = \frac{1}{\cos x} = \sec x$$

Now again differentiating w.r.t. x,

$$\frac{d^2y}{dx^2} = \sec x \tan x$$

d^2

At $x = \frac{\pi}{4}$, $\frac{y}{dx^2} = \sec\frac{\pi}{4}\tan\frac{\pi}{4} = \sqrt{2}$

15. Given $y = \cos^{-1}\left(\dfrac{2^{x+1}}{1+4^x}\right) \Rightarrow y = \cos^{-1}\left[\dfrac{2^x.2^1}{1+4^x}\right]$

Let $2^x = \tan\alpha \Rightarrow \alpha = \tan^{-1}(2^x)$

$$\therefore \quad y = \cos^{-1}\left(\frac{2\tan\alpha}{1+\tan^2\alpha}\right) = \cos^{-1}(\sin 2\alpha) = \cos^{-1}\left[\cos\left(\frac{\pi}{2} - 2\alpha\right)\right]$$

$$\Rightarrow \quad y = \frac{\pi}{2} - 2\alpha = \frac{\pi}{2} - 2\tan^{-1}(2^x)$$

By differentiating w.r.t. x, we get

$$\frac{dy}{dx} = -2\frac{d}{dx}[\tan^{-1}(2^x)] = -\frac{2.2^x \log e^2}{1+2^{2x}} = \frac{-2^{x+1}\log e^2}{1+4^x}$$

16. $\int \sin x \,.\, \sin 2x \,.\, \sin 3x \, dx$

Multiplying and dividing by 2

$$= \frac{1}{2}\int 2\sin x \sin 3x \sin 2x \, dx = \frac{1}{2}\int \sin x [2\sin 3x \sin 2x] \, dx$$

$$= \frac{1}{2}\int \sin x [\cos x - \cos 5x] \, dx \qquad [\because 2\sin a \sin b = \cos(a-b) - \cos(a+b)]$$

$$= \frac{1}{2}\int (\sin x \cos x - \cos 5x \sin x) \, dx = \frac{1}{4}\int (2\sin x \cos x - 2\cos 5x \sin x) \, dx$$

$$= \frac{1}{4}\int (\sin 2x - \sin 6x + \sin 4x) \, dx = \frac{1}{4}\left[-\frac{\cos 2x}{2} + \frac{\cos 6x}{6} - \frac{\cos 4x}{4}\right] + C$$

$$= -\frac{\cos 2x}{8} + \frac{\cos 6x}{24} - \frac{\cos 4x}{16} + C$$

OR

Given $\displaystyle\int \frac{(x^2 - 3x)\,dx}{(x-1)(x-2)} = \int \frac{(x^2 - 3x)\,dx}{x^2 - 3x + 2}$

$$= \int \frac{(x^2 - 3x + 2) - 2}{x^2 - 3x + 2} dx = \int \left[1 - \frac{2}{x^2 - 3x + 2}\right] dx$$

$$= \int dx - 2\int \frac{dx}{x^2 - 3x + 2} = x - 2\int \frac{dx}{\left(x - \frac{3}{2}\right)^2 - \frac{1}{4}}$$

$$= x - 2\left[\log\left|\frac{x - \frac{3}{2} - \frac{1}{2}}{x - \frac{3}{2} + \frac{1}{2}}\right|\right] + C \qquad \left[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= x - 2\log\left|\frac{x-2}{x-1}\right| + C$$

17. Let $I = \int_0^{\pi} \dfrac{x \tan x}{\sec x + \tan x}\, dx$ ...(*i*)

As $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$

$\therefore$ $$I = \int_0^{\pi} \frac{(\pi - x) \tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)}\, dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \tan x}\, dx \quad \text{...}(ii)$$

By adding equations (*i*) and (*ii*), we get

$$2I = \pi \int_0^{\pi} \frac{\tan x}{\sec x + \tan x}\, dx$$

Multiplying and dividing by $(\sec x - \tan x)$, we get

$$2I = \pi \int_0^{\pi} \frac{\tan x(\sec x - \tan x)}{\sec^2 x - \tan^2 x}\, dx$$

$$= \pi \int_0^{\pi} (\sec x \tan x - \tan^2 x)\, dx$$

$$= \pi \int_0^{\pi} \sec x \tan x\, dx - \pi \int_0^{\pi} \sec^2 x\, dx + \int_0^{\pi} dx$$

$$= \pi\left[\sec x\right]_0^{\pi} - \pi\left[\tan x\right]_0^{\pi} + \pi\left[x\right]_0^{\pi} = \pi(-1-1) - 0 + \pi(\pi - 0) = \pi(\pi - 2)$$

$\Rightarrow$ $2I = \pi(\pi - 2)$ $\Rightarrow$ $I = \dfrac{\pi}{2}(\pi - 2)$

18. The family of ellipses having foci on x-axis and centre at the origin, is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Differentiating w.r.t. x, we get

$$\frac{2x}{a^2} + \frac{2y}{b^2}\left(\frac{dy}{dx}\right) = 0 \quad \Rightarrow \quad \frac{2y}{b^2}\frac{dy}{dx} = -\frac{2x}{a^2}$$

$$\Rightarrow \quad \frac{y}{b^2}\frac{dy}{dx} = -\frac{x}{a^2} \quad \Rightarrow \quad y\left(\frac{\left(\frac{dy}{dx}\right)}{x}\right) = \frac{-b^2}{a^2}$$

Again by differentiating w.r.t. x, we get

$$\frac{x\left[y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2\right] - \left(y \cdot \dfrac{dy}{dx}\right)}{x^2} = 0$$

$\therefore$ The required equation is

$$xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

19. We are given

$$(3x^2 + y)\frac{dx}{dy} = x,\ x > 0$$

$$\Rightarrow \qquad \frac{dx}{dy} = \frac{x}{3x^2 + y}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{3x^2 + y}{x} = 3x + \frac{y}{x}$$

$$\Rightarrow \qquad \frac{dy}{dx} - \frac{1}{x}y = 3x$$

This is a linear differential equation of the form $\dfrac{dy}{dx} + Py = Q$

Here $P = -\dfrac{1}{x}, Q = 3x$

$$\text{I.F.} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$$

$$\therefore \qquad \frac{y}{x} = \int \frac{1}{x} 3x\,dx = 3\int dx$$

$$\Rightarrow \qquad \frac{y}{x} = 3x + C \quad \Rightarrow \quad y = 3x^2 + Cx$$

But, it is given when $x = 1, y = 1$

$$\Rightarrow \qquad 1 = 3 + C \qquad \Rightarrow \quad C = -2$$

$$\therefore \qquad y = 3x^2 - 2x$$

OR

Given $\quad y\,dx + x\log\left(\dfrac{y}{x}\right)dy = 2x\,dy$

$$\Rightarrow \qquad \left[x\log\left(\frac{y}{x}\right) - 2x\right]dy = -y\,dx$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)}$$

Let $y = vx, \qquad \Rightarrow \quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$\therefore \qquad v + x\frac{dv}{dx} = \frac{vx}{2x - x\log\left(\frac{vx}{x}\right)}$$

$$\Rightarrow \qquad x\frac{dv}{dx} = \frac{vx}{x(2 - \log v)} - v$$

$$\Rightarrow \qquad x\frac{dv}{dx} = \frac{v - 2v + v\log v}{2 - \log v} = \frac{v\log v - v}{2 - \log v}$$

$$\Rightarrow \qquad \frac{2 - \log v}{v\log v - v}\,dv = \frac{dx}{x}$$

$$\Rightarrow \qquad \int \frac{2 - \log v}{v\log v - v}\,dv = \int \frac{dx}{x}$$

$$\Rightarrow \qquad \int \frac{1 + (1 - \log v)}{v\,(\log v - 1)}\,dv = \int \frac{dx}{x}$$

$$\Rightarrow \qquad \int \frac{dx}{v(\log v - 1)} - \int \frac{dv}{v} = \int \frac{dx}{x} \qquad \ldots(i)$$

Let $\log v - 1 = t \quad \Rightarrow \quad \frac{1}{v}\,dv = dt$

$$\therefore (i) \Rightarrow \qquad \int \frac{1}{t}\,dt - \int \frac{1}{v}\,dv = \int \frac{dx}{x}$$

$$\Rightarrow \qquad \log|t| - \log|v| = \log|x| + \log|c|$$

$$\Rightarrow \qquad \log\left|\frac{t}{v}\right| = \log|cx| \quad \Rightarrow \quad \frac{t}{v} = cx$$

$$\Rightarrow \qquad \frac{\log v - 1}{v} = cx$$

$$\Rightarrow \qquad \frac{\left[\log\left(\frac{y}{x}\right) - 1\right]}{\frac{y}{x}} = cx$$

$$\Rightarrow \qquad \left[\log\left(\frac{y}{x}\right) - 1\right] = cy, \text{ which is the required solution.}$$

20. Given $\vec{a} = \hat{i} - \hat{j}, \quad \vec{b} = 3\hat{j} - \hat{k}, \quad \vec{c} = 7\hat{i} - \hat{k}$

$\because$ vector $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$

$\therefore$ d is along vector $\vec{a} \times \vec{b}$

$$\Rightarrow \qquad \vec{d} = \lambda\,(\vec{a} \times \vec{b}) = \lambda \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 0 & 3 & -1 \end{vmatrix} = \lambda\,(\hat{i} + \hat{j} + 3\hat{k})$$

Also $\vec{c} \cdot \vec{d} = 1 \qquad \Rightarrow \qquad (7\hat{i} - \hat{k}) \cdot \lambda\,(\hat{i} + \hat{j} + 3\hat{k}) = 1$

$$\Rightarrow \qquad \lambda(7 + 0 - 3) = 1 \qquad \Rightarrow \qquad \lambda = \frac{1}{4}$$

$$\therefore \qquad \vec{d} = \frac{1}{4}(\hat{i} + \hat{j} + 3\hat{k})$$

21. Given, pair of lines

$$\frac{x-1}{2}=\frac{y+1}{3}=z \quad \text{and} \quad \frac{x+1}{5}=\frac{y-2}{1}=\frac{z-2}{0}$$

In vector form equations are

$$\vec{r}=(i-\hat{j})+\mu\,(2\hat{i}+3\hat{j}+\hat{k})$$

and $$\vec{r}=(-\hat{i}+2\hat{j}+2\hat{k})+\lambda\,(5\hat{i}+\hat{j})$$

$$\vec{a}_1=\hat{i}-\hat{j}, \qquad \vec{b}_1=2\hat{i}+3\hat{j}+\hat{k}$$

$$\vec{a}_2=-\hat{i}+2\hat{j}+2\hat{k}, \qquad \vec{b}_2=5\hat{i}+\hat{j}$$

$$\vec{a}_2-\vec{a}_1=-2\hat{i}+3\hat{j}+2\hat{k}$$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 1\\ 5 & 1 & 0\end{vmatrix}=-\hat{i}+5\hat{j}-13\hat{k}$$

$\therefore$ $$(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)=(-2\hat{i}+3\hat{j}+2\hat{k}).(-\hat{i}+5\hat{j}-13\hat{k})$$

$$=2+15-26=-9$$

As we know shortest distance $=\left|\dfrac{(\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|$

$$=\left|\frac{-9}{\sqrt{(-1)^2+(5)^2+(-13)^2}}\right|=\left|\frac{-9}{\sqrt{1+25+169}}\right|$$

$$=\left|\frac{-9}{\sqrt{195}}\right|=\frac{9}{\sqrt{195}}\text{ units}$$

Lines are not intersecting as the shortest distance is not zero.

22. An experiment succeeds twice as often as it fails.

$\therefore$ $\quad p=P\text{ (success)}=\dfrac{2}{3}$

and $\quad q=P\text{ (failure)}=\dfrac{1}{3}$

no. of trials $=n=6$

By the help of Binomial distribution,

$$P(r)={}^6C_r\left(\frac{2}{3}\right)^r\left(\frac{1}{3}\right)^{6-r}$$

P (at least four success) $=P(4)+P(5)+P(6)$

$$= {}^6C_4\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4 + {}^6C_5\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^5 + {}^6C_6\left(\frac{2}{3}\right)^6$$

$$= \left(\frac{2}{3}\right)^4\left[\frac{1}{9}\,{}^6C_4 + \frac{2}{9}\,{}^6C_5 + \frac{4}{6}\,{}^6C_6\right]$$

$$= \left(\frac{2}{3}\right)^4\left[\frac{15}{9} + \frac{2}{9}\times 6 + \frac{4}{9}\right] = \frac{16}{81}\times\frac{31}{9} = \frac{496}{729}$$

SECTION–C

23. Let the factory makes x pieces of item A and B by pieces of item.

Time required by item A (one piece)

cutting = 5 minutes, assembling = 10 minutes

Time required by item B (one piece)

cutting = 8 minutes, assembling = 8 minutes

Total time

cutting = 3 hours & 20 minutes,

assembling = 4 hours

Profit on one piece

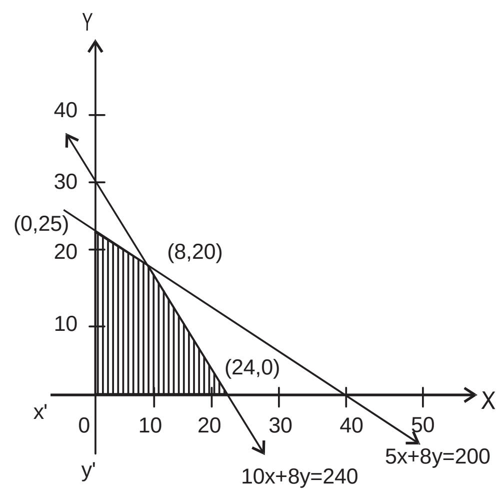
item A = Rs 5, item B = Rs 6

Thus, our problem is maximized

$$z = 5x + 6y$$

Subject to $x \geq 0, y \geq 0$

$$5x + 8y \leq 200$$
$$10x + 8y \leq 240$$



From figure, possible points for maximum value of z are at (24, 0), (8, 20), (0, 25).

at (24, 0), $z = 120$

at (8, 20), $z = 40 + 120 = 160$ (maximum)

at (0, 25), $z = 150$

$\therefore$ 8 pieces of item A and 20 pieces of item B produce maximum profit of Rs 160.

24. Let X be the no. of red balls in a random draw of three balls.

As there are 3 red balls, possible values of X are *0, 1, 2, 3.*

$$P(0) = \frac{{}^3C_0 \times {}^4C_3}{{}^7C_3} = \frac{4\times 3\times 2}{7\times 6\times 5} = \frac{4}{35}$$

$$P(1) = \frac{{}^3C_1 \times {}^4C_2}{{}^7C_3} = \frac{3\times 6\times 6}{7\times 6\times 5} = \frac{18}{35}$$

$$P(2) = \frac{{}^3C_2 \times {}^4C_1}{{}^7C_3} = \frac{3\times 4\times 6}{7\times 6\times 5} = \frac{12}{35}$$

$$P(3) = \frac{{}^3C_3 \times {}^4C_0}{{}^7C_3} = \frac{1 \times 1 \times 6}{7 \times 6 \times 5} = \frac{1}{35}$$

For calculation of Mean & Variance

X	$P(X)$	$XP(X)$	$X^2P(X)$
0	4/35	0	0
1	18/35	18/35	18/35
2	12/35	24/35	48/35
3	1/35	3/35	9/35
Total	1	9/7	15/7

$$\text{Mean} = \Sigma XP(X) = \frac{9}{7}$$

$$\text{Variance} = \Sigma X^2.P(X) - (\Sigma X.P(X))^2 = \frac{15}{7} - \frac{81}{49} = \frac{24}{49}$$

OR

Let A, B and and E be the events defined as follows:

A : Student knows the answer

B : Student guesses the answer

E : Student answers correctly

Then, $P(A) = \frac{3}{5}$, $P(B) = \frac{2}{5}$, $P(E/A) = 1$

$P(E/B) = \frac{1}{3}$

Using Baye's theorem, we get

$$P(A/E) = \frac{P(A).P(E/A)}{P(A).P(E/A) + P(B)P(E/B)} = \frac{\frac{3}{5}}{\frac{3}{5} + \frac{2}{5} \times \frac{1}{3}} = \frac{3 \times 3}{9+2} = \frac{9}{11}$$

25. The line through (3, –4, –5) and (2, –3, 1) is given by

$$\frac{x-3}{2-3} = \frac{y+4}{-3+4} = \frac{z+5}{1+5}$$

$$\Rightarrow \quad \frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} \qquad \ldots(i)$$

The plane determined by points $A(1, 2, 3)$, $B(2, 2, 1)$ and $C(-1, 3, 6)$

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2-1 & 2-2 & 1-3 \\ -1-1 & 3-2 & 6-3 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x-1 & y-2 & z-3 \\ 1 & 0 & -2 \\ -2 & 1 & 3 \end{vmatrix} = 0$$

$$\Rightarrow \quad (x-1)\begin{vmatrix} 0 & -2 \\ 1 & 3 \end{vmatrix} - (y-2)\begin{vmatrix} 1 & -2 \\ -2 & 3 \end{vmatrix} + (z-3)\begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad (x-1)(2) - (y-2)(-1) + (z-3)(1) = 0$

$\Rightarrow \quad 2x - 2 + y - 2 + z - 3 = 0 \quad \Rightarrow \quad 2x + y + z - 7 = 0$ ...(*ii*)

$P(-\mu + 3, \mu - 4, 6\mu - 5)$ is the general point for line (*i*).

If this point lies on plane (*ii*), we get

$$-2\mu + 6 + \mu - 4 + 6\mu - 5 - 7 = 0 \quad \Rightarrow \quad \mu = 2$$

$\therefore$ $P(1, -2, 7)$ is the point of intersection.

26. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} Adj.\, A$$

$$|A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix} = 2(-4+4) + 3(-6+4) + 5(3-2)$$

$$= 2(0) + 3(-2) + 5(1) = -1 \neq 0$$

$$Adj.\, A = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}' = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \qquad \ldots(i)$$

Given equations are

$$2x - 3y + 5z = 16$$
$$3x + 2y - 4z = -4$$
$$x + y - 2z = -3$$

Matrix form is

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ -4 \\ -3 \end{bmatrix}$$

$$AX = B$$

$\Rightarrow \quad X = A^{-1}B$ ...(*ii*)

From equations (i) and (ii), we get

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0&1&-2\\-2&9&-23\\-1&5&-13\end{bmatrix}\begin{bmatrix}16\\-4\\-3\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\1\\3\end{bmatrix}$$

$\Rightarrow \quad x = 2, y = 1, z = 3$

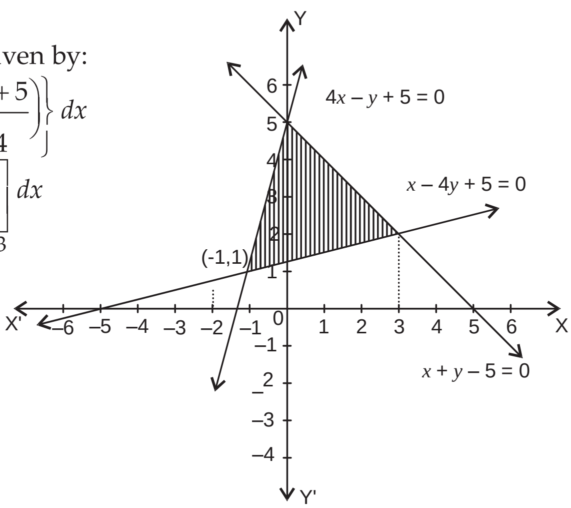
27. We have given

$$4x - y + 5 = 0 \qquad ...(i)$$
$$x + y - 5 = 0 \qquad ...(ii)$$
$$x - 4y + 5 = 0 \qquad ...(iii)$$

By solving equations (i) and (iii), we get (–1, 1)

and by solving (ii) and (iii), we get (3, 2)

$\therefore$ Area of region bounded by the lines is given by:



$$\int_{-1}^{0}\left\{(4x+5)-\left(\frac{x+5}{4}\right)\right\}dx+\int_{0}^{3}\left\{(5-x)-\left(\frac{x+5}{4}\right)\right\}dx$$

$$=\int_{-1}^{0}\left[\frac{15x}{4}+\frac{15}{4}\right]dx+\int_{0}^{3}\left[\frac{15}{4}-\frac{5x}{4}\right]dx$$

$$=\left[\frac{15x^2}{8}+\frac{15x}{4}\right]_{-1}^{0}+\left[\frac{15x}{4}-\frac{5x^2}{8}\right]_{0}^{3}$$

$$=0-\left(\frac{15}{8}-\frac{15}{4}\right)+\left(\frac{45}{4}-\frac{45}{8}\right)-0$$

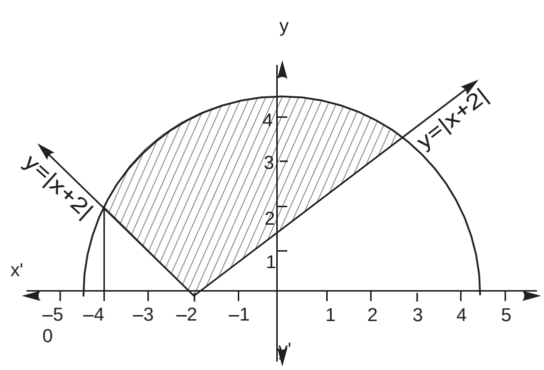
$$=\frac{15}{8}+\frac{45}{8}=\frac{15}{2}\text{ sq. unit.}$$

OR

Given region is $\{(x, y) : |x+2| \le y \le \sqrt{20 - x^2}.\}$

It consists of inequalities $y \ge |x + 2|$
$y \le \sqrt{20 - x^2}$

Plotting these inequalities, we obtain the adjoining shaded region.



Solving $\quad y = x + 2$

and $\quad y^2 = 20 - x^2$

$\Rightarrow \quad (x + 2)^2 = 20 - x^2$

$\Rightarrow \quad 2x^2 + 4x - 16 = 0$

or $\quad (x + 4)(x - 2) = 0$

$\Rightarrow \quad x = -4, 2$

The required area

$$= \int_{-4}^{2} \sqrt{20 - x^2}\,dx - \int_{-4}^{-2} -(x + 2)\,dx - \int_{-2}^{2} (x + 2)\,dx$$

$$= \left[\frac{x}{2}\sqrt{20 - x^2} + \frac{20}{2}\sin^{-1}\frac{x}{\sqrt{20}}\right]_{-4}^{2} + \left[\frac{x^2}{2} + 2x\right]_{-4}^{-2} - \left[\frac{x^2}{2} + 2x\right]_{-2}^{2}$$

$$= 4 + 10\sin^{-1}\frac{1}{\sqrt{5}} + 4 + 10\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) + [2 - 4 - 8 + 8] - [2 + 4 - 2 + 4]$$

$$= 8 + 10\left(\sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{2}{\sqrt{5}}\right) - 2 - 8$$

$$= -2 + 10\left(\sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{2}{\sqrt{5}}\right)$$

$$= -2 + 10\sin^{-1}\left[\frac{1}{\sqrt{5}}\sqrt{1 - \frac{4}{5}} + \frac{2}{\sqrt{5}}\sqrt{1 - \frac{1}{5}}\right]$$

$$= -2 + 10\sin^{-1}\left[\frac{1}{5} + \frac{4}{5}\right] = -2 + 10\sin^{-1} 1$$

$$= -2 + 10\,\frac{\pi}{2} = (5\pi - 2) \text{ sq. units.}$$

28. As given, the lengths of the sides of an isosceles triangle are $9 + x^2$, $9 + x^2$ and $18 - 2x^2$ units.

Using Heron's formula, we get

$$2s = 9 + x^2 + 9 + x^2 + 18 - 2x^2 = 36 \quad \Rightarrow \quad s = 18$$

$$A = \sqrt{18\,(18 - 9 - x^2)\,(18 - 9 - x^2)\,(18 - 18 + 2x^2)} = \sqrt{18(9 - x^2)(9 - x^2)(2x^2)}$$

$$A = 6x\,(9 - x^2)$$

$$A = 6\,(9x - x^3) \qquad \ldots(i)$$

Differentiating (i) w.r.t. x

$$\frac{dA}{dx} = 6(9 - 3x^2)$$

For maximum A, $\dfrac{dA}{dx} = 0$

$\Rightarrow \quad 9 - 3x^2 = 0 \quad \Rightarrow \quad x = \pm\sqrt{3}$

Now again differentiating w.r.t. x

$$\frac{d^2A}{dx^2} = 6\,(-6x) = -36x$$

At $x = \sqrt{3}$, $\quad \dfrac{d^2A}{dx^2} = -36\sqrt{3} < 0$

At $x = -\sqrt{3}$, $\qquad \dfrac{d^2A}{dx^2} = 36\sqrt{3} > 0$

$\therefore$ For $x = \sqrt{3}$, area is maximum.

29. $\displaystyle\int_0^{3/2} |x \cos \pi x|\, dx$

As we know that

$$\cos x = 0 \quad \Rightarrow \quad x = (2n-1)\frac{\pi}{2}, n \in Z$$

$$\therefore \quad \cos \pi x = 0 \quad \Rightarrow \quad x = \frac{1}{2}, \frac{3}{2}$$

For $\quad 0 < x < \dfrac{1}{2}, \qquad x > 0$

$$\cos \pi x > 0 \quad \Rightarrow \quad x \cos \pi x > 0$$

For $\quad \dfrac{1}{2} < x < \dfrac{3}{2}, \quad x > 0$

$$\cos \pi x < 0 \quad \Rightarrow \quad x \cos \pi x < 0$$

$$\therefore \quad \int_0^{3/2} |x \cos \pi x|\, dx$$

$$= \underbrace{\int_0^{1/2} x \cos \pi x\, dx}_{\text{I}} + \underbrace{\int_{1/2}^{3/2} (-x \cos \pi x)\, dx}_{\text{II}} \qquad \ldots(i)$$

$$= \left[x \frac{\sin \pi x}{\pi}\right]_0^{1/2} - \int_0^{1/2} 1 \cdot \frac{\sin \pi x}{\pi}\, dx - \left[\frac{x \sin \pi x}{\pi}\right]_{1/2}^{3/2} - \int_{1/2}^{3/2} \frac{\sin \pi x}{\pi}\, dx$$

$$= \left[\frac{x}{\pi} \sin \pi x + \frac{1}{\pi^2} \cos \pi x\right]_0^{1/2} - \left[\frac{x}{\pi} \sin \pi x + \frac{1}{\pi^2} \cos \pi x\right]_{1/2}^{3/2}$$

$$= \left(\frac{1}{2\pi} + 0 - \frac{1}{\pi^2}\right) - \left(-\frac{3}{2\pi} - \frac{1}{2\pi}\right) = \frac{5}{2\pi} - \frac{1}{\pi^2}$$

Set–II

2. Given $\quad f: R \to R$ and $g: R \to R$ defined by

$$f(x) = \sin x \quad \text{and} \quad g(x) = 5x^2$$

$$\therefore \quad gof(x) = g[f(x)] = g(\sin x) = 5(\sin x)^2 = 5 \sin^2 x$$

3. Given :

$$\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} (1)(x) + (3)(2) \\ 4(x) + (5)(2) \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \qquad \Rightarrow \quad \begin{bmatrix} x + 6 \\ 4x + 10 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Comparing both sides, we get

$$x + 6 = 5 \quad \Rightarrow \quad x = -1$$

Also, $4x + 10 = 6$

$\Rightarrow$ $4x = -4$ or $x = -1$

$\therefore$ $x = -1$

11. We have to prove $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2(3abc - a^3 - b^3 - c^3)$

$$\text{L.H.S} = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= \begin{bmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{bmatrix} \text{[By applying } R_1 \to R_1 + (R_2 + R_3)]$$

$$= 2\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$= 2(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix} = 2(-1)^2 (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2, C_2 \to C_2 \to C_3$

$$= 2(a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ b-c & c-a & a \\ c-a & a-b & b \end{vmatrix} = 2(a+b+c)\begin{vmatrix} b-c & c-a \\ c-a & a-b \end{vmatrix}$$

$$= 2(a+b+c)[(b-c)(a-b) - (c-a)(c-a)]$$

$$= 2(a+b+c)(-a^2 - b^2 - c^2 + ab + bc + ca)$$

$$= 2(3abc - a^3 - b^3 - c^3) = \text{RHS} \qquad \textbf{Hence Proved.}$$

OR

We are given

$$A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$$

$\Rightarrow$ $A = IA$

$$\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \quad \begin{bmatrix} 7 & 5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A \qquad \text{[By applying } R_1 \leftrightarrow R_2\text{]}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} A \qquad \text{[By applying } R_1 \to R_1 - 2R_2\text{]}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 7 & -3 \end{bmatrix} A \qquad \text{[By applying } R_2 \to R_2 - 3R_1\text{]}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 7 & -3 \end{bmatrix} A \qquad \text{[By applying } R_1 \to R_1 + R_2\text{]}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} A \qquad \text{[By applying } R_2 \to - R_2\text{]}$$

Hence, $\quad A^{-1} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

14. $f(x) = \tan^{-1}\left(\dfrac{1-x}{1+x}\right) - \tan^{-1}\left(\dfrac{x+2}{1-2x}\right)$

$$= \tan^{-1}\left(\frac{1-x}{1+x.1}\right) - \tan^{-1}\left(\frac{x+2}{1-2.x}\right)$$

$$= (\tan^{-1} 1 - \tan^{-1} x) - (\tan^{-1} x + \tan^{-1} 2) \qquad \left(\because \tan^{-1}\frac{a-b}{1+ab} = \tan^{-1} a - \tan^{-1} b\right)$$

$$= \tan^{-1} 1 - \tan^{-1} 2 - 2\tan^{-1} x$$

Differentiating w.r.t. x

$$f'(x) = -\frac{2}{1+x^2}$$

17. $|x+2| = \begin{cases} (x+2) & \text{if } x+2 > 0 \quad i.e., \ x > -2 \\ -(x+2) & \text{if } x+2 < 0 \quad i.e., \ x < -2 \end{cases}$

$$\therefore \quad \int_{-5}^{5} |x+2|\,dx = \int_{-5}^{-2} -(x+2)\,dx + \int_{-2}^{5} (x+2)\,dx$$

$$= \left[-\frac{x^2}{2} - 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{5}$$

$$= \left[-\frac{4}{2} + 4\right] - \left[-\frac{25}{2} + 10\right] + \left[\frac{25}{2} + 10\right] - \left[\frac{4}{2} - 4\right]$$

$$= 2 + \frac{5}{2} + \frac{45}{2} + 2 = 29$$

21. Plane passing through the point (0, 0, 0) is

$$a(x-0) + b(y-0) + c(z-0) = 0 \qquad \ldots(i)$$

Plane (i) passes through the point (3, –1, 2)

$$\therefore \quad 3a - b + 2c = 0 \qquad \ldots(ii)$$

Also, Plane (i) is parallel to the line

$$\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$$

$\therefore \qquad a - 4b + 7c = 0 \qquad \ldots(iii)$

From equations (i), (ii) and (iii)

$$\begin{vmatrix} x & y & z \\ 3 & -1 & 2 \\ 1 & -4 & 7 \end{vmatrix} = 0$$

$$\Rightarrow \qquad x\begin{vmatrix} -1 & 2 \\ -4 & 7 \end{vmatrix} - y\begin{vmatrix} 3 & 2 \\ 1 & 7 \end{vmatrix} + z\begin{vmatrix} 3 & -1 \\ 1 & -4 \end{vmatrix} = 0$$

$$\Rightarrow \qquad x[-7+8] - y[21-2] + z[-12+1] = 0$$

$$\Rightarrow \qquad x - 19y - 11z = 0$$

and in vector form, equation is

$$\vec{r}\,.(\hat{i} - 19\hat{j} - 11\hat{k}) = 0$$

SECTION–C

23. $3x - 2y + 3z = -1$

$2x + y - z = 6$

$4x - 3y + 2z = 5$

Now the matrix equation form of above three equations is

$$\begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \\ 5 \end{bmatrix}$$

i.e., $\quad AX = B \Rightarrow \quad X = A^{-1}B$

we know that $\qquad A^{-1} = \dfrac{1}{|A|} Adj.\ A$

$$|A| = \begin{vmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{vmatrix}$$

$$= 3\begin{vmatrix} 1 & -1 \\ -3 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 4 & -3 \end{vmatrix}$$

$$= -3 + 16 - 30 = -17 \neq 0$$

$$Adj.\ A = \begin{bmatrix} -1 & -8 & -10 \\ -5 & -6 & 1 \\ -1 & 9 & 7 \end{bmatrix}' = \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

$$X = A^{-1}B = -\frac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}\begin{bmatrix} -1 \\ 6 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{17}\begin{bmatrix} -34 \\ 17 \\ 51 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$$

By comparing both sides, we get

$$x = 2, y = -1, z = -3$$

24. $\int_{-1}^{3/2} |x \sin \pi x| \, dx$

As we know

$$\sin\theta = 0 \quad \Rightarrow \quad \theta = n\pi, \; n \in Z$$

$$\therefore \quad \sin \pi x = 0 \quad \Rightarrow \quad x = 0, 1, 2, \ldots$$

For $-1 < x < 0$,

$$x < 0, \sin \pi x < 0 \quad \Rightarrow \quad x \sin \pi x > 0$$

For $0 < x < 1$,

$$x > 0, \; \sin \pi x > 0 \quad \Rightarrow \quad x \sin \pi x > 0$$

For $1 < x < \frac{3}{2}$,

$$x > 0, \sin \pi x < 0 \quad \Rightarrow \quad x \sin \pi x < 0$$

$$\therefore \quad \int_{-1}^{3/2} |x \sin \pi x| \, dx$$

$$= \int_{-1}^{1} x \sin \pi x \, dx + \int_{1}^{3/2} (-x \sin \pi x) \, dx$$

$$= \left[x \cdot \frac{(\cos \pi x)}{\pi}\right]_{-1}^{1} - \int_{-1}^{1} 1 \cdot \frac{-\cos \pi x}{\pi} dx - \left[x \cdot \frac{-\cos \pi x}{\pi}\right]_{1}^{3/2} + \int_{1}^{3/2} 1 \cdot \frac{\cos \pi x}{\pi} dx$$

$$= \left[-\frac{x}{\pi}\cos \pi x + \frac{1}{\pi^2}\sin \pi x\right]_{-1}^{1} - \left[-\frac{x}{\pi}\cos \pi x + \frac{1}{\pi^2}\sin \pi x\right]_{1}^{3/2}$$

$$= \left[\frac{1}{\pi} + \frac{1}{\pi} + \frac{1}{\pi^2} + \frac{1}{\pi}\right] = \left[\frac{1}{\pi} + 0 + \frac{1}{\pi} - 0\right] - \left[0 - \frac{1}{\pi^2} - \frac{1}{\pi}\right] = \frac{1}{\pi^2} + \frac{3}{\pi} = \frac{1 + 3\pi}{\pi^2}.$$

Set–III

1. Given $f(x) = 27x^3$ and $g(x) = x^{1/3}$

$$(gof)(x) = g[f(x)] = g[27x^3] = [27x^3]^{1/3} = 3x$$

7. Given,

$$\Rightarrow \quad \begin{bmatrix} 3 & 4 \\ 2 & x \end{bmatrix}\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} 19 \\ 15 \end{bmatrix} \qquad \Rightarrow \quad \begin{bmatrix} 3(x) + 4(1) \\ (2)(x) + (x)(1) \end{bmatrix} = \begin{bmatrix} 19 \\ 15 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 3x + 4 \\ 3x \end{bmatrix} = \begin{bmatrix} 19 \\ 15 \end{bmatrix}$$

Comparing both sides, we get

$$3x + 4 = 19 \quad \text{and} \quad 3x = 15$$

$$\Rightarrow \quad 3x = 19 - 4, \quad 3x = 15$$

$$\Rightarrow \quad 3x = 15, \quad x = 5$$

$$\therefore \quad x = 5$$

13. We have to prove

$$\begin{vmatrix} a+bx^2 & c+dx^2 & p+qx^2 \\ ax^2+b & cx^2+d & px^2+q \\ u & v & w \end{vmatrix} = (x^4-1)\begin{vmatrix} b & d & q \\ a & c & p \\ u & v & w \end{vmatrix}$$

$$\text{L.H.S} = \begin{vmatrix} a+bx^2 & c+dx^2 & p+qx^2 \\ ax^2+b & cx^2+d & px^2+q \\ u & v & w \end{vmatrix}$$

Multiplying R_1 by x^2 and dividing the determinant by x^2

$$= \frac{1}{x^2}\begin{vmatrix} ax^2+bx^4 & cx^2+dx^4 & px^2+qx^4 \\ ax^2+b & cx^2+d & px^2+q \\ u & v & w \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$

$$= \frac{1}{x^2}\begin{vmatrix} b(x^4-1) & d(x^4-1) & q(x^4-1) \\ ax^2+b & cx^2+d & px^2+q \\ u & v & w \end{vmatrix}$$

$$= \frac{x^4-1}{x^2}\begin{vmatrix} b & d & q \\ ax^2+b & cx^2+d & px^2+q \\ u & v & w \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$

$$= \frac{x^4-1}{x^2}\begin{vmatrix} b & & \\ ax^2 & & \\ u & & \end{vmatrix}$$

$$= \frac{x^2(x^4-1)}{x^2}\begin{vmatrix} b & & \\ a & & \end{vmatrix}$$

u

d

q

c
x
2

p
x
2

v w

d q b d

q

c p $= (x^4 - 1)$ a c

p $=$ RHS

v w u v

w

OR

Given $A = \begin{bmatrix} 6 & 5 \\ 5 & 4 \end{bmatrix}$

We can write $A = IA$

$$\begin{bmatrix} 1 & 1 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} A$$

[By applying $R_1 \to R_1 - R_2$]

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -5 & 6 \end{bmatrix} A$$

[By applying $R_2 \to R_2 - 5R_1$]

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -4 & 5 \\ -5 & 6 \end{bmatrix} A$$

[By applying $R_1 \to R_1 + R_2$]

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 5 \\ 5 & -6 \end{bmatrix} A$$

[By applying $R_2 \to -R_2$]

$$\therefore \quad A^{-1} = \begin{bmatrix} -4 & 5 \\ 5 & -6 \end{bmatrix}$$

17. Given
$$x = a\left[\cos t + \log \tan \frac{t}{2}\right] \quad \ldots(i)$$
$$y = a(1 + \sin t) \quad \ldots(ii)$$

Differentiating equation (i) w.r.t. t

$$\frac{dx}{dt} = a\left\{-\sin t + \frac{1}{\tan \frac{t}{2}} \cdot \sec^2 \frac{t}{2} \cdot \frac{1}{2}\right\} = a\left\{-\sin t + \frac{1}{2\sin \frac{t}{2} \cos \frac{t}{2}}\right\}$$

$$\Rightarrow \quad \frac{dx}{dt} = a\left\{-\sin t + \frac{1}{\sin t}\right\} = a\left\{\frac{-\sin^2 t + 1}{\sin t}\right\} = a\,\frac{\cos^2 t}{\sin t}$$

Differentiating equation (ii), w.r.t. t

dy

$$\frac{}{dt} = a\,(0 + \cos t) = a \cos t$$

Now, $$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{a \cos t \times \sin t}{a \cos^2 t} = \tan t$$

Now again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan t) = \sec^2 t \cdot \frac{dt}{dx}$$

$$= \sec^2 t \cdot \frac{\sin t}{a\cos^2 t} = \frac{1}{a}\sec^4 t \cdot \sin t$$

19. Let $I = \int_0^1 x^2 (1-x)^n \, dx$

$\Rightarrow$ $$I = \int_0^1 (1-x)^2 [1-(1-x)]^n \, dx \qquad \left(\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$$

$$= \int_0^1 (1-2x+x^2)x^n \, dx = \int_0^1 (x^n - 2x^{n+1} + x^{n+2})\,dx$$

$$= \left[\frac{x^{n+1}}{n+1} - 2\cdot\frac{x^{n+2}}{n+2} + \frac{x^{n+3}}{n+3}\right]_0^1 = \frac{1}{n+1} - \frac{2}{n+2} + \frac{1}{n+3}$$

$$= \frac{(n+2)(n+3) - 2(n+1)(n+3) + (n+1)(n+2)}{(n+1)(n+2)(n+3)}$$

$$= \frac{n^2 + 5n + 6 - 2n^2 - 8n - 6 + n^2 + 3n + 2}{(n+1)(n+2)(n+3)} = \frac{2}{(n+1)(n+2)(n+3)}$$

21. Sum of given vectors is

$$\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda\hat{i} + 4\hat{j} - 5\hat{k} = (1+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}$$

We have given

$$(\hat{i} + 2\hat{j} + 4\hat{k}) \cdot \hat{r} = 1$$

$\Rightarrow$ $$(\hat{i} + 2\hat{j} + 4\hat{k}) \cdot \frac{[(1+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}]}{\sqrt{(1+\lambda)^2 + 36 + 4}} = 1$$

$\Rightarrow$ $$(1+\lambda) + 12 - 8 = \sqrt{(1+\lambda)^2 + 40}$$

$\Rightarrow$ $$\lambda + 5 = \sqrt{(1+\lambda)^2 + 40}$$

Squaring both sides, we get

$$\lambda^2 + 10\lambda + 25 = 1 + 2\lambda + \lambda^2 + 40$$

$\Rightarrow$ $$8\lambda = 16 \quad \Rightarrow \quad \lambda = 2$$

23. Given $$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & -1 \\ -2 & 1 & 1 \end{bmatrix}$$

and
$$2x + y + 3z = 9 \qquad \ldots(i)$$
$$x + 3y - z = 2 \qquad \ldots(ii)$$
$$-2x + y + z = 7 \qquad \ldots(iii)$$

As we know $\quad A^{-1} = \dfrac{1}{|A|} \, Adj. A$

$$|A| = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 3 & -1 \\ -2 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix}$$

$$= 2\,(4) - 1\,(-1) + 3\,(7) = 30 \neq 0$$

$$Adj.\, A = \begin{bmatrix} 4 & 1 & 7 \\ 2 & 8 & -4 \\ -10 & 5 & 5 \end{bmatrix}' = \begin{bmatrix} 4 & 2 & -10 \\ 1 & 8 & 5 \\ 7 & -4 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{30}\begin{bmatrix} 4 & 2 & -10 \\ 1 & 8 & 5 \\ 7 & -4 & 5 \end{bmatrix}$$

Matrix equation form of equations (*i*), (*ii*), (*iii*), is given by

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & -1 \\ -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 7 \end{bmatrix}$$

i.e., $\quad AX = B \;\Rightarrow\; X = A^{-1}B$

$$\Rightarrow \qquad X = \frac{1}{30}\begin{bmatrix} 4 & 2 & -10 \\ 1 & 8 & 5 \\ 7 & -4 & 5 \end{bmatrix}\begin{bmatrix} 9 \\ 2 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{30}\begin{bmatrix} -30 \\ 60 \\ 90 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

By comparing both sides, we get

$$x = -1, \; y = 2, \; z = 3$$

27. Let side of square be a units and radius of a circle be r units.

It is given,

$\therefore \quad 4a + 2\pi r = k$ where k is a constant $\Rightarrow r = \dfrac{k - 4a}{2\pi}$

Sum of areas, $A = a^2 + \pi r^2$

$$\Rightarrow \qquad A = a^2 + \pi\left[\frac{k - 4a}{2\pi}\right]^2 = a^2 + \frac{1}{4\pi}(k - 4a)^2$$

Differentiating w.r.t. x

$$\frac{dA}{da} = 2a + \frac{1}{4\pi}\cdot 2\,(k - 4a)\,.\,(-4) = 2a - \frac{2\,(k - 4a)}{\pi} \qquad \ldots(i)$$

For minimum area, $\frac{dA}{da} = 0$

$$\Rightarrow \qquad 2a - \frac{2\,(k-4a)}{\pi} = 0$$

$$\Rightarrow \quad 2a = \frac{2\,(k-4a)}{\pi} \qquad \Rightarrow \qquad 2a = \frac{2\,(2\pi r)}{\pi}$$

[As $k = 4a + 2\pi r$ given]

$$\Rightarrow \qquad a = 2r$$

Now again differentiating equation (i) w.r.t. x

$$\frac{d^2A}{da^2} = 2 - \frac{2}{\pi}(-4) = 2 + \frac{8}{\pi}$$

at $a = 2\pi$, $\frac{d^2A}{da^2} = 2 + \frac{8}{\pi} > 0$

$\therefore$ For $ax = 2r$, sum of areas is least.

Hence, sum of areas is least when side of the square is double the radius of the circle.

EXAMINATION PAPERS – 2011

CBSE (Delhi) Set–I

Time allowed: 3 hours **Maximum marks: 100**

General Instructions:

1. ***All*** *questions are compulsory.*
2. *The question paper consists of* ***29*** *questions divided into three Sections* ***A, B*** *and* ***C.*** *Section A comprises of* ***10*** *questions of* ***one*** *mark each, Section B comprises of* ***12*** *questions of* ***four*** *marks each and Section C comprises of* ***7*** *questions of* ***six*** *marks each.*
3. ***All*** *questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.*
4. *There is no overall choice. However, internal choice has been provided in* ***4*** *questions of* ***four*** *marks each and* ***2*** *questions of* ***six*** *marks each. You have to attempt only* ***one*** *of the alternatives in all such questions.*
5. *Use of calculators is* ***not*** *permitted.*

SECTION–A

Question numbers 1 to 10 carry one mark each.

1. State the reason for the relation R in the set {1, 2, 3} given by R = {(1, 2), (2, 1)} not to be transitive.
2. Write the value of $\sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]$
3. For a 2 × 2 matrix, $A = [a_{ij}]$, whose elements are given by $a_{ij} = \frac{i}{j}$, write the value of a_{12}.
4. For what value of x, the matrix $\begin{bmatrix} 5-x & x+1 \\ 2 & 4 \end{bmatrix}$ is singular?
5. Write A^{-1} for $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$
6. Write the value of $\int \sec x\,(\sec x + \tan x)\, dx$
7. Write the value of $\int \frac{dx}{x^2+16}$.
8. For what value of 'a' the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $a\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear?

9. Write the direction cosines of the vector $-2\hat{i}+\hat{j}-5\hat{k}$.

10. Write the intercept cut off by the plane $2x+y-z=5$ on x–axis.

SECTION–B

Question numbers 11 to 22 carry 4 marks each.

11. Consider the binary operation* on the set {1, 2, 3, 4, 5} defined by a * b = min. {a, b}. Write the operation table of the operation *.

12. Prove the following:

$$\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]=\frac{x}{2},\quad x\in\left(0,\frac{\pi}{4}\right)$$

OR

Find the value of $\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)$

13. Using properties of determinants, prove that

$$\begin{vmatrix}-a^2 & ab & ac\\ ba & -b^2 & bc\\ ca & cb & -c^2\end{vmatrix}=4a^2b^2c^2$$

14. Find the value of 'a' for which the function f defined as

$$f(x)=\begin{cases}a\sin\frac{\pi}{2}(x+1), & x\le 0\\ \frac{\tan x-\sin x}{x^3}, & x>0\end{cases}$$

is continuous at $x=0$.

15. Differentiate $x^{x\cos x}+\frac{x^2+1}{x^2-1}$ $w.r.t.\,x$

OR

If $x=a(\theta-\sin\theta)$, $y=a(1+\cos\theta)$, find $\frac{d^2y}{dx^2}$

16. Sand is pouring from a pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

OR

Find the points on the curve $x^2+y^2-2x-3=0$ at which the tangents are parallel to x–axis.

17. Evaluate: $\int\frac{5x+3}{\sqrt{x^2+4x+10}}\,dx$

OR

Evaluate: $\int \frac{2x}{(x^2+1)(x^2+3)}\,dx$

18. Solve the following differential equation:
$e^x \tan y\,dx + (1-e^x)\sec^2 y\,dy = 0$

19. Solve the following differential equation:
$\cos^2 x \frac{dy}{dx} + y = \tan x.$

20. Find a unit vector perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a} = 3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b} = \hat{i}+2\hat{j}-2\hat{k}$.

21. Find the angle between the following pair of lines:
$$\frac{-x+2}{-2}=\frac{y-1}{7}=\frac{z+3}{-3} \quad \text{and} \quad \frac{x+2}{-1}=\frac{2y-8}{4}=\frac{z-5}{4}$$
and check whether the lines are parallel or perpendicular.

22. Probabilities of solving a specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that (i) the problem is solved (ii) exactly one of them solves the problem.

SECTION–C

Question numbers 23 to 29 carry 6 marks each.

23. Using matrix method, solve the following system of equations:
$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4,\ \frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1,\ \frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2;\ x, y, z \neq 0$

OR

Using elementary transformations, find the inverse of the matrix
$$\begin{pmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$$

24. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

25. Using integration find the area of the triangular region whose sides have equations $y=2x+1$, $y=3x+1$ and $x=4$.

26. Evaluate: $\int_0^{\pi/2} 2\sin x \cos x \tan^{-1}(\sin x)\,dx$

OR

Evaluate: $\int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx$

27. Find the equation of the plane which contains the line of intersection of the planes $\vec{\mathbf{r}}\cdot(\hat{i}+2\hat{j}+3\hat{k})-4=0$, $\vec{\mathbf{r}}\cdot(2\hat{i}+\hat{j}-\hat{k})+5=0$ and which is perpendicular to the plane $\vec{\mathbf{r}}\cdot(5\hat{i}+3\hat{j}-6\hat{k})+8=0$.

28. A factory makes tennis rackets and cricket bats. A tennis racket takes $1\cdot5$ hours of machine time and 3 hours of craftman's time in its making while a cricket bat takes 3 hours of machine time and 1 hour of craftman's time. In a day, the factory has the availability of not more than 42 hours of machine time and 24 hours of craftsman's time. If the profit on a racket and on a bat is ` 20 and ` 10 respectively, find the number of tennis rackets and crickets bats that the factory must manufacture to earn the maximum profit. Make it as an L.P.P. and solve graphically.

29. Suppose 5% of men and 0.25% of women have grey hair. A grey haired person is selected at random. What is the probability of this person being male? Assume that there are equal number of males and females.

CBSE (Delhi) Set–II

Only those questions, not included in Set–I, are given.

9. Write the value of $\tan^{-1}\left[\tan\dfrac{3\pi}{4}\right]$.

10. Write the value of $\displaystyle\int\frac{\sec^2 x}{\operatorname{cosec}^2 x}\,dx$.

15. Form the differential equation of the family of parabolas having vertex at the origin and axis along positive y–axis.

16. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+\hat{k}$.

19. If the function $f(x)$ given by $f(x)=\begin{cases}3ax+b, & \text{if } x>1\\ 11, & \text{if } x=1\\ 5ax-2b, & \text{if } x<1\end{cases}$

is continuous at $x=1$, find the values of a and b.

20. Using properties of determinants, prove the following:

$$\begin{vmatrix} x & y & z\\ x^2 & y^2 & z^2\\ x^3 & y^3 & z^3\end{vmatrix}=xyz(x-y)(y-z)(z-x)$$

23. Bag I contains 3 red and 4 black balls and Bag II contains 5 red and 6 black balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that it was drawn from Bag II.

29. Show that of all the rectangles with a given perimeter, the square has the largest area.

CBSE (Delhi) Set–III

Only those questions, not included in Set I and Set II, are given.

1. Write the value of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$.

2. Write the value of $\int \frac{2 - 3\sin x}{\cos^2 x}\,dx$

11. Using properties of determinants, prove the following:

$$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$$

12. Find the value of a and b such that the following function $f(x)$ is a continuous function:

$$f(x) = \begin{cases} 5; x \le 2 \\ ax + b; 2 < x < 10 \\ 21; x \ge 10 \end{cases}$$

13. Solve the following differential equation:

$$(1 + y^2)(1 + \log x)\,dx + x\,dy = 0$$

14. If two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 2$, $|\vec{b}| = 1$ and $\vec{a} \cdot \vec{b} = 1$, then find the value of $(3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + 7\vec{b})$.

23. A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

24. Show that of all the rectangles of given area, the square has the smallest perimeter.

Solutions

CBSE (Delhi) Set–I

SECTION – A

1. R is not transitive as

$$(1, 2) \in R,\ (2, 1) \in R \text{ But } (1, 1) \notin R$$

[**Note :** A relation R in a set A is said to be transitive if $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$ $\forall\ a, b, c \in R$]

2. Let $\sin^{-1}\left(-\frac{1}{2}\right) = \theta$ $\left[\because -\frac{1}{2} \in [-1, 1] \Rightarrow \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right]$

$\Rightarrow \quad \sin\theta = -\frac{1}{2} \quad \Rightarrow \quad \sin\theta = \sin\left(-\frac{\pi}{6}\right)$

$\Rightarrow \quad \theta = -\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad \Rightarrow \quad \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

Now, $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right] = \sin\left[\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right]$

$= \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin\left(\frac{2\pi + \pi}{6}\right) = \sin\frac{3\pi}{6} = \sin\frac{\pi}{2} = 1$

3. $\because \quad a_{ij} = \frac{i}{j} \quad \Rightarrow \quad a_{12} = \frac{1}{2}$ [Here $i = 1$ and $j = 2$]

4. If $\begin{bmatrix} 5-x & x+1 \\ 2 & 4 \end{bmatrix}$ is singular matrix.

then $\begin{vmatrix} 5-x & x+1 \\ 2 & 4 \end{vmatrix} = 0$ [$\because$ A square matrix A is called singular if $|A| = 0$]

$\Rightarrow \quad 4(5-x) - 2(x+1) = 0$

$\Rightarrow \quad 20 - 4x - 2x - 2 = 0 \quad \Rightarrow \quad 18 - 6x = 0$

$\Rightarrow \quad 6x = 18 \quad \Rightarrow \quad x = \frac{18}{6} = 3$

5. For elementary row operations we write

$$A = IA$$

$\Rightarrow \quad \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . A$

$\Rightarrow \quad \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . A$ Applying $R_1 \leftrightarrow R_2$

$$\Rightarrow \quad \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} A \qquad \text{Applying } R_2 \to R_2 - 2R_1$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 1 & -2 \end{bmatrix} A \qquad \text{Applying } R_1 \to R_1 + 3R_2$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} A \qquad \text{Applying } R_2 \to (-1)\, R_2$$

$$\Rightarrow \quad I = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} A \quad \Rightarrow \quad A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

[**Note :** B is called inverse of A if $AB = BA = 1$]

6. $\int \sec x\,(\sec x + \tan x)\, dx$

$$= \int \sec^2 x\, dx + \int \sec x \,.\, \tan x\, dx$$

$$= \tan x + \sec x + C$$

$\left[\because \frac{d}{dx}(\tan x) = \sec^2 x \text{ and } \frac{d}{dx}(\sec x) = \sec x \,.\, \tan x\right]$

7. $\int \frac{dx}{x^2 + 16} = \int \frac{dx}{x^2 + 4^2}$

$$= \frac{1}{4} \cdot \tan^{-1} \frac{x}{4} + C$$

$\left[\because \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C\right]$

8. If $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $a\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear

then $\quad \frac{2}{a} = \frac{-3}{6} = \frac{4}{-8} \Rightarrow a = \frac{2 \times 6}{-3} \quad$ or $\quad a = \frac{2 \times -8}{4}$

$\Rightarrow \quad a = -4$

[**Note :** If $\vec{a}$ and $\vec{b}$ are collinear vectors then the respective components of $\vec{a}$ and $\vec{b}$ are proportional.]

9. Direction cosines of vector $-2\hat{i} + \hat{j} - 5\hat{k}$ are

$$\frac{-2}{\sqrt{(-2)^2 + 1^2 + (-5)^2}}, \quad \frac{1}{\sqrt{(-2)^2 + 1^2 + (-5)^2}}, \quad \frac{-5}{\sqrt{(-2)^2 + 1^2 + (-5)^2}}$$

$$\frac{-2}{\sqrt{30}}, \quad \frac{1}{\sqrt{30}}, \quad \frac{-5}{\sqrt{30}}$$

[**Note :** If l, m, n are direction cosine of $a\hat{i} + b\hat{j} + c\hat{k}$ then

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \quad m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \quad n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$]

10. The equation of given plane is

$$2x + y - z = 5$$

$$\Rightarrow \quad \frac{2x}{5} + \frac{y}{5} - \frac{z}{5} = 1 \quad \Rightarrow \quad \frac{x}{5/2} + \frac{y}{5} + \frac{z}{-5} = 1$$

Hence, intercept cut off by the given plane on x-axis is $\frac{5}{2}\cdot$

[**Note :** If a plane makes intercepts a, b, c on x, y and z-axis respectively then its equation is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \,]$$

SECTION – B

11. Required operation table of the operation * is given as

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

12. L.H.S. $= \cot^{-1}\left[\dfrac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right]$

$$= \cot^{-1}\left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \times \frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} + \sqrt{1-\sin x}}\right]$$

$$= \cot^{-1}\left[\frac{(\sqrt{1+\sin x} + \sqrt{1-\sin x})^2}{(\sqrt{1+\sin x})^2 - (\sqrt{1-\sin x})^2}\right]$$

$$= \cot^{-1}\left[\frac{1+\sin x + 1 - \sin x + 2\sqrt{(1+\sin x)(1-\sin x)}}{1+\sin x - 1 + \sin x}\right]$$

$$= \cot^{-1}\left[\frac{2 + 2\sqrt{1-\sin^2 x}}{2\sin x}\right]$$

$$= \cot^{-1}\left(\frac{1+\cos x}{\sin x}\right)$$

$$= \cot^{-1}\left(\frac{2\cos^2 \frac{x}{2}}{2\sin\frac{x}{2}\,.\cos\frac{x}{2}}\right)$$

$\left[\because\ x \in \left(0, \frac{\pi}{4}\right)\right.$

$\Rightarrow\ 0 < x < \frac{\pi}{4}$

$\Rightarrow\ 0 < \frac{x}{2} < \frac{\pi}{8}$

$\left.\Rightarrow\ \frac{x}{2} \in \left(0, \frac{\pi}{8}\right) \subset (0, \pi)\right]$

$$= \cot^{-1}\left(\cot \frac{x}{2}\right)$$

$$= \frac{x}{2} = \text{R.H.S.}$$

OR

$$\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right) = \tan^{-1}\left(\frac{\frac{x}{y} - \frac{x-y}{x+y}}{1 + \frac{x}{y} \cdot \frac{x-y}{x+y}}\right) \qquad \left[\text{Here } \frac{x}{y} \cdot \frac{x-y}{x+y} > -1\right]$$

$$= \tan^{-1}\left(\frac{x^2 + xy - xy + y^2}{y(x+y)} \times \frac{y(x+y)}{xy + y^2 + x^2 - xy}\right)$$

$$= \tan^{-1}\left(\frac{x^2+y^2}{x^2+y^2}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

13. L.H.S. $= \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$

$$= abc \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$ Taking out factor a, b, c from R_1, R_2 and R_3 respectively

$$= a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$ Taking out factor a, b, c from C_1, C_2 and C_3 respectively.

$$= a^2b^2c^2 \begin{vmatrix} 0 & 0 & 2 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$ Applying $R_1 \to R_1 + R_2$

$$= a^2b^2c^2 \,[0 - 0 + 2(1+1)]$$

$$= 4a^2b^2c^2 = \text{R.H.S.}$$

14. $\because f(x)$ is continuous at $x = 0$.

$\Rightarrow$ (L.H.L. of $f(x)$ at $x = 0$) = (R.H.L. of $f(x)$ at $x = 0$) = $f(0)$

$\Rightarrow$ $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) = f(0)$ …(i)

Now, $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0} a \sin \frac{\pi}{2}(x+1)$ $\left[\because f(x) = a \sin \frac{\pi}{2}(x+1), \text{ if } x \le 0\right]$

$$= \lim_{x \to 0} a \sin\left(\frac{\pi}{2} + \frac{\pi}{2} x\right)$$

$$= \lim_{x \to 0} a \cos\frac{\pi}{2} x = a \,.\, \cos 0 = a$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} \frac{\tan x - \sin x}{x^3} \qquad \left[\because f(x) = \frac{\tan x - \sin x}{x^3} \text{ if } x > 0\right]$$

$$= \lim_{x \to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x^3}$$

$$= \lim_{x \to 0} \frac{\sin x - \sin x \,.\, \cos x}{\cos x \,.\, x^3} = \lim_{x \to 0} \frac{\sin x\,(1 - \cos x)}{\cos x \,.\, x^3}$$

$$= \lim_{x \to 0} \frac{1}{\cos x} \,.\, \lim_{x \to 0} \frac{\sin x}{x} \,.\, \frac{2\sin^2 \frac{x}{2}}{\frac{x^2}{4} \times 4} \qquad \left[\because 1 - \cos x = 2\sin^2 \frac{x}{2}\right]$$

$$= 1 \,.\, 1 \,.\, \frac{1}{2} \lim_{\frac{x}{2} \to 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2$$

$$= \frac{1}{2} \cdot \left(\lim_{\frac{x}{2} \to 0} \frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 = \frac{1}{2} \times 1 = \frac{1}{2}$$

Also, $\quad f(0) = a \sin\frac{\pi}{2}(0 + 1)$

$$= a \sin\frac{\pi}{2} = a$$

Putting above values in (i) we get, $a = \frac{1}{2}$

15. Let $y = x^{x \cos x} + \frac{x^2 + 1}{x^2 - 1}$

Let $y = u + v$ where $u = x^{x \cos x}$, $v = \frac{x^2 + 1}{x^2 - 1}$

$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \ldots(i) \quad$ [Differentiating both sides w.r.t. x]

Now, $\quad u = x^{x \cos x}$

Taking log of both sides we get

$$\log u = \log x^{x \cos x} \quad \Rightarrow \quad \log u = x \cos x \,.\, \log x$$

Differentiating both sides w.r.t. x we get

$$\frac{1}{u} \cdot \frac{du}{dx} = 1 \,.\, \cos x \,.\, \log x + (-\sin x) \,.\, x \log x + \frac{1}{x} \,.\, x \cos x$$

$$\frac{1}{u} \cdot \frac{du}{dx} = \cos x \,.\, \log x - x \,.\, \log x \,.\, \sin x + \cos x$$

$$\frac{du}{dx} = x^{x \cos x} \left\{ \cos x \,.\, \log x - x \log x \sin x + \cos x \right\}$$

Again, $$v = \frac{x^2 + 1}{x^2 - 1}$$

$\therefore$ $$\frac{dv}{dx} = \frac{(x^2 - 1) \,.\, 2x - (x^2 + 1) \,.\, 2x}{(x^2 - 1)^2}$$

$$\frac{dv}{dx} = \frac{2x^3 - 2x - 2x^3 - 2x}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2}$$

Putting the values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in (i) we get

$$\frac{dy}{dx} = x^{x \cos x} \{\cos x \,.\, \log x - x \log x \,.\, \sin x + \cos x\} - \frac{4x}{(x^2 - 1)^2}$$

$$= x^{x \cos x} \{\cos x \,.\, (1 + \log x) - x \log x \,.\, \sin x\} - \frac{4x}{(x^2 - 1)^2}$$

OR

Given, $x = a\,(\theta - \sin \theta)$

Differentiating w.r.t. (θ) we get

$$\frac{dx}{d\theta} = a\,(1 - \cos \theta) \qquad \ldots(i)$$

$$y = a\,(1 + \cos \theta)$$

Differentiating w.r.t. θ we get

$$\frac{dy}{d\theta} = a\,(-\sin \theta) = -\,a \sin \theta \qquad \ldots(ii)$$

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-a \sin \theta}{a\,(1 - \cos \theta)}$$ [Putting values from (i) and (ii)]

$$= \frac{-\,2 \sin \frac{\theta}{2} \,.\, \cos \frac{\theta}{2}}{2 \sin^2 \frac{\theta}{2}} = -\cot \frac{\theta}{2}$$

16. Let V, r and h be the volume, radius and height of the sand-cone at time t respectively.

Given, $\frac{dV}{dt} = 12 \text{ cm}^3/\text{s}$

$$h = \frac{r}{6} \quad \Rightarrow \quad r = 6\text{ h}$$

Now, $V = \frac{1}{3}\pi r^2 h \quad \Rightarrow V = \frac{1}{3}\pi 36 h^3 = 12\pi h^3$

Differentiating w.r.t. t we get

$$\frac{dV}{dt} = 12\pi . 3h^2 . \frac{dh}{dt}$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{12}{36\pi h^2} \qquad \left[\because \frac{dV}{dt} = 12 \text{ cm}^2/\text{s}\right]$$

$$\Rightarrow \quad \left[\frac{dh}{dt}\right]_{t=4} = \frac{12}{36\pi \times 16} = \frac{1}{48\pi} \text{ cm/s.}$$

OR

Let required point be (x_1, y_1) on given curve $x^2 + y^2 - 2x - 3 = 0$.

Now, equation of curve is

$$x^2 + y^2 - 2x - 3 = 0$$

Differentiating w.r.t. x we get

$$2x + 2y . \frac{dy}{dx} - 2 = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-2x + 2}{2y}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-2x_1 + 2}{2y_1} = \frac{-x_1 + 1}{y_1}$$

Since tangent at (x_1, y_1) is parallel to x-axis.

$\therefore$ Slope of tangent = 0

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0 \Rightarrow \quad \frac{-x_1 + 1}{y_1} = 0$$

$$\Rightarrow \quad -x_1 + 1 = 0 \quad \Rightarrow \quad x_1 = 1$$

Since (x_1, y_1) lies on given curve $x^2 + y^2 - 2x - 3 = 0$.

$$\therefore \quad x_1^2 + y_1^2 - 2x_1 - 3 = 0$$

$$\Rightarrow \quad 1^2 + y_1^2 - 2 \times 1 - 3 = 0 \qquad [\because x_1 = 1]$$

$$\Rightarrow \quad y_1^2 = 4 \quad \Rightarrow \quad y_1 = \pm 2$$

Hence, required points are (1, 2) and (1, –2).

[**Note :** Slope of tangent at a point (x_1, y_1) on curve $y = f(x)$ is $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$]

17. Let, $$5x + 3 = A \frac{d}{dx}(x^2 + 4x + 10) + B$$

$\Rightarrow$ $$5x + 3 = A(2x + 4) + B \quad \Rightarrow \quad 5x + 3 = 2Ax + (4A + B)$$

Equating coefficient of x and constant, we get

$$2A = 5 \quad \Rightarrow \quad A = \frac{5}{2} \quad \text{and} \quad 4A + B = 3 \quad \Rightarrow \quad B = 3 - 4 \times \frac{5}{2} = -7$$

Hence, $$\int \frac{5x + 3}{\sqrt{x^2 + 4x + 10}}\, dx = \int \frac{\frac{5}{2}(2x + 4) - 7}{\sqrt{x^2 + 4x + 10}}\, dx$$

$$= \frac{5}{2} \int \frac{2x + 4\, dx}{\sqrt{x^2 + 4x + 10}} - 7 \int \frac{dx}{\sqrt{x^2 + 4x + 10}}$$

$$= \frac{5}{2} I_1 - 7 I_2 \qquad \ldots(i)$$

where $I_1 = \int \frac{(2x + 4)\, dx}{\sqrt{x^2 + 4x + 10}}$ and $I_2 = \int \frac{dx}{\sqrt{x^2 + 4x + 10}}$

Now, $$I_1 = \int \frac{(2x + 4)\, dx}{\sqrt{x^2 + 4x + 10}}$$

$$= \int \frac{dz}{\sqrt{z}} = \int z^{-1/2}\, dz \qquad \left[\begin{array}{l} \text{Let } x^2 + 4x + 10 = z \\ \quad (2x + 4)\, dx = dz \end{array}\right]$$

$$= \frac{z^{-\frac{1}{2} + 1}}{-1/2 + 1} + C_1 = 2\sqrt{z} + C_1$$

$$I_1 = 2\sqrt{x^2 + 4x + 10} + C_1 \qquad \ldots(ii)$$

Again $$I_2 = \int \frac{dx}{\sqrt{x^2 + 4x + 10}}$$

$$= \int \frac{dx}{\sqrt{x^2 + 2 \cdot 2 \cdot x + 4 + 6}} = \int \frac{dx}{\sqrt{(x + 2)^2 + (\sqrt{6})^2}}$$

$$= \log\left|(x + 2) + \sqrt{(x + 2)^2 + (\sqrt{6})^2}\right| + C_2$$

$$I_2 = \log\left|(x + 2) + \sqrt{x^2 + 4x + 10}\right| + C_2 \qquad \ldots(iii)$$

Putting the values of I_1 and I_2 in (i) we get

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}\,dx = \frac{5}{2}\times 2\sqrt{x^2+4x+10} - 7\log|(x+2)+\sqrt{x^2+4x+10}| + C$$

$$\left[\text{where } C = \frac{5}{2}C_1 - 7C_2\right]$$

$$= 5\sqrt{x^2+4x+10} - 7\log|(x+2)+\sqrt{x^2+4x+10}| + C$$

OR

Let $x^2 = z \Rightarrow 2x\,dx = dz$

$\therefore$ $$\int \frac{2x\,dx}{(x^2+1)(x^2+3)} = \int \frac{dz}{(z+1)(z+3)}$$

Now, $$\frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3} \quad \ldots(i)$$

$$\frac{1}{(z+1)(z+3)} = \frac{A(z+3)+B(z+1)}{(z+1)(z+3)}$$

$\Rightarrow$ $$1 = A(z+3)+B(z+1) \;\Rightarrow\; 1 = (A+B)z + (3A+B)$$

Equating the coefficient of z and constant, we get

$$A + B = 0 \quad \ldots(ii)$$

and $$3A + B = 1 \quad \ldots(iii)$$

Substracting (*ii*) from (*iii*) we get

$$2A = 1 \quad \Rightarrow \quad A = \frac{1}{2}$$

$\therefore$ $$B = -\frac{1}{2}$$

Putting the values of A and B in (*i*) we get

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

$\therefore$ $$\int \frac{2x\,dx}{(x^2+1)(x^2+3)} = \int \frac{dz}{(z+1)(z+3)}$$

$$= \int\left(\frac{1}{2(z+1)} - \frac{1}{2(z+3)}\right)dz = \frac{1}{2}\int\frac{dz}{z+1} - \frac{1}{2}\int\frac{dz}{z+3}$$

$$= \frac{1}{2}\log|z+1| - \frac{1}{2}\log|z+3| + C = \frac{1}{2}\log|x^2+1| - \frac{1}{2}\log|x^2+3|$$

$$= \frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + C$$

$\left[\textbf{Note:} \log m + \log n = \log m.n \text{ and } \log m - \log n = \log \frac{m}{n}\right]$

$$= \log\sqrt{\frac{x^2+1}{x^2+3}} + C$$

18. $e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0$

$$\Rightarrow \quad (1 - e^x) \sec^2 y\, dy = -e^x \tan y\, dx \quad \Rightarrow \quad \frac{\sec^2 y\, dy}{\tan y} = \frac{-e^x}{1 - e^x} dx$$

Integrating both sides we get

$$\left[\begin{array}{ll} \text{Let} & \tan y = z \\ \Rightarrow & \sec^2 y\, dy = dz \\ \text{Also,} & 1 - e^x = t \\ \Rightarrow & -e^x dx = dt \end{array}\right.$$

$$\Rightarrow \quad \int \frac{\sec^2 y\, dy}{\tan y} = \int \frac{-e^x\, dx}{1 - e^x}$$

$$\Rightarrow \quad \int \frac{dz}{z} = \int \frac{dt}{t}$$

$$\Rightarrow \quad \log z = \log t + \log C \quad \Rightarrow \quad z = t\,C$$

$$\Rightarrow \quad \tan y = (1 - e^x)\,.\,C \qquad \text{[Putting the value of } z \text{ and } t]$$

19. $\cos^2 x\,.\,\frac{dy}{dx} + y = \tan x$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{1}{\cos^2 x}\,.\,y = \frac{\tan x}{\cos^2 x} \quad \Rightarrow \quad \frac{dy}{dx} + \sec^2 xy = \sec^2 x\,.\,\tan x$$

The above equation is in the form of, $\frac{dy}{dx} + Py = Q$

where $P = \sec^2 x, Q = \sec^2 x\,.\,\tan x$

$$\therefore \quad \text{I.F.} = e^{\int P\, dx} = e^{\int \sec^2 x\, dx} = e^{\tan x}$$

Hence, required solution is

$$y \times \text{I. F.} = \int Q \times \text{I. F.}\, dx + C$$

$$\Rightarrow \quad y\,.\,e^{\tan x} = \int \sec^2 x\,.\,\tan x\,.\,e^{\tan x}\, dx + C$$

$$\Rightarrow \quad y.e^{\tan x} = \int z.e^z dz + C \qquad \left[\begin{array}{ll} \text{Let} & \tan x = z \\ \Rightarrow & \sec^2 x\, dx = dz \end{array}\right]$$

$$\Rightarrow \quad y.e^{\tan x} = z.e^z - \int e^z dz + C$$

$$\Rightarrow \quad y.e^{\tan x} = z.e^z - e^z + C$$

$$y.e^{\tan x} = \tan x\,.\,e^{\tan x} - e^{\tan x} + C$$

$$\Rightarrow \quad y = \tan x - 1 + C.e^{-\tan x}$$

20. Given $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$

$\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$

$\therefore \quad \vec{a} + \vec{b} = 4\hat{i} + 4\hat{j}$

$\vec{a} - \vec{b} = 2\hat{i} + 4\hat{k}$

Now, vector perpendicular to $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ is

$$(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix}$$

$$= (16-0)\,\hat{i} - (16-0)\,\hat{j} + (0-8)\,\hat{k} = 16\hat{i} - 16\hat{j} - 8\hat{k}$$

$\therefore$ Unit vector perpendicular to $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ is given by

$$\pm \frac{(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})}{|(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})|}$$

$$= \pm \frac{16\hat{i}-16\hat{j}-8\hat{k}}{\sqrt{16^2+(-16)^2+(-8)^2}} = \pm \frac{8(2\hat{i}-2\hat{j}-\hat{k})}{8\sqrt{2^2+2^2+1^2}}$$

$$= \pm \frac{2\hat{i}-2\hat{j}-\hat{k}}{\sqrt{9}} = \pm\left(\frac{2}{3}\hat{i}-\frac{2}{3}\hat{j}-\frac{\hat{k}}{3}\right)$$

$$= \pm \frac{2}{3}\hat{i} \mp \frac{2}{3}\hat{j} \mp \frac{1}{3}\hat{k}$$

21. The equation of given lines can be written in standard form as

$$\frac{x-2}{2} = \frac{y-1}{7} = \frac{z-(-3)}{-3} \qquad \ldots(i)$$

and

$$\frac{x-(-2)}{-1} = \frac{y-4}{2} = \frac{z-5}{4} \qquad \ldots(ii)$$

If $\vec{b}_1$ and $\vec{b}_2$ are vectors parallel to lines (*i*) and (*ii*) respectively, then

$$\vec{b}_1 = 2\hat{i} + 7\hat{j} - 3\hat{k} \text{ and } \vec{b}_2 = -\hat{i} + 2\hat{j} + 4\hat{k}$$

Obviously, if θ is the angle between lines (*i*) and (*ii*) then θ is also the angle between $\vec{b}_1$ and $\vec{b}_2$.

$\therefore$
$$\cos\theta = \left|\frac{\vec{b}_1 . \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$$

$$= \left|\frac{(2\hat{i}+7\hat{j}-3\hat{k}).(-\hat{i}+2\hat{j}+4\hat{k})}{\sqrt{2^2+7^2+(-3)^2}\,.\sqrt{(-1)^2+2^2+4^2}}\right|$$

$$= \left| \frac{-2+14-12}{\sqrt{62}\,.\,\sqrt{21}} \right| = 0$$

$\Rightarrow$ $$\theta = \frac{\pi}{2}$$

Angle between both lines is 90°.

Hence, given lines are perpendicular to each other.

22. Let A and B be the events that the problem is solved independently by A and B respectively.

$\therefore$ $$P(A) = \frac{1}{2} \quad \text{and} \quad P(B) = \frac{1}{3}$$

$\therefore$ $P(A')$ = Probability of event that the problem is not solved by A

$$= 1 - P(A)$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

$P(B')$ = Probability of event that the problem is not solved by B

$$= 1 - P(B)$$

$$= 1 - \frac{1}{3} = \frac{2}{3}$$

(*i*) P (event that the problem is not solved) = P (event that the problem is not solved by A and B)

$$= P(A' \cap B')$$

$$= P(A') \times P(B') \qquad [\because A \text{ and } B \text{ are independent events}]$$

$$= \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$$

$\therefore$ P (event that the problem is solved) = 1 – P (event that the problem is not solved)

$$= 1 - \frac{1}{3} = \frac{2}{3}$$

(*ii*) P (event that exactly one of them solves the problem)

= P (solved by A and not solved by B or not solved by A and solved by B)

$$= P(A \cap B') + P(A' \cap B)$$

$$= P(A) \times P(B') + P(A') \times P(B)$$

$$= \frac{1}{2} \times \frac{2}{3} + \frac{1}{2} \times \frac{1}{3} = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

[**Note :** If A and B are independent events of same experiment then

(*i*) A' and B are independent (*ii*) A and B' are independent (iii) A' and B' are independent]

SECTION – C

23. Let $\frac{1}{x}=u, \frac{1}{y}=v, \frac{1}{z}=w$

Now the given system of linear equation may be written as

$$2u+3v+10w=4, \quad 4u-6v+5w=1 \quad \text{and} \quad 6u+9v-20w=2$$

Above system of equation can be written in matrix form as

$$AX=B \quad \Rightarrow \quad X=A^{-1}B \qquad \ldots(i)$$

where $A=\begin{bmatrix}2 & 3 & 10\\ 4 & -6 & 5\\ 6 & 9 & -20\end{bmatrix}, X=\begin{bmatrix}u\\ v\\ w\end{bmatrix}, B=\begin{bmatrix}4\\ 1\\ 2\end{bmatrix}$

$$|A|=\begin{vmatrix}2 & 3 & 10\\ 4 & -6 & 5\\ 6 & 9 & -20\end{vmatrix}=2(120-45)-3(-80-30)+10(36+36)$$

$$=150+330+720=1200\neq 0$$

For adj A :

$A_{11}=120-45=75$ $\quad A_{12}=-(-80-30)=110$ $\quad A_{13}=36+36=72$

$A_{21}=-(-60-90)=150,$ $\quad A_{22}=-40-60=-100$ $\quad A_{23}=-(18-18)=0$

$A_{31}=15+60=75$ $\quad A_{32}=-(10-40)=30$ $\quad A_{33}=-12-12=-24$

$$\therefore \quad adj.\,A=\begin{bmatrix}75 & 110 & 72\\ 150 & -100 & 0\\ 75 & 30 & -24\end{bmatrix}'=\begin{bmatrix}75 & 150 & 75\\ 110 & -100 & 30\\ 72 & 0 & -24\end{bmatrix}$$

$$\therefore \quad A^{-1}=\frac{1}{|A|}.\,adj.\,A=\frac{1}{1200}\begin{bmatrix}75 & 150 & 75\\ 110 & -100 & 30\\ 72 & 0 & -24\end{bmatrix}$$

Putting the value of A^{-1}, X and B in (i), we get

$$\begin{bmatrix}u\\ v\\ w\end{bmatrix}=\frac{1}{1200}\begin{bmatrix}75 & 150 & 75\\ 110 & -100 & 30\\ 72 & 0 & -24\end{bmatrix}\begin{bmatrix}4\\ 1\\ 2\end{bmatrix}$$

$$\begin{bmatrix}u\\ v\\ w\end{bmatrix}=\frac{1}{1200}\begin{bmatrix}300+150+150\\ 440-100+60\\ 288+0-48\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/3 \\ 1/5 \end{bmatrix}$$

Equating the corresponding elements of matrix we get

$$u = \frac{1}{2}, v = \frac{1}{3}, w = \frac{1}{5} \quad \Rightarrow \quad x = 2, y = 3, z = 5$$

OR

Let $A = \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

For finding the inverse, using elementary row operation we write

$$A = IA$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 + 3R_1$ and $R_3 \to R_3 - 2R_1$, we get

$$\Rightarrow \quad \begin{bmatrix} 1 & 3 & -2 \\ 0 & 9 & -7 \\ 0 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 - \frac{1}{3} R_2$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 9 & -7 \\ 0 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -1/3 & 0 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to \frac{1}{9} R_2$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -7/9 \\ 0 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -1/3 & 0 \\ 1/3 & 1/9 & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \to R_3 + 5R_2$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -7/9 \\ 0 & 0 & 1/9 \end{bmatrix} = \begin{bmatrix} 0 & -1/3 & 0 \\ 1/3 & 1/9 & 0 \\ -1/3 & 5/9 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 - 3R_3$, $R_2 \to R_2 + 7R_3$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/9 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -1/3 & 5/9 & 1 \end{bmatrix} A$$

Applying $R_3 \to 9R_3$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix} A \quad \Rightarrow \quad I = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix} A$$

Hence,
$$A^{-1} = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix}$$

24. Let x and y be the length and breadth of a rectangle inscribed in a circle of radius r. If A be the area of rectangle then

$$A = x.y$$

$$A = x \,.\, \sqrt{4r^2 - x^2}$$

$$\Rightarrow \quad \frac{dA}{dx} = x \frac{1}{2\sqrt{4r^2 - x^2}} \times (-2x) + \sqrt{4r^2 - x^2}$$

$$\frac{dA}{dx} = -\frac{2x^2}{2\sqrt{4r^2 - x^2}} + \sqrt{4r^2 - x^2}$$

$$\frac{dA}{dx} = \frac{-x^2 + 4r^2 - x^2}{\sqrt{4r^2 - x^2}}$$

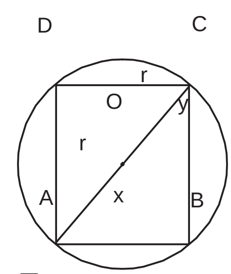
$$\frac{dA}{dx} = \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}}$$

$\left[\because \Delta ABC\right.$ is right angled triangle

$\Rightarrow \quad 4r^2 = x^2 + y^2$

$\Rightarrow \quad y^2 = 4r^2 - x^2$

$\Rightarrow \quad y = \sqrt{4r^2 - x^2}$ $\qquad \ldots(i)$



For maximum or minimum, $\dfrac{dA}{dx} = 0$

$$\Rightarrow \quad \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}} = 0 \quad \Rightarrow \quad 2x^2 = 4r^2 \quad \Rightarrow \quad x = \sqrt{2}r$$

Now,
$$\frac{d^2A}{dx^2} = \frac{\sqrt{4r^2 - x^2}\,.(-4x) - (4r^2 - 2x^2)\,.\,\dfrac{1 \times -2x}{2\sqrt{4r^2 - x^2}}}{(\sqrt{4r^2 - x^2})^2}$$

$$= \frac{-4x\,(4r^2 - x^2) + x\,(4r^2 - 2x^2)}{(4r^2 - x^2)^{3/2}} = \frac{x\{-16r^2 + 4x^2 + 4r^2 - 2x^2\}}{(4r^2 - x^2)^{3/2}}$$

$$= \frac{x(-12r^2 + 2x^2)}{(4r^2 - x^2)^{3/2}}$$

$$\left[\frac{d^2A}{dx^2}\right]_{x=\sqrt{2}r} = \frac{\sqrt{2}r\,(-12r^2 + 2.2r^2)}{(4r^2 - 2r^2)^{3/2}}$$

$$= \frac{\sqrt{2}r \times -8r^2}{(2r^2)^{3/2}} = \frac{-8\sqrt{2}r^3}{2\sqrt{2}r^3} = -4 < 0$$

Hence, A is maximum when $x = \sqrt{2}r$.

Putting $x = \sqrt{2}r$ in (i) we get

$$y = \sqrt{4r^2 - 2r^2} = \sqrt{2}r$$

i.e., $x = y = \sqrt{2}r$

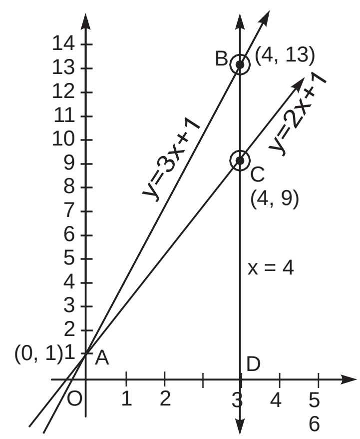
Therefore, Area of rectangle is maximum when $x = y$ *i.e.,* rectangle is square.

25. The given lines are

$$y = 2x + 1 \qquad \ldots(i)$$

$$y = 3x + 1 \qquad \ldots(ii)$$

$$x = 4 \qquad \ldots(iii)$$



For intersection point of (i) and (iii)

$$y = 2 \times 4 + 1 = 9$$

Coordinates of intersecting point of (i) and (iii) is (4, 9)

For intersection point of (ii) and (iii)

$$y = 3 \times 4 + 1 = 13$$

i.e., Coordinates of intersection point of (ii) and (iii) is (4, 13)

For intersection point of (i) and (ii)

$$2x + 1 = 3x + 1 \quad \Rightarrow \quad x = 0$$

$$\therefore \qquad y = 1$$

i.e., Coordinates of intersection point of (i) and (ii) is (0, 1).

Shaded region is required triangular region.

$\therefore$ Required Area = Area of trapezium $OABD$ – Area of trapezium $OACD$

$$= \int_0^4 (3x + 1)\,dx - \int_0^4 (2x + 1)\,dx$$

$$= \left[3\frac{x^2}{2} + x\right]_0^4 - \left[\frac{2x^2}{2} + x\right]_0^4$$

$$= [(24 + 4) - 0] - [(16 + 4) - 0] = 28 - 20$$

$$= 8 \text{ sq. units}$$

26. Let $I = 2\int_0^{\pi/2} \sin x \,.\, \cos x \,.\, \tan^{-1}(\sin x)\, dx$

Let $\sin x = z$, $\cos x\, dx = dz$

If $x = 0$, $z = \sin 0 = 0$

If $x = \frac{\pi}{2}$, $z = \sin\frac{\pi}{2} = 1$

$\therefore \quad I = 2\int_0^1 z \tan^{-1}(z)\, dz$

$$= 2\left[\tan^{-1} z \,.\, \frac{z^2}{2}\right]_0^1 - 2\int_0^1 \frac{1}{1+z^2} \,.\, \frac{z^2}{2}\, dz$$

$$= 2\left[\frac{\pi}{4} \,.\, \frac{1}{2} - 0\right] - \frac{2}{2}\int_0^1 \frac{z^2}{1+z^2}\, dz$$

$$= \frac{\pi}{4} - \int_0^1 \frac{1+z^2-1}{1+z^2}\, dz = \frac{\pi}{4} - \int_0^1 dz + \int_0^1 \frac{dz}{1+z^2}$$

$$= \frac{\pi}{4} - [z]_0^1 + [\tan^{-1} z]_0^1 = \frac{\pi}{4} - 1 + \left[\frac{\pi}{4} - 0\right] = \frac{\pi}{2} - 1$$

OR

Let $\quad I = \int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x}\, dx$

$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) . \sin\left(\frac{\pi}{2} - x\right) . \cos\left(\frac{\pi}{2} - x\right)}{\sin^4\left(\frac{\pi}{2} - x\right) + \cos^4\left(\frac{\pi}{2} - x\right)}\, dx \qquad \left[\text{By Property } \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right.$

$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) \cos x \,.\, \sin x}{\cos^4 x + \sin^4 x}\, dx$

$\Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\cos x \,.\, \sin x}{\sin^4 x + \cos^4 x}\, dx - \int_0^{\pi/2} \frac{x \sin x \,.\, \cos x}{\sin^4 x + \cos^4 x}\, dx$

$\Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x \,.\, \cos x\, dx}{\sin^4 x + \cos^4 x} - I$

$$\Rightarrow \qquad 2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x \,.\cos x\, dx}{\sin^4 x + \cos^4 x} = \frac{\pi}{2}\int_0^{\pi/2} \frac{\dfrac{\sin x\,.\cos x}{\cos^4 x}\,dx}{\tan^4 x + 1}$$

[Dividing numerator and denominator by $\cos^4 x$]

$$= \frac{\pi}{2\times 2}\int_0^{\pi/2} \frac{2\tan x\,.\sec^2 x\, dx}{1+(\tan^2 x)^2}$$

Let $\tan^2 x = z;\ 2\tan x\,.\sec^2 x\,dx = dz$

If $x = 0, z = 0\,;\, x = \frac{\pi}{2}, z = \infty$

$$= \frac{\pi}{4}\int_0^{\infty} \frac{dz}{1+z^2}$$

$$= \frac{\pi}{4}[\tan^{-1} z]_0^{\infty}$$

$$= \frac{\pi}{4}(\tan^{-1}\infty - \tan^{-1} 0)$$

$$\therefore \qquad 2I = \frac{\pi}{4}\left(\frac{\pi}{2} - 0\right) \qquad \Rightarrow \qquad I = \frac{\pi^2}{16}$$

27. The given two planes are

$$\vec{r}\,(\hat{i} + 2\hat{j} + 3\hat{k}) - 4 = 0 \qquad \ldots(i)$$

and $$\vec{r}\,(2\hat{i} + \hat{j} - \hat{k}) + 5 = 0 \qquad \ldots(ii)$$

The equation of a plane passing through line of intersection of the planes (*i*) and (*ii*) is given by

$$\vec{r}\,.(\hat{i} + 2\hat{j} + 3\hat{k}) - 4 + \lambda\,[\mathbf{r}\,.(2\hat{i} + \hat{j} - \hat{k}) + 5] = 0$$

$$\vec{r}\,[(1+2\lambda)\,\hat{i} + (2+\lambda)\,\hat{j} + (3-\lambda)\,\hat{k}] - 4 + 5\lambda = 0 \qquad \ldots(iii)$$

Since, the plane (*iii*) is perpendicular to the plane

$$\vec{r}\,.(5\hat{i} + 3\hat{j} - 6\hat{k}) + 8 = 0 \qquad \ldots(iv)$$

$\Rightarrow$ Normal vector of (*iii*) is perpendicular to normal vector of (*iv*)

$$\Rightarrow \qquad \{(1+2\lambda)\,\hat{i} + (2+\lambda)\,\hat{j} + (3-\lambda)\,\hat{k}\}\,.\{5\hat{i} + 3\hat{j} - 6\hat{k}\} = 0$$

$$\Rightarrow \qquad (1+2\lambda)\times 5 + (2+\lambda)\times 3 + (3-\lambda)\times(-6) = 0$$

$$\Rightarrow \qquad 5 + 10\lambda + 6 + 3\lambda - 18 + 6\lambda = 0$$

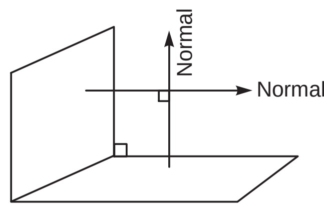
$$\Rightarrow \qquad 19\lambda - 7 = 0$$

$$\Rightarrow \qquad \lambda = \frac{7}{19}$$

Putting the value of λ in (*iii*) we get equation of required plane

$$\vec{r}\cdot\left[\left(1+2\times\frac{7}{19}\right)\hat{i}+\left(2+\frac{7}{19}\right)\hat{j}+\left(3-\frac{7}{19}\right)\hat{k}\right]-4+5\times\frac{7}{19}=0$$

$$\Rightarrow \quad \vec{r}\cdot\left(\frac{33}{19}\hat{i}+\frac{45}{19}\hat{j}+\frac{50}{19}\hat{k}\right)-\frac{41}{19}=0 \quad \Rightarrow \quad \vec{r}\cdot(33\hat{i}+45\hat{j}+50\hat{k})-41=0$$



[**Note :** Normals of two perpendicular planes are perpendicular to each other.

28. Let the number of tennis rackets and cricket bats manufactured by factory be x and y respectively.

Here, profit is the objective function Z.

$\therefore \quad Z = 20x + 10y \quad \ldots(i)$

We have to maximise z subject to the constraints

$1\cdot 5x + 3y \le 42 \quad \ldots(ii)$ [Constraint for machine hour]

$3x + y \le 24 \quad \ldots(iii)$ [Constraint for Craft man's hour]

$x \ge 0$

$y \ge 0$ [Non-negative constraint]

Graph of $x = 0$ and $y = 0$ is the y-axis and x-axis respectively.

$\therefore$ Graph of $x \ge 0, y \ge 0$ is the Ist quadrant.

Graph of $1\cdot 5x + 3y = 42$

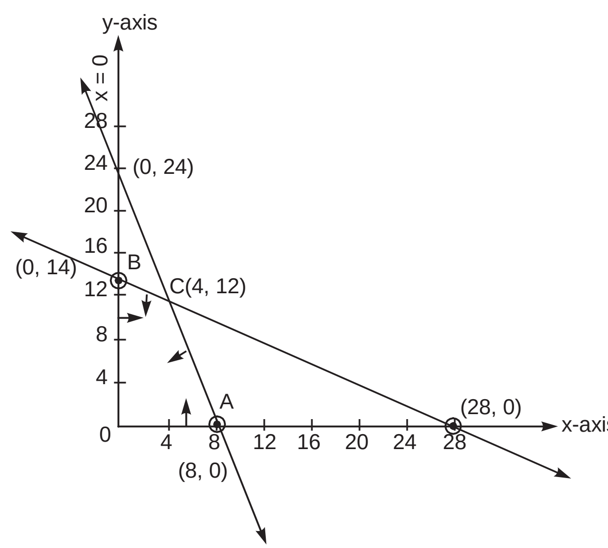
x	0	28
y	14	0

$\therefore$ Graph for $1\cdot 5x + 3y \le 42$ is the part of Ist quadrant which contains the origin.

Graph for $3x + y \le 24$

Graph of $3x + y = 24$

x	0	8
y	24	0



$\therefore$ Graph of $3x + y \le 24$ is the part of Ist quadrant in which origin lie

Hence, shaded area $OACB$ is the feasible region.

For coordinate of C equation $1\cdot 5x + 3y = 42$ and $3x + y = 24$ are solved as

$$1\cdot 5x + 3y = 42 \qquad \ldots(iv)$$

$$3x + y = 24 \qquad \ldots(v)$$

$$2 \times (iv) - (v) \quad \Rightarrow \quad \begin{array}{r} 3x + 6y = 84 \\ \underline{_{-}3x \pm y \;=\; _{-}24} \\ 5y = 60 \end{array}$$

$$\Rightarrow \qquad y = 12$$

$$\Rightarrow \qquad x = 4 \qquad \text{(Substituting } y = 12 \text{ in } (iv))$$

Now value of objective function Z at each corner of feasible region is

Corner Point	$Z = 20x + 10y$
$O\,(0, 0)$	0
$A\,(8, 0)$	$20 \times 8 + 10 \times 0 = 160$
$B\,(0, 14)$	$20 \times 0 + 10 \times 14 = 140$
$C\,(4, 12)$	$20 \times 4 + 10 \times 12 = 200$ ← Maximum

Therefore, maximum profit is ₹ 200, when factory makes 4 tennis rackets and 12 cricket bats.

29. Let E_1, E_2 and A be event such that

E_1 = Selecting male person

E_2 = Selecting women (female person)

A = Selecting grey haired person.

Then $\qquad P(E_1) = \frac{1}{2}, \qquad P(E_2) = \frac{1}{2}$

$$P\left(\frac{A}{E_1}\right) = \frac{5}{100}, \qquad P\left(\frac{A}{E_2}\right) = \frac{0\cdot 25}{100}$$

Here, required probability is $P\left(\frac{E_1}{A}\right)$.

$$\therefore \qquad P\left(\frac{E_1}{A}\right) = \frac{P(E_1)\,.\,P\left(\frac{A}{E_1}\right)}{P(E_1)\,.\,P\left(\frac{A}{E_1}\right) + P(E_2)\,.\,P\left(\frac{A}{E_2}\right)}$$

$$\therefore \qquad P\left(\frac{E_1}{A}\right) = \frac{\frac{1}{2} \times \frac{5}{100}}{\frac{1}{2} \times \frac{5}{100} + \frac{1}{2} \times \frac{0\cdot 25}{100}} = \frac{5}{5 + 0\cdot 25} = \frac{500}{525} = \frac{20}{21}$$

CBSE (Delhi) Set–II

9. $\tan^{-1}\left[\tan \frac{3\pi}{4}\right] = \tan^{-1}\left(\tan\left(\pi - \frac{\pi}{4}\right)\right)$

$$= \tan^{-1}\left(\tan \frac{\pi}{4}\right) = \frac{\pi}{4}$$

$\left[\because \tan^{-1}(\tan x) = x \text{ if } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \text{ Here } \frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$

10. $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x}\, dx = \int \frac{1}{\cos^2 x} \times \frac{\sin^2 x}{1}\, dx$

$$= \int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx$$

$$= \int \sec^2 x\, dx - \int dx = \tan x - x + c$$

15. The equation of parabola having vertex at origin and axis along +ve y-axis is

$$x^2 = 4ay \quad \ldots(i) \qquad \text{where } a \text{ is parameters.}$$

Differentiating w.r.t. x we get, $\quad 2x = 4a \cdot \frac{dy}{dx}$

i.e., $\quad x = 2ay' \qquad \left[\text{where } y' = \frac{dy}{dx}\right]$

$\Rightarrow \quad a = \frac{x}{2y'}$

Putting $a = \frac{x}{2y'}$ in (i) we get

$$x^2 = 4 \cdot \frac{x}{2y'} \cdot y$$

$\Rightarrow \quad y' = \frac{2y}{x} \quad \Rightarrow \quad xy' = 2y$

$\Rightarrow \quad xy' - 2y = 0$

It is required differential equation.

16. Given two vectors are

$$\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k} \quad \text{and} \quad \vec{b} = \hat{i} - 2\hat{j} + \hat{k}$$

If $\vec{c}$ is the resultant vector of $\vec{a}$ and $\vec{b}$ then

$$\vec{c} = \vec{a} + \vec{b}$$

$$= (2\hat{i} + 3\hat{j} - \hat{k}) + (\hat{i} - 2\hat{j} + \hat{k})$$

$$= 3\hat{i} + \hat{j} + 0.\hat{k}$$

Now a vector having magnitude 5 and parallel to $\vec{c}$ is given by

$$= \frac{5\,\vec{c}}{|\vec{c}|} = \frac{5\,(3\hat{i} + \hat{j} + 0\hat{k})}{\sqrt{3^2 + 1^2 + 0^2}} = \frac{15}{\sqrt{10}}\,\hat{i} + \frac{5}{\sqrt{10}}\,\hat{j}$$

It is required vector.

[**Note :** A vector having magnitude l and parallel to $\vec{a}$ is given by $l \cdot \frac{\vec{a}}{|\vec{a}|}$.]

19. $\because f(x)$ is continuous at $x = 1$.

$\Rightarrow$ (L.H.L. of $f(x)$ at $x = 1$) = (R.H.L. of $f(x)$ at $x = 1$) = $f(1)$

$$\Rightarrow \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1) \qquad \ldots(i)$$

Now, $\displaystyle \lim_{x \to 1^-} f(x) = \lim_{x \to 1} 5ax - 2b$ $\qquad$ [$\because f(x) = 5ax - 2f$ if $x < 1$]

$$= 5a - 2b$$

$\displaystyle \lim_{x \to 1^+} f(1) = \lim_{x \to 1} 3ax + b$ $\qquad$ [$\because f(x) = 3ax + b$ if $x > 1$]

$$= 3a + b$$

Also, $\qquad f(1) = 11$

Putting these values in (i) we get

$$5a - 2b = 3a + b = 11$$

$$\Rightarrow \quad 5a - 2b = 11 \qquad \ldots(ii)$$

$$3a + b = 11 \qquad \ldots(iii)$$

On solving (ii) and (iii), we get

$$a = 3, \; b = 2$$

20. L.H.S. $= \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$

$$= xyz \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} \qquad \text{[Taking } x, y, z \text{ common from } C_1, C_2, C_3 \text{ respectively]}$$

$$= xyz \begin{vmatrix} 1 & 0 & 0 \\ x & y - x & z - x \\ x^2 & y^2 - x^2 & z^2 - x^2 \end{vmatrix} \qquad C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$$

$$= xyz\,(y-x)\,(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^2 & y+x & z+x \end{vmatrix}$$ [Taking common $(y-x)$ and $(z-x)$ from C_2 and C_3 respectively]

$= xyz\,(y-x)\,(z-x)\,[1\,(z+x-y-x)]$ [Expanding along R_1]

$= xyz\,(y-x)\,(z-x)\,(z-y)$

$= xyz\,(x-y)\,(y-z)\,(z-x)$

23. Let E_1, E_2 and A be event such that

E_1 = choosing the bag I

E_2 = choosing the bag II

A = drawing red ball

Then, $P(E_1) = \frac{1}{2}$, $P(E_2) = \frac{1}{2}$ and $P\left(\frac{A}{E_1}\right) = \frac{3}{7}$, $P\left(\frac{A}{E_2}\right) = \frac{5}{11}$

$P\left(\frac{E_2}{A}\right)$ is required.

By Baye's theorem, $$P\left(\frac{E_2}{A}\right) = \frac{P(E_2)\,.\,P\left(\frac{A}{E_2}\right)}{P(E_1)\,.\,P\left(\frac{A}{E_1}\right) + P(E_2)\,.\,P\left(\frac{A}{E_2}\right)}$$

$$= \frac{\frac{1}{2} \times \frac{5}{11}}{\frac{1}{2} \times \frac{3}{7} + \frac{1}{2} \times \frac{5}{11}} = \frac{\frac{5}{11}}{\frac{3}{7} + \frac{5}{11}}$$

$$= \frac{5}{11} \times \frac{77}{68} = \frac{35}{68}$$

29. Let the length and breadth of rectangle be x and y.

If A and P are the area and perimeter of rectangle respectively then

$A = x\,.\,y$ and $P = 2(x+y)$

$\Rightarrow$ $A = x\left(\frac{P}{2} - x\right)$ $\left(\because y = \frac{P}{2} - x\right)$

$\Rightarrow$ $A = \frac{P}{2}x - x^2$ $\frac{dA}{dx} = \frac{P}{2} - 2x$

$\Rightarrow$ For maximum and minimum of A. $\frac{dA}{dx} = 0$

$\Rightarrow$ $\frac{P}{2} - 2x = 0$ $\Rightarrow$ $x = \frac{P}{4}$

Again $\dfrac{d^2A}{dx^2} = -2$

$\Rightarrow \qquad \left(\dfrac{d^2A}{dx^2}\right)_{x=\frac{P}{4}} = 0$

Hence, A is maximum for $x = \dfrac{P}{4}$

$\therefore \qquad y = \dfrac{P}{2} - \dfrac{P}{4} = \dfrac{P}{4}$

Therefore, for largest area of rectangle $x = y = \dfrac{P}{4}$ *i.e.*, with given perimeter, rectangle having largest area must be square.

CBSE (Delhi) Set–III

1. $\cos^{-1}\left(\cos\dfrac{7\pi}{6}\right) = \cos^{-1}\left(\cos\left(2\pi - \dfrac{5\pi}{6}\right)\right)$ $\qquad \left[\because \dfrac{5\pi}{6} \in [0, \pi]\right]$

$= \cos^{-1}\left(\cos\left(\dfrac{5\pi}{6}\right)\right)$ $\qquad [\because \cos(2\pi - \theta) = \cos\theta]$

$= \dfrac{5\pi}{6}$ $\qquad \left[\because \cos^{-1}(\cos x) = x \text{ if } x \in [0, \pi] \text{ Here } \dfrac{5\pi}{6} \in [0, \pi]\right]$

2. Let $I = \displaystyle\int \frac{2 - 3\sin x}{\cos^2 x}\, dx$

$= \displaystyle\int \frac{2}{\cos^2 x}\, dx - \int \frac{3\sin x\, dx}{\cos^2 x}$

$= 2\displaystyle\int \sec^2 x\, dx - 3\int \frac{-dz}{z^2}$ $\qquad$ [Let $\cos x = z \; - \sin x\, dx = dz$]

$= 2\tan x + 3\dfrac{z^{-2+1}}{-2+1} + c$

$= 2\tan x - \dfrac{3}{\cos x} + c$

11. L.H.S. $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$

$= \begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$ $\qquad R_1 \to R_1 + R_2 + R_3$

$$= (5x+4)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} \qquad \text{[Taking } (5x+4) \text{ common from } R_1]$$

$$= (5x+4)\begin{vmatrix} 1 & 0 & 0 \\ 2x & 4-x & 0 \\ 2x & 0 & 4-x \end{vmatrix} \qquad \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$$

$$= (5x+4)\,[1\,\{(4-x)^2 - 0\} + 0 + 0] \qquad \text{[Expanding along } R_1]$$

$$= (5x+4)\,(4-x)^2 = \text{R.H.S.}$$

12. Since $f(x)$ is continuous.

$\Rightarrow$ $f(x)$ is continuous at $x = 2$ and $x = 10$.

$\Rightarrow$ (L.H.L. of $f(x)$ at $x = 2$) = (R.H.L. of $f(x)$ at $x = 2$) = $f(x)$

$$\Rightarrow \quad \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2) \qquad \ldots(i)$$

Similarly,
$$\lim_{x \to 10^-} f(x) = \lim_{x \to 10^+} f(x) = f(10) \qquad \ldots(ii)$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2} 5 \qquad [\because f(x) = 5 \text{ if } x \le 2]$$

$$= 5$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2} ax + b \qquad [\because f(x) = ax + b \text{ if } x > 2]$$

$$= 2a + b$$

$$f(2) = 5$$

Putting these values in (*i*) we get

$$2a + b = 5 \qquad \ldots(iii)$$

Again
$$\lim_{x \to 10^-} f(x) = \lim_{x \to 10} ax + b \qquad [\because f(x) = ax + b \text{ if } x < 10]$$

$$= 10a + b$$

$$\lim_{x \to 10^+} f(x) = \lim_{x \to 10} 21 \qquad [\because f(x) = 21 \text{ if } x > 10]$$

$$= 21$$

$$f(10) = 21$$

Putting these values in (*ii*) we get

$$10a + b = 21 = 21$$

$$\Rightarrow \quad 10a + b = 21 \qquad \ldots(iv)$$

Substracting (*iii*) from (*iv*) we get

$$10a + b = 21$$
$$\underline{\_\ 2a \pm b = \_\ 5}$$
$$8a = 16$$
$$a = 2$$
$\therefore$ $$b = 5 - 2 \times 2 = 1$$
$$a = 2, b = 1$$

13. $(1 + y^2)(1 + \log x)\, dx + x dy = 0$

$$x\, dy = -(1 + y^2)(1 + \log x)\, dx$$

$\Rightarrow$ $$\frac{dy}{1 + y^2} = -\frac{1 + \log x}{x}\, dx$$

Integrating both sides we get

$$\int \frac{dy}{1 + y^2} = -\int \frac{1 + \log x}{x}\, dx$$

$\Rightarrow$ $$\tan^{-1} y = -\int z\, dz \qquad \begin{bmatrix} \text{Let } 1 + \log x = z \\ \frac{1}{x} dx = dz \end{bmatrix}$$

$\Rightarrow$ $$\tan^{-1} y = -\frac{z^2}{2} + c$$

$\Rightarrow$ $$\tan^{-1} y = -\frac{1}{2}(1 + \log x)^2 + c$$

14. Given $|\vec{a}| = 2, |\vec{b}| = 1$ and $\vec{a} . \vec{b} = 1$

Now, $$(3\vec{a} - 5\vec{b}).(2\vec{a} + 7\vec{b}) = 3\vec{a}.2\vec{a} + 3\vec{a}.7\vec{b} - 5\vec{b}.2\vec{a} - 5\vec{b}.7\vec{b}$$
$$= 6\vec{a}.\vec{a} + 21\vec{a}.\vec{b} - 10\vec{b}.\vec{a} - 35\vec{b}.\vec{b}$$
$$= 6|\vec{a}|^2 + 11\vec{a}.\vec{b} - 35|\vec{b}|^2$$
$$= 6(2)^2 + 11 \times 1 - 35(1)^2$$
$$= 24 + 11 - 35 = 0$$

[**Note :** $\vec{\mathbf{a}} . \vec{\mathbf{a}} = |\vec{\mathbf{a}}|.|\vec{\mathbf{a}}| \cos 0° = |\vec{\mathbf{a}}|^2 \times 1 = |\vec{\mathbf{a}}|^2$

Also, scalar product of vectors is commutative

$\therefore$ $$\vec{a}.\vec{b} = \vec{b}.\vec{a}$$

23. Let E_1, E_2 and A be event such that

E_1 = Occurring six on die.

E_2 = Not occurring six on die.

A = Reporting six by man on die.

Here $P(E_1) = \frac{1}{6}$, $P(E_2) = \frac{5}{6}$

$P\left(\frac{A}{E_1}\right) = P$ (Speaking truth *i.e.*, man reports six on die when six has occurred on the die)

$= \frac{3}{4}$

$P\left(\frac{A}{E_2}\right) = P$ (Not speaking truth *i.e.*, man report six on die when six has not occurred on die)

$= 1 - \frac{3}{4} = \frac{1}{4}$

Required probability is $P\left(\frac{E_1}{A}\right)$.

By Baye's theorem, $P\left(\frac{E_1}{A}\right) = \dfrac{P(E_1) \cdot P\left(\frac{A}{E_1}\right)}{P(E_1) \cdot P\left(\frac{A}{E_1}\right) + P(E_2) \cdot P\left(\frac{A}{E_2}\right)}$

$$= \frac{\frac{1}{6} \times \frac{3}{4}}{\frac{1}{6} \times \frac{3}{4} + \frac{5}{6} \times \frac{1}{4}} = \frac{3}{24} \times \frac{24}{3+5} = \frac{3}{8}$$

24. Let x, y be the length and breadth of rectangle whose area is A and perimeter is P.

$\therefore \quad P = 2(x + y)$

$\Rightarrow \quad P = 2\left(x + \frac{A}{x}\right) \qquad \left[\because A = x \cdot y \quad y = \frac{A}{x}\right]$

For maximum or minimum value of perimeter P

$$\frac{dP}{dx} = 2\left(1 - \frac{A}{x^2}\right) = 0$$

$\Rightarrow \quad 1 - \frac{A}{x^2} = 0 \quad \Rightarrow \quad x^2 = A$

$\Rightarrow \quad x = \sqrt{A}$ [Dimensions of rectangle is always positive]

Now, $\quad \frac{d^2P}{dx^2} = 2\left(0 - A \times \frac{-1}{x^3}\right) = \frac{2A}{x^3}$

$\therefore \quad \left[\frac{d^2P}{dx^2}\right]_{x = \sqrt{A}} = \frac{2a}{(\sqrt{A})^3} > 0$

i.e., for $x = \sqrt{A}$, P (perimeter of rectangle) is smallest.

$\therefore \quad y = \frac{A}{x} = \frac{A}{\sqrt{A}} = \sqrt{A}$

Hence, for smallest perimeter, length and breadth of rectangle are equal ($x = y = \sqrt{A}$) *i.e.*, rectangle is square.

EXAMINATION PAPERS –2011

CBSE (All India) Set–I

Time allowed: 3 hours **Maximum marks: 100**

General Instructions: *As given in Examination Paper (Delhi) – 2011.*

SECTION–A

Question numbers 1 to 10 carry 1 mark each.

1. Let A = {1, 2, 3}, B = {4, 5, 6, 7} and let f = {(1, 4), (2,5), (3, 6)} be a function from A to B. State whether f is one-one or not.

2. What is the principal value of $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)+\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$?

3. Evaluate:

$$\begin{vmatrix} \cos 15^\circ & \sin 15^\circ \\ \sin 75^\circ & \cos 75^\circ \end{vmatrix}$$

4. If $A=\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, write A^{-1} in terms of A.

5. If a matrix has 5 elements, write all possible orders it can have.

6. Evaluate: $\int (ax+b)^3\, dx$

7. Evaluate: $\int \frac{dx}{\sqrt{1-x^2}}$

8. Write the direction-cosines of the line joining the points (1, 0, 0) and (0, 1, 1).

9. Write the projection of the vector $\hat{i}-\hat{j}$ on the vector $\hat{i}+\hat{j}$.

10. Write the vector equation of the line given by $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$.

SECTION–B

Question numbers 11 to 22 carry 4 marks each.

11. Let f : R→R be defined as $f(x)=10x+7$. Find the function g : R→R such that $gof = fog = I_R$.

OR

A binary operation $*$ on the set {0, 1, 2, 3, 4, 5} is defined as:

$$a * b = \begin{cases} a+b, & \text{if } a+b<6 \\ a+b-6, & \text{if } a+b \geq 6 \end{cases}$$

Show that zero is the identity for this operation and each element 'a' of the set is invertible with 6–a, being the inverse of 'a'.

12. Prove that:

$$\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right] = \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x, -\frac{1}{\sqrt{2}} \leq x \leq 1$$

13. Using properties of determinants, solve the following for x:

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$$

14. Find the relationship between 'a' and 'b' so that the function 'f' defined by:

$f(x) = \begin{cases} ax+1, & \text{if } x \leq 3 \\ bx+3, & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$.

OR

If $x^y = e^{x-y}$, show that $\dfrac{dy}{dx} = \dfrac{\log x}{\{\log(xe)\}^2}$.

15. Prove that $y = \dfrac{4\sin\theta}{(2+\cos\theta)} - \theta$ is an increasing function in $\left[0, \dfrac{\pi}{2}\right]$.

OR

If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its surface area.

16. If $x = \tan\left(\dfrac{1}{a}\log y\right)$, show that

$$(1+x^2)\frac{d^2y}{dx^2} + (2x-a)\frac{dy}{dx} = 0$$

17. Evaluate: $\displaystyle\int_0^{\pi/2} \frac{x+\sin x}{1+\cos x}\,dx$

18. Solve the following differential equation:

$$x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$$

19. Solve the following differential equation:

$$(y+3x^2)\frac{dx}{dy} = x.$$

20. Using vectors, find the area of the triangle with vertices A (1, 1, 2), B (2, 3, 5) and C (1, 5, 5).

21. Find the shortest distance between the following lines whose vector equations are:

$$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k} \text{ and } \vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}.$$

22. A random variable X has the following probability distribution:

X	0	1	2	3	4	5	6	7
P (X)	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2 + K$

Determine:

(i) K (ii) P (X < 3) (iii) P (X > 6) (iv) P (0 < X < 3)

OR

Find the probability of throwing at most 2 sixes in 6 throws of a single die.

SECTION–C

Question numbers 23 to 29 carry 6 marks each.

23. Using matrices, solve the following system of equations:

$$4x + 3y + 3z = 60, \quad x + 2y + 3z = 45 \text{ and } 6x + 2y + 3z = 70$$

24. Show that the right-circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base.

OR

A window has the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is 12 m, find the dimensions of the rectangle that will produce the largest area of the window.

25. Evaluate: $\displaystyle\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

OR

Evaluate: $\displaystyle\int \frac{6x+7}{\sqrt{(x-5)(x-4)}}\,dx$

26. Sketch the graph of $y = |x+3|$ and evaluate the area under the curve $y = |x+3|$ above x-axis and between $x = -6$ to $x = 0$.

27. Find the distance of the point (–1, –5, –10), from the point of intersection of the line $\vec{r} = (2\hat{i} - \hat{j} + 2k) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r}\cdot(\hat{i} - \hat{j} + \hat{k}) = 5$.

28. Given three identical boxes I, II and III each containing two coins. In box I, both coins are gold coins, in box II, both are silver coins and in box III, there is one gold and one silver coin. A person chooses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold?

29. A merchant plans to sell two types of personal computers — a desktop model and a portable model that will cost ` 25,000 and ` 40,000 respectively. He estimates that the total monthly demand of computers will not exceed 250 units. Determine the number of units of each type of computers which the merchant should stock to get maximum profit if he does not want to invest more than ` 70 lakhs and his profit on the desktop model is ` 4,500 and on the portable model is ` 5,000. Make an L.P.P. and solve it graphically.

CBSE (All India) Set–II

Only those questions, not included in Set-I, are given.

9. Evaluate:

$$\int \frac{(\log x)^2}{x}\, dx.$$

10. Write the unit vector in the direction of the vector $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$.

19. Prove the following:

$$2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{31}{17}\right)$$

20. Using properties of determinants, solve the following for x:

$$\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$$

21. Evaluate:

$$\int_0^{\pi/4} \log(1+\tan x)\, dx$$

22. Solve the following differential equation:

$$x\,dy - (y + 2x^2)\,dx = 0$$

28. Using matrices, solve the following system of equations:

$$x + 2y + z = 7, \quad x + 3z = 11 \quad \text{and} \quad 2x - 3y = 1$$

29. Find the equation of the plane passing through the line of intersection of the planes $\vec{r}\cdot(\hat{i} + \hat{j} + \hat{k}) = 1$ and $\vec{r}\cdot(2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$ and parallel to x-axis.

CBSE (All India) Set–III

Only those questions, not included in Set I and Set II, are given.

1. Evaluate: $\displaystyle\int \frac{e^{\tan^{-1} x}}{1+x^2}\, dx$

2. Write the angle between two vectors $\vec{a}$ and $\vec{b}$, with magnitudes $\sqrt{3}$ and 2 respectively having $\vec{a}\cdot\vec{b} = \sqrt{6}$.

11. Prove that : $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$

12. Using properties of determinants, solve the following for x:

$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

13. Evaluate: $\displaystyle\int_0^1 \log\left(\frac{1}{x}-1\right)dx$

14. Solve the following differential equation: $x\,dx + (y - x^3)\,dx = 0$

23. Using matrices, solve the following system of equations:

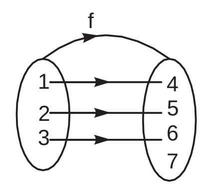
$$x + 2y - 3z = -4, \quad 2x + 3y + 2z = 2 \quad \text{and} \quad 3x - 3y - 4z = 11$$

24. Find the equation of the plane passing through the line of intersection of the planes $2x + y - z = 3$ and $5x - 3y + 4z + 9 = 0$ and parallel to the line $\dfrac{x-1}{2} = \dfrac{y-3}{4} = \dfrac{z-5}{5}$.

Solutions

CBSE (All India) Set–I

SECTION – A



1. f is one-one because

$$f(1) = 4;\ f(2) = 5;\ f(3) = 6$$

No two elements of A have same f image.

2. $\cos^{-1}\left(\cos\dfrac{2\pi}{3}\right) + \sin^{-1}\left(\sin\dfrac{2\pi}{3}\right) = \cos^{-1}\left(\cos\dfrac{2\pi}{3}\right) + \sin^{-1}\left(\sin\left(\pi - \dfrac{\pi}{3}\right)\right)$ $\left[\because \dfrac{2\pi}{3} \notin \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)\right]$

$$= \cos^{-1}\left(\cos\frac{2\pi}{3}\right) + \sin^{-1}\left(\sin\frac{\pi}{3}\right)$$

$$= \frac{2\pi}{3} + \frac{\pi}{3}$$

$$= \frac{3\pi}{3} = \pi$$

[**Note :** By Property of inverse functions

$\sin^{-1}(\sin x) = x \quad \text{if } x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

and $\cos^{-1}(\cos x) = x \quad \text{if } x \in [0, \pi]$]

3. Expanding the determinant, we get

$$\cos 15^\circ \cdot \cos 75^\circ - \sin 15^\circ \cdot \sin 75^\circ$$

$$= \cos(15^\circ + 75^\circ) = \cos 90^\circ = 0$$

[**Note :** $\cos(A + B) = \cos A \cdot \cos B - \sin \cdot \sin B$]

4. $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$

$\therefore \qquad |A| = \begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix} = -4 - 15 = -19 \neq 0$

$\Rightarrow A$ is invertible matrix.

Here, $C_{11} = -2, C_{12} = -5, C_{21} = -3, C_{22} = 2$

$\therefore \qquad adj\ A = \begin{bmatrix} -2 & -5 \\ -3 & 2 \end{bmatrix}^T = \begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$

$\therefore \qquad A^{-1} = \dfrac{1}{|A|} . adj\ A$

$$= \frac{1}{-19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix} = \frac{1}{19}\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$$

$$= \frac{1}{19} A$$ [**Note :** C_{ij} is cofactor a_{ij} of $A = [a_{ij}]$]

5. Possible orders are 1×5 and 5×1.

6. $\int (ax+b)^3\ dx$

Let $\qquad ax + b = z$

$adx = dz \quad \Rightarrow \quad dx = \dfrac{dz}{a}$

$\therefore \qquad \int (ax+b)^3\ dx = \int z^3 . \dfrac{dz}{a}$

$$= \frac{1}{a}\frac{z^4}{4} + c = \frac{1}{4a}(ax+b)^4 + c$$

7. $\int \dfrac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$. Because $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$.

8. Direction ratios of line joining (1, 0, 0) and (0, 1, 1) are

$0-1, \quad 1-0, \quad 1-0$

i.e., $\quad -1, \quad 1, \quad 1$

$\therefore$ Direction cosines of line joining (1, 0, 0) and (0, 1, 1) are

$$\frac{-1}{\sqrt{(-1)^2+(1)^2+(1)^2}}, \frac{1}{\sqrt{(-1)^2+(1)^2+(1)^2}}, \frac{1}{\sqrt{(-1)^2+(1)^2+(1)^2}}$$

$$-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$$

9. Let $\vec{a} = \hat{i} - \hat{j}$, $\vec{b} = \hat{i} + \hat{j}$

Now, projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

$$= \frac{(\hat{i} - \hat{j}) \cdot (\hat{i} + \hat{j})}{|\hat{i} + \hat{j}|} = \frac{1 - 1}{\sqrt{1^2 + 1^2}} = 0$$

10. The given equation of line may written as

$$\frac{x-5}{3} = \frac{y-(-4)}{7} = \frac{z-6}{2}$$

Here, $\vec{a} = 5\hat{i} - 4\hat{j} + 6\hat{k}$ and $\vec{b} = 3\hat{i} + 7\hat{j} + 2\hat{k}$

Hence, required vector equation is

$$\vec{r} = \vec{a} + \lambda \vec{b}$$

i.e., $$\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda (3\hat{i} + 7\hat{j} + 2\hat{k})$$

SECTION – B

11. $\because$ $gof = fog = I_R$

$\Rightarrow$ $fog = I_R$

$\Rightarrow$ $fog(x) = I(x)$

$\Rightarrow$ $f(g(x)) = x$ [$\because I(x) = x$ being identity function]

$\Rightarrow$ $10(g(x)) + 7 = x$ [$\because f(x) = 10x + 7$]

$\Rightarrow$ $g(x) = \dfrac{x-7}{10}$

i.e., $g : R \to R$ is a function defined as $g(x) = \dfrac{x-7}{10}$.

OR

For Identity Element :

Let a be an arbitrary element of set {0, 1, 2, 3, 4, 5}

Now, $a * 0 = a + 0 = a$ …(*i*)

$0 * a = 0 + a = a$ …(*ii*) [$\because a + 0 = 0 + a < 6 \ \forall \ a \in \{0, 1, 2, 3, 4, 5\}$]

Eq. (*i*) and (*ii*) $\Rightarrow$ $a * 0 = 0 * a = a \ \ \forall a \in \{0, 1, 2, 3, 4, 5\}$

Hence, 0 is identity for binary operation $*$.

For Inverse :

Let a be an arbitrary element of set {0, 1, 2, 3, 4, 5}.

Now, $a * (6 - a) = a + (6 - a) - 6$ [$\because a + (6 - a) \geq 6$]

$$= a + 6 - a - 6$$

$$= 0 \text{ (identity)} \qquad \ldots(i)$$

Also, $$(6 - a) * a = (6 - a) + a - 6 \qquad [\because a + (6 - a) \geq 6]$$

$$= 6 - a + a - 6$$

$$= 0 \text{ (identity)} \qquad \ldots(ii)$$

Eq. (*i*) and (*ii*) $\Rightarrow a * (6 - a) = (6 - a) * a = 0$ (identity) $\forall\ a \in \{0, 1, 2, 3, 4, 5\}$

Hence, each element '*a*' of given set is invertible with inverse $6 - a$.

12. Let $x = \sin\theta \Rightarrow \theta = \sin^{-1} x$

$$\left[\begin{array}{l} \because -\frac{1}{\sqrt{2}} \leq x \leq 1 \\ \Rightarrow \sin\left(-\frac{\pi}{4}\right) \leq \sin\theta \leq \sin\frac{\pi}{2} \\ \Rightarrow \theta \in \left[-\frac{\pi}{4}, \frac{\pi}{2}\right] \end{array}\right]$$

Now, $$\tan^{-1}\left[\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right]$$

$$= \tan^{-1}\left[\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \times \frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} - \sqrt{1-x}}\right]$$

$$= \tan^{-1}\left[\frac{(\sqrt{1+x} - \sqrt{1-x})^2}{(\sqrt{1+x})^2 - (\sqrt{1-x})^2}\right]$$

$$= \tan^{-1}\left[\frac{(\sqrt{1+x})^2 + (\sqrt{1-x})^2 - 2 . \sqrt{1+x} . \sqrt{1-x}}{1 + x - 1 + x}\right]$$

$$= \tan^{-1}\left[\frac{1 + x + 1 - x - 2\sqrt{1-x^2}}{2x}\right] = \tan^{-1}\left[\frac{1 - \sqrt{1-x^2}}{x}\right]$$

$$= \tan^{-1}\left[\frac{1 - \sqrt{1-\sin^2\theta}}{\sin\theta}\right] = \tan^{-1}\left[\frac{1 - \cos\theta}{\sin\theta}\right]$$

$$= \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2} . \cos\frac{\theta}{2}}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= \frac{\theta}{2} = \frac{1}{2}\sin^{-1} x$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - \cos^{-1} x\right) \qquad \left[\begin{array}{l} Q \ \sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2} \\ \text{and} \quad x \in \left[-\dfrac{1}{2}, 1\right] \subset [-1, 1] \end{array}\right]$$

13. Given,
$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x-2 & 1 & 2 \\ x-4 & -1 & -4 \\ x-8 & -11 & -40 \end{vmatrix} = 0 \qquad \begin{matrix} C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 - 3C_1 \end{matrix}$$

$$\Rightarrow \quad \begin{vmatrix} x-2 & 1 & 2 \\ -2 & -2 & -6 \\ -6 & -12 & -42 \end{vmatrix} = 0 \qquad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

expanding along R_1 we get

$\Rightarrow \quad (x-2)(84-72) - 1(84-36) + 2(24-12) = 0$

$\Rightarrow \quad 12x - 24 - 48 + 24 = 0 \quad \Rightarrow \quad 12x = 48$

$\Rightarrow \quad x = 4$

14. Since, $f(x)$ is continuous at $x = 3$.

$$\Rightarrow \quad \lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3) \qquad \ldots(i)$$

Now,
$$\lim_{x \to 3^-} f(x) = \lim_{h \to 0} f(3-h) \qquad \left[\begin{matrix} \text{Let } x = 3-h \\ x \to 3^- \Rightarrow h \to 0 \end{matrix}\right.$$

$$= \lim_{h \to 0} a(3-h) + 1 \qquad [\because f(x) = ax + 1 \ \forall \ x \le 3]$$

$$= \lim_{h \to 0} 3a - ah + 1 = 3a + 1$$

$$\lim_{x \to 3^+} f(x) = \lim_{h \to 0} f(3+h) \qquad \left[\begin{matrix} \text{Let } x = 3+h \\ x \to 3^+ \Rightarrow h \to 0 \end{matrix}\right]$$

$$= \lim_{h \to 0} b(3+h) + 3 \qquad [\because f(x) = bx + 3 \ \forall \ x > 3]$$

$$= 3b + 3$$

From (i),
$$3a + 1 = 3b + 3$$
$$3a - 3b = 2$$
$$a - b = \frac{2}{3} \quad \text{or} \quad 3a - 3b = 2 \quad \text{which is the required relation.}$$

OR

Given, $x^y = e^{x-y}$

Taking log of both sides

$$\Rightarrow \quad \log x^y = \log e^{x-y}$$

$\Rightarrow \quad y \,.\, \log x = (x - y) \log e \qquad [\because \log e = 1]$

$\Rightarrow \quad y \,.\, \log x = (x - y) \quad \Rightarrow \quad y \log x + y = x$

$\Rightarrow \quad y = \dfrac{x}{1 + \log x}$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{(1 + \log x) \,.\, 1 - x \,.\left(0 + \dfrac{1}{x}\right)}{(1 + \log x)^2}$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1 + \log x - 1}{(1 + \log x)^2} = \dfrac{\log x}{(\log e + \log x)^2} \qquad [\because 1 = \log e]$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{\log x}{(\log ex)^2} \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{\log x}{\{\log (ex)\}^2}.$

15. Given, $\quad y = \dfrac{4 \sin \theta}{2 + \cos \theta} - \theta$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{(2 + \cos\theta) \,.\, 4 \cos\theta - 4 \sin\theta \,.\, (0 - \sin\theta)}{(2 + \cos\theta)^2} - 1$

$= \dfrac{8 \cos\theta + 4 \cos^2\theta + 4 \sin^2\theta - (2 + \cos\theta)^2}{(2 + \cos\theta)^2}$

$= \dfrac{8 \cos\theta + 4 - 4 - \cos^2\theta - 4 \cos\theta}{(2 + \cos\theta)^2}$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{4 \cos\theta - \cos^2\theta}{(2 + \cos\theta)^2} = \dfrac{\cos\theta\,(4 - \cos\theta)}{(2 + \cos\theta)^2}$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{+ve \times +ve}{+ve} \qquad \left[\begin{array}{l} \because \theta \in [0, \pi/2] \Rightarrow \cos\theta > 0 \\ 4 - \cos\theta \text{ is } +ve \text{ as } -1 \le \cos\theta \le 1 \end{array}\right]$

$\Rightarrow \quad \dfrac{dy}{dx} > 0$

i.e., $\quad y = \dfrac{4 \sin\theta}{2 + \cos\theta} - \theta$ is increasing function in $\left[0, \dfrac{\pi}{2}\right]$.

OR

Here, radius of the sphere $r = 9$ cm.

Error in calculating radius, $\delta r = 0 \cdot 03$ cm.

Let δs be approximate error in calculating surface area.

If S be the surface area of sphere, then

$$S = 4\pi r^2$$

$$\Rightarrow \quad \frac{ds}{dr} = 4\pi \,.\, 2r = 8\pi r$$

Now by definition, approximately

$$\frac{ds}{dr} = \frac{\delta s}{\delta r} \qquad \left[\because \frac{ds}{dr} = \lim_{\delta r \to 0} \frac{\delta s}{\delta r}\right]$$

$$\Rightarrow \quad \delta s = \left(\frac{ds}{dr}\right) . \,\delta r$$

$$\Rightarrow \quad \delta s = 8\pi r \,.\, \delta r$$

$$= 8\pi \times 9 \times 0 \cdot 03 \text{ cm}^2 \qquad [\because r = 9 \text{ cm}]$$

$$= 2 \cdot 16\ \pi \text{ cm}^2$$

16. Given $\quad x = \tan\left(\frac{1}{a}\log y\right)$

$$\Rightarrow \quad \tan^{-1} x = \frac{1}{a}\log y$$

$$\Rightarrow \quad a\tan^{-1} x = \log y$$

Differentiating w.r.t. x, we get

$$\Rightarrow \quad \frac{a}{1+x^2} = \frac{1}{y} \cdot \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{ay}{1+x^2}$$

$$\Rightarrow \quad (1+x^2)\frac{dy}{dx} = ay$$

Differentiating w.r.t. x, we get

$$(1+x^2)\frac{d^2y}{dx^2} + 2x \,.\, \frac{dy}{dx} = a \,.\, \frac{dy}{dx}$$

$$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} + (2x - a)\frac{dy}{dx} = 0$$

17. $I = \int_0^{\pi/2} \frac{x + \sin x}{1 + \cos x}\, dx$

$$= \int_0^{\pi/2} \frac{x}{1+\cos x}\, dx + \int_0^{\pi/2} \frac{\sin x}{1+\cos x}\, dx$$

$I = I_1 + I_2$ $\qquad\qquad$ ...(i)

where $I_1 = \int_0^{\pi/2} \frac{x\,dx}{1+\cos x}$ and $I_2 = \int_0^{\pi/2} \frac{\sin x}{1+\cos x}\,dx$

Now, $I_1 = \int_0^{\pi/2} \frac{x\,dx}{1+\cos x}$

$$= \int_0^{\pi/2} \frac{x\,dx}{2\cos^2 \frac{x}{2}} = \frac{1}{2}\int_0^{\pi/2} x\,.\sec^2 \frac{x}{2}\,dx$$

$$= \frac{1}{2}\left[\left\{2x\,.\tan\frac{x}{2}\right\}_0^{\pi/2} - 2\int_0^{\pi/2} \tan\frac{x}{2}\,dx\right] \qquad \left[\because \int \sec^2 x\,dx = \tan x + c\right]$$

$$= \left\{\frac{\pi}{2}\,.1 - 0\right\} - \left\{2\log\left|\sec\frac{x}{2}\right|\right\}_0^{\pi/2} \qquad \left[\because \int \tan x\,dx = \log\sec x + c\right]$$

$$= \frac{\pi}{2} - 2\left\{\log\left|\sec\frac{\pi}{4}\right| - \log\sec 0\right\} \qquad [\because \log 1 = 0]$$

$$= \frac{\pi}{2} - \log(\sqrt{2})^2$$

$$I_1 = \frac{\pi}{2} - \log 2$$

Again, $I_2 = \int_0^{\pi/2} \frac{\sin x\,dx}{1+\cos x}$

Let $1+\cos x = z$ Also, if $x = \frac{\pi}{2}$, $z = 1+\cos\frac{\pi}{2} = 1+0 = 1$

$-\sin x\,dx = dz$ if $x = 0$, $z = 1+1 = 2$

$\Rightarrow$ $\sin x\,dx = -dz$

$\therefore$

$$I_2 = \int_2^1 \frac{-dz}{z}$$

$$= \int_1^2 \frac{dz}{z} \qquad \left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$$

$$= [\log z]_1^2$$

$$= \log 2 - \log 1 = \log 2$$

Puting the values of I_1 and I_2 in (i), we get

$$\int_0^{\pi/2} \frac{x+\sin x}{1+\cos x}\,dx = \frac{\pi}{2} - \log 2 + \log 2 = \frac{\pi}{2}$$

18. Given $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$

$\Rightarrow$ $x\,dy = (y + \sqrt{x^2+y^2})\,dx \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}$

Let $$F(x, y) = \frac{y + \sqrt{x^2 + y^2}}{x}$$

$\therefore$ $$F(\lambda x, \lambda y) = \frac{\lambda y + \sqrt{\lambda^2 x^2 + \lambda^2 y^2}}{\lambda x}$$

$$= \frac{\lambda \{y + \sqrt{x^2 + y^2}\}}{\lambda x} = \lambda^{\circ} . F(x, y)$$

$\Rightarrow F(x, y)$ is a homogeneous function of degree zero.

Now, $$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}$$

Let $$y = vx$$

$\Rightarrow$ $$\frac{dy}{dx} = v + x . \frac{dv}{dx}$$

Putting above value, we have

$$v + x . \frac{dv}{dx} = \frac{vx + \sqrt{x^2 + v^2 x^2}}{x}$$

$\Rightarrow$ $$v + x . \frac{dv}{dx} = v + \sqrt{1 + v^2} \quad \Rightarrow \quad x . \frac{dv}{dx} = \sqrt{1 + v^2}$$

$\Rightarrow$ $$\frac{dx}{x} = \frac{dv}{\sqrt{1 + v^2}}$$

Integrating both sides, we get

$$\int \frac{dx}{x} = \int \frac{dv}{\sqrt{1 + v^2}}$$

$\Rightarrow$ $$\log x + \log c = \log |v + \sqrt{1 + v^2}| \qquad \left[\because \int \frac{dx}{\sqrt{x^2 + a^2}} = \log|x + \sqrt{x^2 + a^2}| + c\right]$$

$\Rightarrow$ $$cx = v + \sqrt{1 + v^2} \quad \Rightarrow \quad cx = \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}$$

$\Rightarrow$ $$cx = \frac{y}{x} + \frac{\sqrt{x^2 + y^2}}{x} \quad \Rightarrow \quad cx^2 = y + \sqrt{x^2 + y^2}$$

19. $(y + 3x^2) \dfrac{dx}{dy} = x$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{y + 3x^2}{x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} + 3x$$

$$\Rightarrow \quad \frac{dy}{dx} + \left(-\frac{1}{x}\right).y = 3x$$

It is in the form of $\frac{dy}{dx} + Py = Q$

Here $P = -\frac{1}{x}$ and $Q = 3x$

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int -\frac{1}{x}dx}$$

$$= e^{-\log x} = e^{\log\frac{1}{x}} = \frac{1}{x} \qquad [\because e^{\log z} = z]$$

Hence, general solution is

$$y.\frac{1}{x} = \int 3x.\frac{1}{x}\,dx + c \quad [\text{General solution } y \times 1.F = \int Q \times I.F.\,dx + C]$$

$$\Rightarrow \quad \frac{y}{x} = 3x + c$$

$$\Rightarrow \quad y = 3x^2 + cx$$

20. Given, $A \equiv (1, 1, 2); B \equiv (2, 3, 5); C \equiv (1, 5, 5)$

$\therefore \quad \vec{AB} = (2-1)\hat{i} + (3-1)\hat{j} + (5-2)\hat{k}$

$\vec{AB} = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{AC} = (1-1)\hat{i} + (5-1)\hat{j} + (5-2)\hat{k}$

$= 0.\hat{i} + 4\hat{j} + 3\hat{k}$

$\therefore$ The area of required triangle $= \frac{1}{2}|\vec{AB} \times \vec{AC}|$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix}$$

$$= \{(6-12)\hat{i} - (3-0)\hat{j} + (4-0)\hat{k}\}$$

$$= -6\hat{i} - 3\hat{j} + 4\hat{k}$$

$$\therefore \quad |\vec{AB} \times \vec{AC}| = \sqrt{(-6)^2 + (-3)^2 + (4)^2} = \sqrt{61}$$

$\therefore$ Required area $= \frac{1}{2}\sqrt{61} = \frac{\sqrt{61}}{2}$ sq. units.

21. The given equation of lines may be written as

$$\vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + t(-\hat{i} + \hat{j} - 2\hat{k}) \qquad \ldots(i)$$

$$\vec{r} = (\hat{i} - \hat{j} - \hat{k}) + s(\hat{i} + 2\hat{j} - 2\hat{k}) \qquad \ldots(ii)$$

Comparing given equation (*i*) and (*ii*) with $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$, we get

$$\vec{a}_1 = \hat{i} - 2\hat{j} + 3\hat{k}, \qquad \vec{b}_1 = -\hat{i} + \hat{j} - 2\hat{k}$$

$$\vec{a}_2 = \hat{i} - \hat{j} - \hat{k}, \qquad \vec{b}_2 = \hat{i} + 2\hat{j} - 2\hat{k}$$

$$\vec{a}_2 - \vec{a}_1 = \hat{j} - 4\hat{k}$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix}$$

$$= (-2+4)\,\hat{i} - (2+2)\,\hat{j} + (-2-1)\,\hat{k}$$

$$= 2\hat{i} - 4\hat{j} - 3\hat{k}$$

$\therefore$ $$|\vec{b}_1 \times \vec{b}_2| = \sqrt{2^2 + (-4)^2 + (-3)^2} = \sqrt{29}$$

$\therefore$ Required shortest distance $= \left| \dfrac{(\vec{a}_2 - \vec{a}_1)\,.\,(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right|$

$$= \left| \frac{(\hat{j} - 4\hat{k})\,.\,(2\hat{i} - 4\hat{j} - 3\hat{k})}{\sqrt{29}} \right| = \left| \frac{-4+12}{\sqrt{29}} \right|$$

$$= \frac{8}{\sqrt{29}} \text{ units.}$$

22. Q $\displaystyle\sum_{j=1}^{n} P_i = 1$

$\therefore$ $0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$\Rightarrow$ $10k^2 + 9k - 1 = 0$

$\Rightarrow$ $10k^2 + 10k - k - 1 = 0 \quad \Rightarrow \quad 10k\,(k+1) - 1\,(k+1) = 0$

$\Rightarrow$ $(k+1)\,(10k-1) = 0 \quad \Rightarrow \quad k = -1$ and $k = \dfrac{1}{10}$

But k can never be negative as probability is never negative.

$\therefore$ $k = \dfrac{1}{10}$

Now,

(*i*) $k = \dfrac{1}{10}$

(*ii*) $P\,(X < 3) = P\,(X = 0) + P\,(X = 1) + P\,(X = 2)$

$$= 0 + k + 2k = 3k = \frac{3}{10}.$$

(iii) $P(X > 6) = P(X = 7) = 7k^2 + k$

$$= 7 \times \frac{1}{100} + \frac{1}{10} = \frac{17}{100}$$

(iv) $P(0 < X < 3) = P(X = 1) + P(X = 2)$

$$= k + 2k = 3k = \frac{3}{10}.$$

OR

The repeated throws of a die are Bernoulli trials.

Let X denotes the number of sixes in 6 throws of die.

Obviously, X has the binomial distribution with $n = 6$

and $\quad p = \frac{1}{6},\ q = 1 - \frac{1}{6} = \frac{5}{6}$

where p is probability of getting a six

and q is probability of not getting a six

Now, Probability of getting at most 2 sixes in 6 throws $= P(X = 0) + P(X = 1) + P(X = 2)$

$$= {}^6C_0 . p^0 . q^6 + {}^6C_1 p^1 q^5 + {}^6C_2 p^2 q^4$$

$$= \left(\frac{5}{6}\right)^6 + \frac{6!}{1!5!} . \frac{1}{6} . \left(\frac{5}{6}\right)^5 + \frac{6!}{2!4!} . \left(\frac{1}{6}\right)^2 . \left(\frac{5}{6}\right)^4$$

$$= \left(\frac{5}{6}\right)^6 + 6 . \frac{1}{6}\left(\frac{5}{6}\right)^5 + \frac{6 \times 5}{2} \times \left(\frac{1}{6}\right)^2 . \left(\frac{5}{6}\right)^4$$

$$= \left(\frac{5}{6}\right)^4 \left[\frac{25}{36} + \frac{5}{6} + \frac{5}{12}\right]$$

$$= \left(\frac{5}{6}\right)^4 \times \frac{25 + 30 + 15}{36} = \left(\frac{5}{6}\right)^4 \times \frac{70}{36}$$

$$= \frac{21875}{23328}$$

SECTION – C

23. The system can be written as

$$AX = B \quad \Rightarrow \quad X = A^{-1}B \qquad \ldots(i)$$

where $A = \begin{bmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \\ 6 & 2 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 60 \\ 45 \\ 70 \end{bmatrix}$

$|A| = 4(6 - 6) - 3(3 - 18) + 2(2 - 12)$

$$= 0 + 45 - 20 = 25 \neq 0$$

For adj A

$A_{11} = 6 - 6 = 0$ $\quad A_{21} = -(9-4) = -5$ $\quad A_{31} = (9-4) = 5$

$A_{12} = -(3-18) = 15$ $\quad A_{22} = (12-12) = 0$ $\quad A_{32} = -(12-2) = -10$

$A_{13} = (2-12) = -10$ $\quad A_{23} = -(8-18) = 10$ $\quad A_{33} = (8-3) = 5$

$$\therefore \quad adj\,A = \begin{bmatrix} 0 & 15 & -10 \\ -5 & 0 & 10 \\ 5 & -10 & 5 \end{bmatrix}^T = \begin{bmatrix} 0 & -5 & 5 \\ 15 & 0 & -10 \\ -10 & 10 & 5 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{25}\begin{bmatrix} 0 & -5 & 5 \\ 15 & 0 & -10 \\ -10 & 10 & 5 \end{bmatrix}$$

$$= \frac{5}{25}\begin{bmatrix} 0 & -1 & 1 \\ 3 & 0 & -2 \\ -2 & 2 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 0 & -1 & 1 \\ 3 & 0 & -2 \\ -2 & 2 & 1 \end{bmatrix}$$

Now putting values in (i), we get

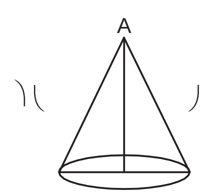
$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 0 & -1 & 1 \\ 3 & 0 & -2 \\ -2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 60 \\ 45 \\ 70 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 0 - 45 + 70 \\ 180 + 0 - 140 \\ -120 + 90 + 70 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 25 \\ 40 \\ 40 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

Hence, $x = 5$, $y = 8$, $z = 8$.

24. Let ABC be right-circular cone having radius 'r' and height 'h'. If V and S are its volume and surface area (curved) respectively, then



$$S = \pi r l$$

$$S = \pi r\sqrt{h^2 + r^2} \qquad \ldots(i)$$

Putting the value of h in (i), we get

$$S = \pi r\sqrt{\frac{9V^2}{\pi^2 r^4} + r^2}$$

$$\Rightarrow \quad S^2 = \pi^2 r^2\left(\frac{9V^2 + \pi^2 r^6}{\pi^2 r^4}\right)$$

$\because V = \frac{1}{3}\pi r^2 h$

$h = \frac{3V}{\pi r^2}$

l

h

B D r C

[Maxima or Minima is same for S or S^2]

$\Rightarrow \quad S^2 = \dfrac{9V^2}{r^2} + \pi^2 r^4$

$\Rightarrow \quad (S^2)' = \dfrac{-18V^2}{r^3} + 4\pi^2 r^3$ ...(*ii*) [Differentiating w.r.t. '*r*']

Now, $\quad (S^2)' = 0$

$\Rightarrow \quad -18\dfrac{V^2}{r^3} + 4\pi^2 r^3 = 0$

$\Rightarrow \quad 4\pi^2 r^6 = 18V^2$

$\Rightarrow \quad 4\pi^2 r^6 = 18 \times \dfrac{1}{9}\pi^2 r^4 h^2$ [Putting value of *V*]

$\Rightarrow \quad 2r^2 = h^2 \quad \Rightarrow \quad r = \dfrac{h}{\sqrt{2}}$

Differentiating (*ii*) w.r.t. '*r*' again

$$(S^2)'' = \frac{54V^2}{r^4} + 12\pi^2 r^2$$

$\Rightarrow \quad (S^2)''\big]_{r=\frac{h}{\sqrt{2}}} > 0$ (for any value of *r*)

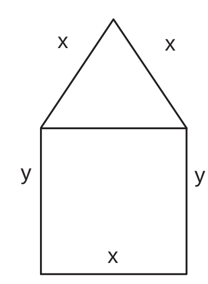
Hence, S^2 *i.e.*, S is minimum for $r = \dfrac{h}{\sqrt{2}}$ or $h = \sqrt{2}\,r$.

i.e., For least curved surface, altitude is equal to $\sqrt{2}$ times the radius of the base.

OR

Let x and y be the dimensions of rectangular part of window and x be side of equilateral part. If A be the total area of window, then

$$A = x.y + \frac{\sqrt{3}}{4}x^2$$



Also $\quad x + 2y + 2x = 12$

$\Rightarrow \quad 3x + 2y = 12$

$\Rightarrow \quad y = \dfrac{12 - 3x}{2}$

$\therefore \quad A = x\dfrac{(12 - 3x)}{2} + \dfrac{\sqrt{3}}{4}x^2$

$\Rightarrow \quad A = 6x - \dfrac{3x^2}{2} + \dfrac{\sqrt{3}}{4}x^2$

$\Rightarrow \quad A' = 6 - 3x + \dfrac{\sqrt{3}}{2}x$ [Differentiating w.r.t. x]

Now, for maxima or minima

$$A' = 0$$

$$6 - 3x + \frac{\sqrt{3}}{2}x = 0$$

$\Rightarrow \qquad x = \dfrac{12}{6 - \sqrt{3}}$

Again $\qquad A'' = -3 + \dfrac{\sqrt{3}}{2} < 0$ (for any value of x)

$$A''\Big]_{x = \frac{12}{6-\sqrt{3}}} < 0$$

i.e., A is maximum if $x = \dfrac{12}{6 - \sqrt{3}}$ and $y = \dfrac{12 - 3\left(\dfrac{12}{6-\sqrt{3}}\right)}{2}$.

i.e., For largest area of window, dimensions of rectangle are

$$x = \frac{12}{6 - \sqrt{3}} \quad \text{and} \quad y = \frac{18 - 6\sqrt{3}}{6 - \sqrt{3}}.$$

25. Let $\qquad I = \displaystyle\int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan x}} = \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \dfrac{\sqrt{\sin x}}{\sqrt{\cos x}}}$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}\, dx}{\sqrt{\cos x} + \sqrt{\sin x}} \qquad \ldots(i)$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left(\dfrac{\pi}{3} + \dfrac{\pi}{6} - x\right)}\, dx}{\sqrt{\cos\left(\dfrac{\pi}{3} + \dfrac{\pi}{6} - x\right)} + \sqrt{\sin\left(\dfrac{\pi}{3} + \dfrac{\pi}{6} - x\right)}}$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}\, dx}{\sqrt{\sin x} + \sqrt{\cos x}} \qquad \ldots(ii)$$

Adding (i) and (ii), $2I = \displaystyle\int_{\pi/6}^{\pi/3} \frac{\cancel{\sqrt{\sin x} + \sqrt{\cos x}}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$

$$2I = \int_{\pi/6}^{\pi/3} dx = [x]_{\pi/6}^{\pi/3}$$

$\therefore \qquad I = \dfrac{1}{2}\left[\dfrac{\pi}{3} - \dfrac{\pi}{6}\right] = \dfrac{1}{2}\left[\dfrac{2\pi - \pi}{6}\right]$

$$I = \frac{\pi}{12}$$

OR

Let $$I=\int \frac{6x+7}{\sqrt{(x-5)(x-4)}}\,dx=\int \frac{6x+7}{\sqrt{x^2-9x+20}}\,dx$$

Now, Let $$6x+7=A.\frac{d}{dx}(x^2-9x+20)+B$$

$$6x+7=A(2x-9)+B$$

$\Rightarrow$ $$6x+7=2Ax-9A+B$$

Comparing the coefficient of x, we get

$$2A=6 \quad \text{and} \quad -9A+B=7$$

$$A=3 \quad \text{and} \quad B=34$$

$\therefore$ $$I=\int \frac{3(2x-9)+34}{\sqrt{x^2-9x+20}}\,dx$$

$$=3\int \frac{(2x-9)\,dx}{\sqrt{x^2-9x+20}}+34\int \frac{dx}{\sqrt{x^2-9x+20}}$$

$$I=3I_1+34I_2 \qquad \ldots(i)$$

where $$I_1=\int \frac{(2x-9)\,dx}{\sqrt{x^2-9x+20}} \quad \text{and} \quad I_2=\int \frac{dx}{\sqrt{x^2-9x+20}}$$

Now, $$I_1=\int \frac{(2x-9)\,dx}{\sqrt{x^2-9x+20}}$$

Let $x^2-9x+20=z^2$

$(2x-9)\,dx=2z\,dz$

$\therefore$ $$I_1=2\int \frac{z\,dz}{z}=2z+c_1$$

$$I_1=2\sqrt{x^2-9x+20}+c_1$$

$$I_2=\int \frac{dx}{\sqrt{x^2-9x+20}}=\int \frac{dx}{\sqrt{x^2-2.\frac{9}{2}x+\left(\frac{9}{2}\right)^2-\frac{81}{4}+20}}$$

$$=\int \frac{dx}{\sqrt{\left(x-\frac{9}{2}\right)^2-\frac{1}{4}}}$$

$$I_2=\int \frac{dx}{\sqrt{\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2}}$$

$$= \log\left|\left(x-\frac{9}{2}\right)+\sqrt{\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right|+C_2$$

$$\left[\because \int \frac{dx}{\sqrt{x^2-a^2}} = \log|x+\sqrt{x^2-a^2}|+x\right]$$

$$= \log\left|\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right|+C_2$$

Putting the value of I_1 and I_2 in (i)

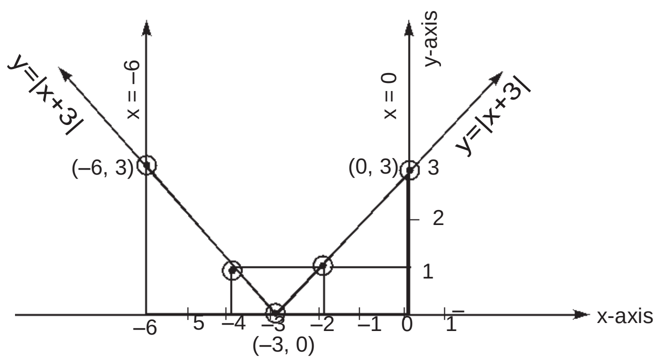
$$\therefore \quad I = 6\sqrt{x^2-9x+20}+3c_1+34\left\{\left|\log\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right|\right\}+34C_2$$

$$= 6\sqrt{x^2-9x+20}+34\log\left|\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right|+C$$

where $\quad C = 3c_1+34c_2$.

26. For graph of $y = |x+3|$

x	0	–3	–6	–2	–4
y	3	0	3	1	1



Shaded region is the required region.

Hence, Required area $= \int_{-6}^{0} |x+3|\,dx$

$$= \int_{-6}^{-3} |x+3|\,dx + \int_{-3}^{0} |x+3|\,dx \quad \text{[By Property of definite integral]}$$

$$= \int_{-6}^{-3} -(x+3)\,dx + \int_{-3}^{0} (x+3)\,dx \left[\begin{array}{l} x+3\ge 0 \text{ if } -3\le x\le 0 \\ x+3\le 0 \text{ if } -6\le x\le -3 \end{array}\right.$$

$$= -\left[\frac{x^2}{2}+3x\right]_{-6}^{-3} + \left[\frac{x^2}{2}+3x\right]_{-3}^{0}$$

$$= -\left[\left(\frac{9}{2} - 9\right) - \left(\frac{36}{2} - 18\right)\right] + \left[0 - \left(\frac{9}{2} - 9\right)\right]$$

$$= \frac{9}{2} + \frac{9}{2} = 9 \text{ sq. units.}$$

27. Given line and plane are

$$\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda\,(3\hat{i} + 4\hat{j} + 2\hat{k}) \qquad \ldots(i)$$

$$\vec{r}\,.\,(\hat{i} - \hat{j} + \hat{k}) = 5 \qquad \ldots(ii)$$

For intersection point, we solve equations (*i*) and (*ii*) by putting the value of $\vec{r}$ from (*i*) in (*ii*).

$$[(2\hat{i} - \hat{j} + 2\hat{k}) + \lambda\,(3\hat{i} + 4\hat{j} + 2\hat{k})]\,.\,(\hat{i} - \hat{j} + \hat{k}) = 5$$

$$\Rightarrow \quad (2 + 1 + 2) + \lambda\,(3 - 4 + 2) = 5 \quad \Rightarrow 5 + \lambda = 5 \quad \Rightarrow \lambda = 0$$

Hence, position vector of intersecting point is $2\hat{i} - \hat{j} + 2\hat{k}$.

i.e., coordinates of intersection of line and plane is (2, –1, 2).

Hence, Required distance $= \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2}$

$$= \sqrt{9 + 16 + 144} = \sqrt{169} = 13$$

28. Let E_1, E_2, E_3 be events such that

$E_1 \equiv$ Selection of Box I ; $E_2 \equiv$ Selection of Box II ; $E_3 \equiv$ Selection of Box III

Let A be event such that

$A \equiv$ the coin drawn is of gold

Now, $P(E_1) = \frac{1}{3}, P(E_2) = \frac{1}{3}, P(E_3) = \frac{1}{3}$, $\quad P\left(\frac{A}{E_1}\right) = P$ (a gold coin from box I) $= \frac{2}{2} = 1$

$P\left(\frac{A}{E_2}\right) = P$ (a gold coin from box II) $= 0$, $\quad P\left(\frac{A}{E_3}\right) = P$ (a gold coin from box III) $= \frac{1}{2}$

the probability that the other coin in the box is also of gold $= P\left(\frac{E_1}{A}\right)$

$$\therefore \qquad P\left(\frac{E_1}{A}\right) = \frac{P(E_1)\,.\,P\left(\frac{A}{E_1}\right)}{P(E_1)\,.\,P\left(\frac{A}{E_1}\right) + P(E_2)\,.\,P\left(\frac{A}{E_2}\right) + P(E_3)\,.\,P\left(\frac{A}{E_3}\right)}$$

$$= \frac{\frac{1}{3} \times 1}{\frac{1}{3} \times 1 + \frac{1}{3} \times 0 + \frac{1}{3} \times \frac{1}{2}} = \frac{2}{3}.$$

29. Let the number of desktop and portable computers to be sold be x and y respectively.

Here, Profit is the objective function Z.

$\therefore \quad Z = 4500x + 5000y \quad \ldots(i)$

we have to maximise z subject to the constraints

$x + y \leq 250 \quad \ldots(ii)$ (Demand Constraint)

$25000x + 40000y \leq 70,00,000 \quad \ldots(iii)$ (Investment constraint)

$\Rightarrow \quad 5x + 8y \leq 1400$

$x \geq 0, \; y \geq 0 \quad \ldots(iv)$ (Non-negative constraint)

Graph of $x = 0$ and $y = 0$ is the y-axis and x-axis respectively.

$\therefore$ Graph of $x \geq 0$, $y \geq 0$ is the Ist quadrant.

Graph of $x + y \leq 250$:

Graph of $x + y = 250$

x	0	250
y	250	0

$\therefore$ Graph of $x + y \leq 250$ is the part of Ist quadrant where origin lies.

Graph of $5x + 8y \leq 1400$:

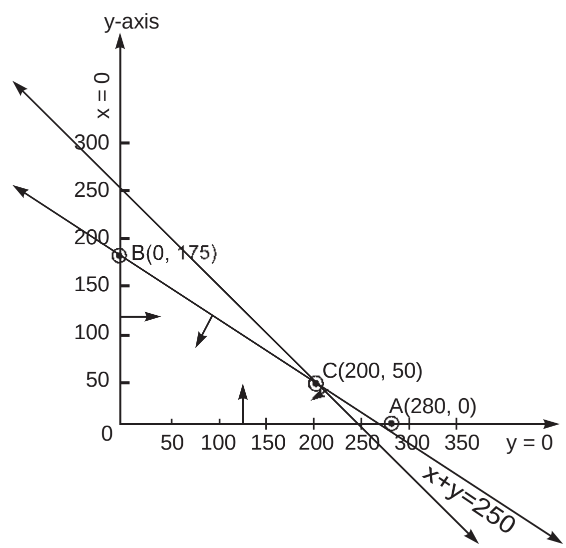
Graph of $5x + 8y = 1400$

x	0	280
y	175	0

$\therefore$ Graph of $5x + 8y \leq 1400$ is the part of Ist quadrant where origin lies.

For cooridnates of C, equation $x + y = 250$ and $5x + 8y = 1400$ are solved and we get

$x = 200, \; y = 50$



Now, we evaluate objective function Z at each corner

Corner Point	$Z = 4500x + 5000y$
$O(0, 0)$	0
A (250, 0)	1125000
C (200, 50)	1150000 ← maximum
B (0, 175)	875000

Maximum profit is ` 11,50,000 when he plan to sell 200 unit desktop and 50 portable computers.

CBSE (All India) Set–II

9. Let $\log x = z$

$\Rightarrow \quad \frac{1}{x}\,dx = z$ (differentiating both sides)

Now,
$$\int \frac{(\log x)^2}{x}\,dx = \int z^2\,dz$$
$$= \frac{z^3}{3} + c = \frac{1}{3}(\log x)^3 + c$$

10. Required unit vector in the direction of $\vec{\mathbf{a}}$
$$= \frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{2^2 + 1^2 + 2^2}} = \pm\frac{1}{3}(2\hat{i} + \hat{j} + 2\hat{k})$$

19. L.H.S. $= 2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

$$= \tan^{-1}\frac{2 \times \frac{1}{2}}{1 - \left(\frac{1}{2}\right)^2} + \tan^{-1}\left(\frac{1}{7}\right) \qquad \left[\text{By Property } -1 \le \frac{1}{2} < 1\right]$$

$$= \tan^{-1}\frac{4}{3} + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{\frac{4}{3} + \frac{1}{7}}{1 - \frac{4}{3} \times \frac{1}{7}}\right) \qquad \left[\because \frac{4}{3} \times \frac{1}{7} < 1\right]$$

$$= \tan^{-1}\left(\frac{31}{17}\right) = \text{R.H.S.}$$

20. Given,
$$\Delta = \begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$$

Now,
$$\Delta = \begin{vmatrix} 3a-x & 3a-x & 3a-x \\ a-x & x+a & a-x \\ a-x & a-x & a+x \end{vmatrix} \quad R_1 \to R_1 + R_2 + R_3$$

$$= (3a-x)\begin{vmatrix} 1 & 1 & 1 \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix}$$

$$= (3a - x)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 2x & a - x \\ -2x & -2x & a + x \end{vmatrix} \qquad \begin{matrix} C_1 \to C_1 - C_3 \\ C_2 \to C_2 - C_3 \end{matrix}$$

$$= (3a - x)\,[1\,(0 + 4x^2)] \qquad \text{[Expanding along } R_1\text{]}$$

$$= 4x^2\,(3a - x)$$

$\therefore \quad 4x^2\,(3a - x) = 0$

$\Rightarrow \quad x = 0 \quad \text{or} \quad x = 3a$

21. Let $\displaystyle I = \int_0^{\pi/4} \log(1 + \tan x)\,dx$

$$= \int_0^{\pi/4} \log\left[1 + \tan\left(\frac{\pi}{4} - x\right)\right] dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx\right]$$

$$= \int_0^{\pi/4} \log\left(1 + \frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x}\right) dx$$

$$= \int_0^{\pi/4} \log\left(1 + \frac{1 - \tan x}{1 + \tan x}\right) dx = \int_0^{\pi/4} \log\left(\frac{1 + \tan x + 1 - \tan x}{1 + \tan x}\right) dx$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1 + \tan x}\right) dx$$

$$= \int_0^{\pi/4} \log 2\,dx - \int_0^{\pi/4} \log(1 + \tan x)\,dx$$

$$I = \log 2\,[x]_0^{\pi/4} - I$$

$\Rightarrow \quad 2I = \dfrac{\pi}{4} \log 2$

$\Rightarrow \quad I = \dfrac{\pi}{8} \log 2$

22. $x\,dy - (y + 2x^2)\,dx = 0$

The given differential equation can be written as

$$x\frac{dy}{dx} - y = 2x^2 \quad \text{or} \quad \frac{dy}{dx} - \frac{1}{x}\,.\,y = 2x$$

$$\text{I.F.} = e^{-\int \frac{1}{x}\,dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$$

$\therefore$ Solution is $\qquad y \cdot \frac{1}{x} = \int 2x \cdot \frac{1}{x}\, dx$

$\Rightarrow \qquad y \cdot \frac{1}{x} = 2x + C \quad \text{or} \quad y = 2x^2 + Cx$

28. The given system can be written as

$$AX = B \quad \Rightarrow \quad X = A^{-1}B \qquad \ldots(i)$$

where $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{vmatrix} = 1\,(0+9) - 2\,(0-6) + 1\,(-3-0) = 18 \neq 0$$

For adj A

$A_{11} = 0 + 9 = 9 \qquad A_{12} = -(0-6) = 6 \qquad A_{13} = -3 - 0 = -3$

$A_{21} = -(0+3) = -3 \qquad A_{22} = 0 - 2 = -2 \qquad A_{23} = -(-3-4) = 7$

$A_{31} = 6 - 0 = 6 \qquad A_{32} = -(3-1) = -2 \qquad A_{33} = 0 - 2 = -2$

$$\therefore \qquad adj.\, A = \begin{bmatrix} 9 & 6 & -3 \\ -3 & -2 & 7 \\ 6 & -2 & -2 \end{bmatrix}^T = \begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

$$\therefore \qquad A^{-1} = \frac{1}{|A|} \cdot adj.\, A$$

$$= \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

Now putting above values in (i), we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}\begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 63 - 33 + 6 \\ 42 - 22 - 2 \\ -21 + 77 - 2 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$\Rightarrow \qquad x = 2,\ y = 1,\ z = 3$$

$\begin{bmatrix} 36 \end{bmatrix} \Rightarrow = \left| 18 \right|$

$\begin{bmatrix} 54 \end{bmatrix}$

[From equality of matrices]

29. Two given planes are

$$\vec{r}\,.(\hat{i}+\hat{j}+\hat{k})-1=0$$

$$\vec{r}\,.(2\hat{i}+3\hat{j}-\hat{k})+4=0$$

It's cartesian forms are

$$x+y+z-1=0 \qquad \ldots(i)$$

and $$2x+3y-z+4=0 \qquad \ldots(ii)$$

Now, equation of plane passing through line of intersection of plane (*i*) & (*ii*) is given by

$$(x+y+z-1)+\lambda\,(2x+3y-z+4)=0$$

$$(1+2\lambda)\,x+(1+3\lambda)\,y+(1-\lambda)\,z-1+4\lambda=0 \qquad \ldots(iii)$$

Since (*iii*) is parallel to x-axis

$\Rightarrow$ Normal of plane (*iii*) is perpendicular to x-axis.

$\Rightarrow \quad (1+2\lambda).1+(1+3\lambda).0+(1-\lambda).0=0$ [$\because$ Direction ratios of x-axis are (1, 0, 0)]

$\Rightarrow \quad 1+2\lambda=0 \quad \Rightarrow \quad \lambda=-\dfrac{1}{2}$

Hence, required equation of plane is

$$0x+\left(1-\frac{3}{2}\right)y+\left(1+\frac{1}{2}\right)z-1+4\times-\frac{1}{2}=0$$

$$\Rightarrow \quad -\frac{1}{2}y+\frac{3}{2}z-1-2=0$$

$$\Rightarrow \quad y-3z+6=0 \text{ or } \vec{r}\,.(\hat{j}-3\hat{k})+6=0$$

CBSE (All India) Set–III

1. Let $\tan^{-1}x=z$

$$\frac{1}{1+x^2}\,dx=dz \qquad \text{[Differentiating we get]}$$

$$\therefore \quad \int\frac{e^{\tan^{-1}x}}{1+x^2}\,dx=\int e^z\,.\,dz$$

$$=e^z+c=e^{\tan^{-1}x}+c$$

2. If θ be the angle between $\vec{a}$ and $\vec{b}$, then

$$\vec{a}\,.\,\vec{b}=|\vec{a}|.|\vec{b}|\cos\theta$$

$$\sqrt{6} = \sqrt{3}\,.\,2\cos\theta$$

$$\therefore \qquad \cos\theta = \frac{\sqrt{6}}{2\times\sqrt{3}} = \frac{\sqrt{3}\,.\sqrt{2}}{2\sqrt{3}} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\therefore \qquad \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

11. L.H.S. $= \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right)$

$$= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}\times\frac{1}{5}}\right) + \tan^{-1}\left(\frac{1}{8}\right) \qquad \left[\because \frac{1}{2}\times\frac{1}{5} = \frac{1}{10} < 1\right]$$

$$= \tan^{-1}\left(\frac{7}{9}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \tan^{-1}\left(\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{9}\times\frac{1}{8}}\right) = \tan^{-1}\left(\frac{65}{72}\times\frac{72}{65}\right)$$

$$= \tan^{-1}(1) = \frac{\pi}{4}\cdot$$

12. Let $\Delta = \begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix}$

$$= \begin{vmatrix} 3x+a & 3x+a & 3x+a \\ x & x+a & x \\ x & x & x+a \end{vmatrix} \qquad R_1 \to R_1 + R_2 + R_3$$

$$= \begin{vmatrix} 0 & 0 & 3x+a \\ 0 & a & x \\ -a & -a & x+a \end{vmatrix} \qquad \begin{matrix} C_1 \to C_1 - C_3 \\ C_2 \to C_2 - C_3 \end{matrix}$$

$= (3x+a)\,(0+a)$ [Expanding along R_1]

$= a\,(3x+a) = 3ax + a^2$

Given $\Delta = 0$

$$\therefore \qquad 3ax + a^2 = 0$$

$$x = -\frac{a^2}{3a} = -\frac{a}{3}$$

13. Let $I = \int_0^1 \log\left(\frac{1}{x} - 1\right) dx$

$$= \int_0^1 \log\left(\frac{1-x}{x}\right) dx \qquad \ldots(i)$$

$$I = \int_0^1 \log\left(\frac{1-(1-x)}{1-x}\right) dx \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)\,dx\right]$$

$$I = \int_0^1 \log\left(\frac{x}{1-x}\right) dx \qquad \ldots(ii)$$

Adding (*i*) and (*ii*), we get

$$2I = \int_0^1 \log\left(\frac{1-x}{x}\right) dx + \int_0^1 \log\left(\frac{x}{1-x}\right) dx$$

$$= \int_0^1 \log\left(\frac{1-x}{x} \cdot \frac{x}{1-x}\right) dx \qquad [\because \log A + \log B = \log(A \times B)]$$

$$= \int_0^1 \log 1\, dx$$

$$2I = 0 \quad \therefore \quad I = 0$$

14. $x\,dy + (y - x^3)\,dx = 0$

$$\Rightarrow \quad x\,dy = -(y - x^3)\,dx \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-y + x^3}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-y}{x} + x^2 \quad \Rightarrow \quad \frac{dy}{dx} + \left(\frac{1}{x}\right).y = x^2$$

It is in the form of $\frac{dy}{dx} + Py = Q$

where $P = \frac{1}{x}$ and $Q = x^2$

$\therefore$ I.F. $= e^{\int \frac{1}{x} dx} = e^{\log x} = x$

Hence, solution is

$$y\,.x = \int x\,.x^2\,dx + C$$

$$xy = \frac{x^4}{4} + C \quad \Rightarrow \quad y = \frac{x^3}{4} + \frac{C}{x}$$

23. The given system of equation can be written as

$$AX = B \quad \Rightarrow \quad X = A^{-1}B \qquad \ldots(i)$$

where $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{vmatrix} = 1(-12+6) - 2(-8-6) - 3(-6-9) = 67 \neq 0$

For adj A :

$A_{11} = -6 \qquad A_{21} = 17 \qquad A_{31} = 13$

$A_{12} = 14 \qquad A_{22} = 5 \qquad A_{32} = -8$

$A_{13} = -15 \qquad A_{23} = 9 \qquad A_{33} = -1$

$$\therefore \qquad adj.\, A = \begin{bmatrix} -6 & 14 & -15 \\ 17 & 5 & 9 \\ 13 & -8 & -1 \end{bmatrix}^T = \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$$

$$\therefore \qquad A^{-1} = \frac{1}{|A|} . \, adj.\, A = \frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$$

Putting the value of X, A^{-1} and B in (i), we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}\begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{67}\begin{bmatrix} 24+34+143 \\ -56+10-88 \\ 60+18-11 \end{bmatrix} = \frac{1}{67}\begin{bmatrix} 201 \\ -134 \\ 67 \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$\Rightarrow \qquad x = 3, y = -2, z = 1$

24. The given planes are

$$2x + y - z - 3 = 0 \qquad \ldots(i)$$

and

$$5x - 3y + 4z + 9 = 0 \qquad \ldots(ii)$$

The equation of the plane passing through the line of intersection of (i) and (ii) is given by

$$(2x + y - z - 3) + \lambda\,(5x - 3y + 4z + 9) = 0$$

$$\Rightarrow \qquad (2 + 5\lambda)\, x + (1 - 3\lambda)\, y + (4\lambda - 1)\, z + (9\lambda - 3) = 0 \qquad \ldots(iii)$$

It is given that plane (iii) is parallel to $\dfrac{x-1}{2} = \dfrac{y-3}{4} = \dfrac{z-5}{5}$.

$\Rightarrow$ Normal of (iii) is perpendicular to given line.

$$\therefore \qquad (2 + 5\lambda) . 2 + (1 - 3\lambda) . 4 + (4\lambda - 1) . 5 = 0$$

$$\Rightarrow \qquad 18\lambda + 3 = 0$$

$$\Rightarrow \qquad \lambda = -\frac{1}{6}$$

Putting the value of λ in (iii), we get the required plane.

$$(2x + y - z - 3) - \frac{1}{6}(5x - 3y + 4z + 9) = 0$$

$$\Rightarrow \qquad 12x + 6y - 6z - 18 - 5x + 3y - 4z - 9 = 0$$

$$\Rightarrow \qquad 7x + 9y - 10z - 27 = 0$$

EXAMINATION PAPERS –2011

CBSE (Foreign) Set–I

Time allowed: 3 hours **Maximum marks: 100**

General Instructions : *As given in Examination Paper (Delhi) – 2011.*

SECTION – A

Question numbers 1 to 10 carry one mark each.

1. If $f : R \rightarrow R$ is defined by $f(x) = 3x + 2$, define $f[f(x)]$.

2. Write the principal value of $\tan^{-1}(-1)$.

3. Write the values of $x - y + z$ from the following equation :

$$\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$$

4. Write the order of the product matrix :

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [2 \quad 3 \quad 4]$$

5. If $\begin{vmatrix} x & x \\ 1 & x \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$, write the positive value of x.

6. Evaluate :

$$\int \frac{(1+\log x)^2}{x}\, dx.$$

7. Evaluate :

$$\int_1^{\sqrt{3}} \frac{dx}{1+x^2}.$$

8. Write the position vector of the mid-point of the vector joining the points $P(2, 3, 4)$ and $Q(4, 1, -2)$.

9. If $\vec{a} \cdot \vec{a} = 0$ and $\vec{a} \cdot \vec{b} = 0$, then what can be concluded about the vector $\vec{b}$?

10. What are the direction cosines of a line, which makes equal angles with the co-ordinates axes?

SECTION – B

Question numbers 11 to 22 carry 4 marks each.

11. Consider $f : R_+ \to [4, \infty]$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse (f^{-1}) of f given by $f^{-1}(y) = \sqrt{y-4}$, where R_+ is the set of all non-negative real numbers.

12. Prove the following :

$$\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

OR

Solve the following equation for x :

$$\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}(x), \quad x > 0$$

13. Prove, using properties of determinants :

$$\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k)$$

14. Find the value of k so that the function f defined by

$$f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$$

is continuous at $x = \dfrac{\pi}{2}$.

15. Find the intervals in which the function f given by

$$f(x) = \sin x + \cos x, \qquad 0 \le x \le 2\pi$$

is strictly increasing or strictly decreasing.

OR

Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.

16. Prove that :

$$\frac{d}{dx}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right] = \sqrt{a^2 - x^2}$$

OR

If $y = \log\left[x + \sqrt{x^2 + 1}\right]$, prove that $(x^2 + 1)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = 0$.

17. Evaluate : $\int e^{2x} \sin x \, dx$

OR

Evaluate : $\displaystyle\int \frac{3x+5}{\sqrt{x^2 - 8x + 7}}\, dx$

18. Find the particular solution of the differential equation :

$$(1 + e^{2x})\, dy + (1 + y^2)\, e^x\, dx = 0, \text{ given that } y = 1, \text{ when } x = 0.$$

19. Solve the following differential equation :

$$\frac{dy}{dx} + y \cot x = 4x \operatorname{cosec} x, \text{ given that } y = 0 \text{ when } x = \frac{\pi}{2}.$$

20. If vectors $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda \vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ.

21. Find the shortest distance between the lines :

$$\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k}) \text{ and}$$
$$\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$$

22. Find the mean number of heads in three tosses of a fair coin.

SECTION – C

Question numbers 23 to 29 carry 6 marks each.

23. Use product $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$ to solve the system of equations :

$$\begin{aligned} x - y + 2z &= 1 \\ 2y - 3z &= 1 \\ 3x - 2y + 4z &= 2 \end{aligned}$$

OR

Using elementary transformations, find the inverse of the matrix :

$$\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

24. A window is in the form of a rectangle surmounted by a semi-circular opening. The total perimeter of the window is 10 metres. Find the dimensions of the rectangle so as to admit maximum light through the whole opening.

25. Using the method of integration, find the area of the region bounded by the lines :

$$\begin{aligned} 2x + y &= 4 \\ 3x - 2y &= 6 \\ x - 3y + 5 &= 0 \end{aligned}$$

26. Evaluate $\int_1^4 (x^2 - x)\, dx$ as a limit of sums.

OR

Evaluate :

$$\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x}\, dx$$

27. Find the equation of the plane passing through the point (–1, 3, 2) and perpendicular to each of the planes :

$$x + 2y + 3z = 5 \quad \text{and} \quad 3x + 3y + z = 0$$

28. A cottage industry manufactures pedestal lamps and wooden shades, each requiring the use of grinding/cutting machine and a sprayer. It takes 2 hours on the grinding/cutting machine and 3 hours on the sprayer to manufacture a pedestal lamp. It takes one hour on the grinding/cutting machine and 2 hours on the sprayer to manufacture a shade. On any day, the sprayer is available for at the most 20 hours and the grinding/cutting machine for at the most 12 hours. The profit from the sale of a lamp is ₹ 5 and that from a shade is ₹ 3. Assuming that the manufacturer can sell all the lamps and shades that he produces, how should he schedule his daily production in order to maximise his profit? Make an L.P.P. and solve it graphically.

29. A factory has two machines A and B. Past record shows that machine A produced 60% of the items of output and machine B produced 40% of the items. Futher, 2% of the items produced by machine A and 1% produced by machine B were defective. All the items are put into one stockpile and then one item is chosen at random from this and is found to be defective. What is the probability that it was produced by machine B?

CBSE (Foreign) Set–II

9. Write *fog*, if $f : R \to R$ and $g : R \to R$ are given by

$$f(x) = |x| \quad \text{and} \quad g(x) = |5x - 2|.$$

10. Evaluate :

$$\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\, dx$$

19. Prove, using properties of determinants :

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

20. Find the value of k so that the function f, defined by

$$f(x) = \begin{cases} kx + 1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases}$$

is continuous at $x = \pi$.

21. Solve the following differential equation:

$$\frac{dy}{dx} + 2y \tan x = \sin x. \text{ given that } y = 0, \text{ when } x = \frac{\pi}{3}.$$

22. Find the shortest distance between the lines :

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k}) \quad \text{and}$$

$$\vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu(2\hat{i} + 3\hat{j} + \hat{k})$$

28. Find the vector equation of the plane, passing through the points $A(2, 2, -1)$, $B(3, 4, 2)$ and $C(7, 0, 6)$. Also, find the cartesian equation of the plane.

29. Bag I contains 3 red and 4 black balls and Bag II contains 4 red and 5 black balls. One ball is transferred from Bag I to Bag II and then a ball is drawn from Bag II at random. The ball so drawn is found to be red in colour. Find the probability that the transferred ball is black.

CBSE (Foreign) Set–III

1. Write fog, if $f: R \to R$ and $g: R \to R$ are given by $f(x) = 8x^3$ and $g(x) = x^{1/3}$.

2. Evaluate :

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$$

11. Prove, using properties of determinants :

$$\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$$

12. For what value of λ is the function

$$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \le 0 \\ 4x+1, & \text{if } x > 0 \end{cases}$$

continuous at $x = 0$?

13. Solve the following differential equation :

$$(1+x^2)\frac{dy}{dx} + 2xy = \frac{1}{1+x^2}, \text{ given } y = 0 \text{ when } x = 1.$$

14. Find the shortest distance betwen the lines :

$$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$$

and
$$\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$$

23. Find the equation of the palne passing through the point $(1, 1, -1)$ and perpendicular to the planes $x + 2y + 3z - 7 = 0$ and $2x - 3y + 4z = 0$.

24. There are three coins. One is a two headed coin (having heads on both faces), another is a biased coin that comes up heads 75% of the times and the third is an unbiased coin. One of the three coins is chosen at random and tossed, and it shows heads. What is the probability that it was the two-headed coin?

Solutions

CBSE (Foreign) Set–I

Section – A

1. $$f(f(x)) = f(3x+2)$$
$$= 3.(3x+2)+2 = 9x+6+2$$
$$= 9x+8$$

2. Let $\tan^{-1}(-1) = \theta$

$\Rightarrow \tan\theta = -1$

$\Rightarrow \tan\theta = -\tan\frac{\pi}{4}$

$\Rightarrow \tan\theta = \tan\left(-\frac{\pi}{4}\right)$

$\Rightarrow \theta = -\frac{\pi}{4} \Rightarrow \tan^{-1}(-1) = -\frac{\pi}{4}$

$\left[\because -\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right.$ range of the principal value branch of $\tan^{-1}$ function and $\tan\left(-\frac{\pi}{4}\right) = -1$

$\therefore$ Principal value of $\tan^{-1}(-1)$ is $-\frac{\pi}{4}$.

3. We have

$$\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$$

By definition of equality of matrices, we have

$$x+y+z = 9 \quad \text{... }(i)$$
$$x+z = 5 \quad \text{.... }(ii)$$
$$y+z = 7 \quad \text{... }(iii)$$

$(i) - (ii)$ $\quad x+y+z-x-z = 9-5$

$\Rightarrow \Rightarrow \quad y = 4 \quad$ ... (iv)

$(ii) - (iv) \Rightarrow \quad x-y+z = 5-4$

$\Rightarrow \quad x-y+z = 1$

4. Order is 3×3 because it is product of two matrices having order 3×1 and 1×3.

5. We have

Q $$\begin{vmatrix} x & x \\ 1 & x \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$$

$\Rightarrow \quad x^2 - x = 6-4 \quad \Rightarrow \quad x^2 - x - 2 = 0$

$\Rightarrow \quad x^2 - 2x + x - 2 = 0 \Rightarrow x(x-2) + 1(x-2) = 0$

$\Rightarrow \quad (x-2)(x+1) = 0$

$\Rightarrow \quad x = 2$ or $x = -1$ (Not accepted)

$\Rightarrow \quad x = 2$

6. $$I = \int \frac{(1+\log x)^2}{x}\, dx$$

Let $1 + \log x = z$

$$\frac{1}{x}\, dx = dz \Rightarrow I = \int z^2\, dz$$

$$= \frac{z^3}{3} + C = \frac{1}{3}(1+\log x)^3 + C$$

7. $$I = \int_1^{\sqrt{3}} \frac{dx}{1+x^2}$$

$$= [\tan^{-1} x]_1^{\sqrt{3}} \qquad \left[\because \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}\right]$$

$$= \tan^{-1}(\sqrt{3}) - \tan^{-1}(1)$$

$$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

8. Let $\vec{a}, \vec{b}$ be position vector of points $P(2, 3, 4)$ and $Q(4, 1, -2)$ respectively.

$\therefore \quad \vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$\vec{b} = 4\hat{i} + \hat{j} - 2\hat{k}$

$\therefore$ Position vector of mid point of P and $Q = \dfrac{\vec{a} + \vec{b}}{2} = \dfrac{6\hat{i} + 4\hat{j} + 2\hat{k}}{2}$

$$= 3\hat{i} + 2\hat{j} + \hat{k}$$

9. $\therefore \quad \vec{a} \cdot \vec{a} = 0$

$\Rightarrow \quad |\vec{a}| \cdot |\vec{a}| \cdot \cos\theta = 0$

$\Rightarrow \quad |\vec{a}| \cdot |\vec{a}| = 0 \qquad [\because \cos 0 = 1]$

$\Rightarrow \quad |\vec{a}|^2 = 0 \Rightarrow |\vec{a}| = 0$

$\vec{b} \Rightarrow$ may be any vector $\left[\text{as } \vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\theta = 0 \cdot |\vec{b}| \cdot \cos\theta = 0\right]$

10. Let α be the angle made by line with coordinate axes.

$\Rightarrow$ Direction cosines of line are $\cos\alpha, \cos\alpha, \cos\alpha$

$$\Rightarrow \quad \cos^2 \alpha + \cos^2 \alpha + \cos^2 \alpha = 1$$

$$\Rightarrow \quad 3\cos^2 \alpha = 1 \Rightarrow \cos^2 \alpha = \frac{1}{3}$$

$$\Rightarrow \quad \cos \alpha = \pm \frac{1}{\sqrt{3}}$$

Hence, the direction cosines, of the line equally inclined to the coordinate axes are $\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}$

[**Note :** If l, m, n are direction cosines of line, then $l^2 + m^2 + n^2 = 1$]

Section – B

11. For one-one

Let $\quad x_1, x_2 \in R$ (Domain)

$$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1^2 + 4 = x_2^2 + 4$$

$$\Rightarrow \quad x_1^2 = x_2^2$$

$$\Rightarrow \quad x_1 = x_2 \qquad [\therefore x_1, x_2 \text{ are +ve real number}]$$

f is one-one function.

For onto

Let $y \in [4, \infty)$ s.t.

$$y = f(x) \quad \forall\, x \in R_t \qquad \text{(set of non-negative reals)}$$

$$\Rightarrow \quad y = x^2 + 4$$

$$\Rightarrow \quad x = \sqrt{y-4} \qquad [\therefore x \text{ is + ve real number}]$$

Obviously, $\forall\, y \in [4, \alpha]$, x is real number $\in R$ (domain)

i.e., all elements of codomain have pre image in domain.

$\Rightarrow$ f is onto.

Hence f is invertible being one-one onto.

For inverse function : If f^{-1} is inverse of f, then

$$fof^{-1} = I \qquad \text{(Identity function)}$$

$$\Rightarrow \quad fof^{-1}(y) = y \quad \forall\, y \in [4, \infty)$$

$$\Rightarrow \quad f(f^{-1}(y)) = y$$

$$\Rightarrow \quad (f^{-1}(y))^2 + 4 = y \qquad [\text{Q} f(x) = x^2 + 4]$$

$$\Rightarrow \quad f^{-1}(y) = \sqrt{y-4}$$

Therefore, required inverse function is f^{-1} $[4, \infty]$ ® R defined by

$$f^{-1}(y) = \sqrt{y-4} \quad \forall\, y \in [4, \alpha).$$

12. L.H.S.
$$= \frac{9\pi}{8} - \frac{9}{4} \sin^{-1} \frac{1}{3}$$
$$= \frac{9}{4}\left(\frac{\pi}{2} - \sin^{-1} \frac{1}{3}\right)$$

$$= \frac{9}{4}\cos^{-1}\left(\frac{1}{3}\right) \qquad \left[\because \frac{1}{3} \in [-1, 1]\right]$$

Let $\cos^{-1}\left(\frac{1}{3}\right) = \theta$

$$\Rightarrow \quad \cos\theta = \frac{1}{3} \qquad [\,\theta \in [0, \pi]\,]$$

$$\therefore \quad \sin\theta = +\sqrt{1 - \left(\frac{1}{3}\right)^2} \qquad \left[\because \theta \in [0, \pi] \Rightarrow \sin\theta \text{ is } +\text{ve}\right]$$

$$\Rightarrow \quad \sin\theta = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) \Rightarrow \cos^{-1}\left(\frac{1}{3}\right) = \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

Putting the value of θ]

$$\therefore \quad \text{L.H.S} = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \text{R.H.S.}$$

OR

$$\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x \Rightarrow 2\tan^{-1}\frac{1-x}{1+x} = \tan^{-1}x$$

$$\Rightarrow \quad \tan^{-1}\frac{2\left(\frac{1-x}{1+x}\right)}{1-\left(\frac{1-x}{1+x}\right)^2} = \tan^{-1}x \qquad \left[\begin{array}{l}\text{By property} \\ \text{Here } \frac{1-x}{1+x} < 1 \text{ as } x > 0\end{array}\right]$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2(1-x^2)}{(1+x)^2 - (1-x)^2}\right) = \tan^{-1}x$$

$$\Rightarrow \quad \frac{1-x^2}{2x} = x \Rightarrow 3x^2 = 1$$

$$\Rightarrow \quad x = \frac{1}{\sqrt{3}} \qquad [\because x > 0]$$

13. L.H.S.
$$= \begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$$

$$= \begin{vmatrix} 3y+k & y & y \\ 3y+k & y+k & y \\ 3y+k & y & y+k \end{vmatrix} \qquad [\text{Applying } C_1 \to C_1 + C_2 + C_3$$

$$= (3y+k)\begin{vmatrix} 1 & y & y \\ 1 & y+k & y \\ 1 & y & y+k \end{vmatrix} \qquad [\text{Taking common } (3y+k) \text{ from } C_1$$

$$= (3y+k)\begin{vmatrix} 1 & y & y \\ 0 & k & 0 \\ 0 & 0 & k \end{vmatrix} \qquad \left[\begin{array}{l}\text{Applying} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1\end{array}\right]$$

Expanding along C_1 we get

$$= (3y+k)\,\{1(k^2 - 0) - 0 + 0\}$$

$$= (3y+k).\,k^2$$

$$= k^2\,(3y+k)$$

14. $$\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{h \to 0} f\left(\frac{\pi}{2} - h\right) \qquad \left[\begin{array}{l}\text{Let } x = \frac{\pi}{2} - h \\ x \to \frac{\pi}{2}^- \Rightarrow h = 0\end{array}\right]$$

$$= \lim_{h \to 0} \frac{k\cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)} = \lim_{h \to 0} \frac{k \sin h}{2h} \qquad \left[\because f(x) = \frac{k\cos x}{\pi - 2x} \text{ if } x \neq \frac{\pi}{2}\right]$$

$$= \frac{k}{2} \lim_{h \to 0} \frac{\sin h}{h} = \frac{k}{2} \qquad \ldots(i)$$

$$\lim_{x \to \frac{\pi}{2}^+} f(x) = \lim_{h \to 0} f\left(\frac{\pi}{2} + h\right) \qquad \left[\begin{array}{l}\text{Let } x = \frac{\pi}{2} + h \\ x \to \frac{\pi}{2}^+ \Rightarrow h = 0\end{array}\right]$$

$$= \lim_{h \to 0} \frac{k\cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)}$$

$$= \lim_{h \to 0} \frac{-k \sin h}{\pi - \pi - 2h} \qquad \left[\because f(x) = \frac{k\cos x}{\pi - 2x} \text{ if } x \neq \frac{\pi}{2}\right]$$

$$= \lim_{h \to 0} \frac{k \sin h}{2h} = \frac{k}{2} \lim_{h \to 0} \frac{\sin h}{h} = \frac{k}{2} \qquad \ldots(ii)$$

Also $$f\left(\frac{\pi}{2}\right) = 3 \qquad \left[\because f(x) = 3 \text{ if } x = \frac{\pi}{2}\right]$$

Since $f(x)$ is continuous at $x = \frac{\pi}{2}$

$$\therefore \quad \lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^+} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\frac{k}{2} = \frac{k}{2} = 3 \quad \Rightarrow \quad k = 6.$$

15. $f(x) = \sin x + \cos x$

Differentiating w.r.t. x, we get

$$f'(x) = \cos x - \sin x$$

For critical points

$$f'(x) = 0$$

$\Rightarrow \quad \cos x - \sin x = 0 \;\Rightarrow\; \cos x = \sin x$

$\Rightarrow \quad \cos x = \cos\left(\frac{\pi}{2} - x\right)$

$\Rightarrow \quad x = 2n\pi \pm \left(\frac{\pi}{2} - x\right) \qquad$ where $n = 0, \pm 1, \pm 2, \ldots$

$\Rightarrow \quad x = 2n\pi + \frac{\pi}{2} - x \quad$ or $\quad x = 2n\pi - \frac{\pi}{2} + x$

$\Rightarrow \quad 2x = 2n\pi + \frac{\pi}{2} \qquad$ (Not exist)

$\Rightarrow \quad x = n\pi + \frac{\pi}{4}$

$$x = \frac{\pi}{4}, \frac{5\pi}{4} \qquad [\because 0 \le x \le 2\pi]$$

The critical value of $f(x)$ are $\frac{\pi}{4}, \frac{5\pi}{4}$.

Therefore, required intervals are $\left[0, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right]$

Obviously, $\quad f'(x) > 0$ if $x \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$

and $\quad f'(x) < 0$ if $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

i.e., $f(x)$ is strictly increasing in $\left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$ and strictly decreasing in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

OR

Let (x_1, y_1) be the required point on the curve $y = x^3$,

Now $\quad y = x^3$

$\therefore \quad \frac{dy}{dx} = 3x^2 \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 3x_1^2$

$\Rightarrow$ Slope of tangent at point (x_1, y_1) on curve $(y = x^3)$ *is* $\left(\frac{dy}{dx}\right)_{(x_1\, y_1)}$

From question

$$3x_1^2 = y_1 \qquad \ldots (i)$$

Also since (x_1, y_1) lies on curve $y = x^3$

$$\therefore \qquad y_1 = x_1^3 \qquad \ldots (ii)$$

From (i) and (ii)

$$3x_1^2 = x_1^3 \Rightarrow 3x_1^2 - x_1^3 = 0$$

$$\Rightarrow \qquad x_1^2(3 - x_1) = 0 \Rightarrow x_1 = 0, x_1 = 3$$

If $x_1 = 0, y_1 = 0$

If $x_1 = 3, y_1 = 27$

Hence, required points are (0, 0) and (3, 27).

16. Prove that

$$\frac{d}{dx}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right] = \sqrt{a^2 - x^2}$$

L.H.S.

$$= \frac{d}{dx}\left(\frac{x}{2}\sqrt{a^2 - x^2}\right) + \frac{d}{dx}\left(\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right)$$

$$= \frac{1}{2}\left\{x \cdot \frac{1}{2\sqrt{a^2 - x^2}} \times -2x + \sqrt{a^2 - x^2}\right\} + \frac{a^2}{2} \cdot \frac{1}{\sqrt{1 - \frac{x^2}{a^2}}} \times \frac{1}{a}$$

$$= \frac{-x^2}{2\sqrt{a^2 - x^2}} + \frac{\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2\sqrt{a^2 - x^2}}$$

$$= \frac{-x^2 + a^2 - x^2 + a^2}{2\sqrt{a^2 - x^2}}$$

$$= \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} = \sqrt{a^2 - x^2} = \text{R.H.S.}$$

OR

Given $\qquad y = \log\left[x + \sqrt{x^2 + 1}\right]$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + 1}} \times \left[1 + \frac{2x}{2\sqrt{x^2 + 1}}\right] \qquad \text{[Differentiating]}$$

$$= \frac{2(x + \sqrt{x^2 + 1})}{(x + \sqrt{x^2 + 1}) \times 2\sqrt{x^2 + 1}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 1}}$$

Differentiating again, we get

$\therefore \quad \frac{d^2y}{dx^2} = -\frac{1}{2}(x^2+1)^{-3/2}.\,2x = \frac{-x}{(x^2+1)^{3/2}} \Rightarrow (x^2+1)\frac{d^2y}{dx^2} = \frac{-x}{\sqrt{x^2+1}}$

$\Rightarrow \quad (x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$

17. Let $\quad I = \int e^{2x} \sin x \, dx$

$= -e^{2x}\cos x - \int 2e^{2x}(-\cos x)\,dx$

$= -e^{2x}\cos x + 2\int e^{2x}\cos x \, dx$

$= -e^{2x}\cos x + 2[e^{2x}\sin x - \int 2e^{2x}\sin x \, dx]$

$= -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin x \, dx$

$+ C' = e^{2x}(2\sin x - \cos x) - 4I + C'$

$\Rightarrow \quad I = \frac{e^{2x}}{5}[2\sin x - \cos x] + C \qquad \left[\text{where } C = \frac{C'}{5}\right]$

OR

Now $\quad 3x + 5 = A.\frac{d}{dx}(x^2 - 8x + 7) + B$

$\Rightarrow \quad 3x + 5 = A(2x - 8) + B$

$\Rightarrow \quad 3x + 5 = 2Ax - 8A + B$

Equating the coefficient of x and constant, we get

$2A = 3$ and $-8A + B = 5$

$A = \frac{3}{2}$ and $-8 \times \frac{3}{2} + B = 5$

$\Rightarrow \quad B = 5 + 12 = 17$

Hence $\quad \int \frac{3x+5}{\sqrt{x^2-8x+7}}\,dx = \int \frac{\frac{3}{2}(2x-8)+17}{\sqrt{x^2-8x+7}}\,dx$

$= \frac{3}{2}\int \frac{(2x-8)}{\sqrt{x^2-8x+7}}\,dx + 17\int \frac{dx}{\sqrt{x^2-8x+7}}$

$= \frac{3}{2} I_1, + 17\, I_2$ $\quad \ldots(i)$

Where $\quad I_1 = \int \frac{2x-8}{\sqrt{= x^2 - 8x + 7}}\,dx, \; I_2 \quad \int \frac{dx}{\sqrt{x^2-8x+7}}$

Now $\quad I_1 = \int \frac{2x-8}{\sqrt{x^2-8x+7}}\,dx$

Let $\quad x^2 - 8x + 7 = z^2 \quad \Rightarrow \quad (2x-8)\,dx = 2zdz$

$$\therefore \qquad I_1 = \int \frac{2z\,dz}{z}$$

$$= 2\int dz = 2z + C_1$$

$$I_1 = 2\sqrt{x^2 - 8x + 7} + C_1 \qquad \ldots(ii)$$

$$I_2 = \int \frac{dx}{\sqrt{x^2 - 8x + 7}}$$

$$= \int \frac{dx}{\sqrt{x^2 - 2.x.4 + 16 - 16 + 7}} = \int \frac{dx}{\sqrt{(x-4)^2 - 3^2}}$$

$$= \log\left|(x-4) + \sqrt{(x-4)^2 - 3^2}\right| + C_2$$

$$I_2 = \log\left|(x-4) + \sqrt{x^2 - 8x + 7}\right| + C_2 \qquad \ldots(iii)$$

Putting the value of I_1 and I_2 in (i)

$$\int \frac{(3x+5)\,dx}{\sqrt{x^2 - 8x + 7}} = \frac{3}{2}\cdot 2\sqrt{x^2 - 8x + 7} + 17\log\left|(x-4) + \sqrt{x^2 - 8x + 7}\right| + (C_1 + C_2)$$

$$= 3\sqrt{x^2 - 8x + 7} + 17\log\left|(x-4) + \sqrt{x^2 - 8x + 7}\right| + C.$$

$$\left[\textbf{Note:} \int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C\right]$$

18. Given equation is

$$(1 + e^{2x})\,dy + (1 + y^2)\,e^x\,dx = 0$$

$$\Rightarrow \qquad (1 + e^{2x})\,dy = -(1 + y^2)\,e^x\,dx \qquad \Rightarrow \qquad \frac{dy}{1+y^2} = -\frac{e^x\,dx}{1 + e^{2x}}$$

Integrating both sides, we get

$$\tan^{-1} y = -\int \frac{e^x\,dx}{1 + (e^x)^2}$$

$$\Rightarrow \qquad \tan^{-1} y = -\int \frac{dz}{1 + z^2} \qquad \text{Let } e^x = z, \quad e^x\,dx = dz$$

$$\Rightarrow \qquad \tan^{-1} y = -\tan^{-1} z + C \qquad \Rightarrow \qquad \tan^{-1} y + \tan^{-1} e^x = c$$

For particular solution :

Putting $y = 1$ and $x = 0$, we get

$$\tan^{-1}(1) + \tan^{-1} e^0 = C \qquad \Rightarrow \qquad \tan^{-1}(1) + \tan^{-1}(1) = c$$

$$\Rightarrow \qquad \frac{\pi}{4} + \frac{\pi}{4} = C \qquad \Rightarrow \qquad C = \frac{\pi}{2}$$

Therefore, required particular solution is

π

$$\tan^{-1} y + \tan^{-1} e^{x} = \frac{}{2}$$

19. Given differential equation is

$$\frac{dy}{dx} + y \cot x = 4x \operatorname{cosec} x$$

$$\Rightarrow \qquad \frac{dy}{dx} + \cot x \,.\, y = 4x \operatorname{cosec} x$$

Comparing the given equation with $\frac{dy}{dx} + Py = Q$, we get

$$P = \cot x,\ Q = 4x \operatorname{cosec} x$$

$$\therefore \qquad \text{I.F.} = e^{\int \cot x \, dx}$$

$$= e^{\log(\sin x)} = \sin x$$

Hence the General solution is

$$y.\sin x = \int 4x.\operatorname{cosec} x.\sin x \, dx + C$$

$$\Rightarrow \qquad y \sin x = \int 4x \, dx + C \qquad [\operatorname{cosec} x \,.\, \sin x = 1]$$

$$\Rightarrow \qquad y \sin x = 2x^2 + C$$

Putting $y = 0$ and $x = \frac{\pi}{2}$, we get

$$0 = 2\frac{\pi^2}{4} + C \ \Rightarrow\ C = -\frac{\pi^2}{2}$$

Therefore, required solution is $y \sin x = 2x^2 - \frac{\pi^2}{2}$

Note: When the given differential equation is in the form of $\frac{dy}{dx} + Py = Q$, where P, Q are constant or function of x only, then general solution is

$$y \times (\text{I.F.}) = \int (Q \times \text{I.F.})\, dx + C$$

where $\qquad \text{I.F.} = e^{\int P dx}$

20. Here

$$\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k},\ \vec{b} = -\hat{i} + 2\hat{j} + \hat{k},\ \vec{c} = 3\hat{i} + \hat{j}$$

$$\vec{a} + \lambda \vec{b} = (2\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-\hat{i} + 2\hat{j} + \hat{k}) = (2-\lambda)\hat{i} + (2+2\lambda)\hat{j} + (3+\lambda)\hat{k}$$

Since $(\vec{a} + \lambda \vec{b})$ is perpendicular to $\vec{c}$

$$\Rightarrow \qquad (\vec{a} + \lambda \vec{b}) \,.\, \vec{c} = 0 \quad \Rightarrow \quad (2-\lambda).3 + (2+2\lambda).1 + (3+\lambda).0 = 0$$

$$\Rightarrow \qquad 6 - 3\lambda + 2 + 2\lambda = 0 \ \Rightarrow\ \lambda = 8$$

[**Note :** If $\vec{a}$ is perpendicular to $\vec{b}$, then $\vec{a}\,.\,\vec{b} = |\vec{a}|.|\vec{b}|.\cos 90° = 0$]

21. Given equation of lines are

$$\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k}) \quad \ldots (i)$$

$$\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k}) \quad \ldots (ii)$$

Comparing (i) and (ii) with $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$, we get

$$\vec{a}_1 = 6\hat{i} + 2\hat{j} + 2\hat{k} \qquad \vec{a}_2 = -4\hat{i} - \hat{k}$$

$$\vec{b}_1 = \hat{i} - 2\hat{j} + 2\hat{k} \qquad \vec{b}_2 = 3\hat{i} - 2\hat{j} - 2\hat{k}$$

$$\vec{a}_1 - \vec{a}_2 = (6\hat{i} + 2\hat{j} + 2\hat{k}) - (-4\hat{i} - \hat{k}) = 10\hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix}$$

$$= (4 + 4)\hat{i} - (-2 - 6)\hat{j} + (-2 + 6)\hat{k}$$

$$= 8\hat{i} + 8\hat{j} + 4\hat{k}$$

$$\therefore \quad |\vec{b}_1 \times \vec{b}_2| = \sqrt{8^2 + 8^2 + 4^2} = \sqrt{144} = 12$$

Therefore, required shortest distance

$$= \left| \frac{(\vec{a}_1 - \vec{a}_2) \cdot (\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

$$= \left| \frac{(10\hat{i} + 2\hat{j} + 3\hat{k}) \cdot (8\hat{i} + 8\hat{j} + 4\hat{k})}{12} \right|$$

$$= \left| \frac{80 + 16 + 12}{12} \right|$$

$$= \frac{108}{12} = 9$$

Note : Shortest distance (S.D) between two skew lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$ is given by

$$\text{S.D.} = \left| \frac{(\vec{a}_1 - \vec{a}_2) \cdot (\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_2 \times \vec{b}_2|} \right|$$

22. The sample space of given experiment is

$$S = \{(HHH), (HHT), (HTT), (TTT), (TTH), (THH), (HTH), (THT)\}$$

Let X denotes the no. of heads in three tosses of a fair coin Here, X is random which may have values 0, 1, 2, 3.

Now, $P(X=0)=\frac{1}{8}$, $P(X=1)=\frac{3}{8}$

$P(X=2)=\frac{3}{8}$, $P(X=3)=\frac{1}{8}$

Therefore, Probability distibution is

X	0	1	2	3
P(X)	1/8	3/8	3/8	1/8

$\therefore$ Mean number $(E(x)) = 0\times\frac{1}{8}+1\times\frac{3}{8}+2\times\frac{3}{8}+3\times\frac{1}{8}$

$$=0+\frac{3}{8}+\frac{6}{8}+\frac{3}{8}=\frac{12}{8}=\frac{3}{2}$$

Section – C

23. Given system of equation is

$$x-y+2z=1,\quad 2y-3z=1,\quad 3x-2y+4z=2$$

Above system of equation can be written in matrix form

as $\quad AX=B \Rightarrow X=A^{-1}B \qquad \ldots(i)$

whrere $A=\begin{bmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}, B=\begin{bmatrix}1\\ 1\\ 2\end{bmatrix}$

Let $C=\begin{bmatrix}-2 & 0 & 1\\ 9 & 2 & -3\\ 6 & 1 & -2\end{bmatrix}$

Now $AC=\begin{bmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}\begin{bmatrix}-2 & 0 & 1\\ 9 & 2 & -3\\ 6 & 1 & -2\end{bmatrix}$

$$=\begin{bmatrix}-2-9+12 & 0-2+2 & 1+3-4\\ 0+18-18 & 0+4-3 & 0-6+6\\ -6-18+24 & 0-4+4 & 3+6-8\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$\Rightarrow \quad AC=I$

$\Rightarrow \quad A^{-1}(AC)=A^{-1}I \qquad$ [Pre multiplication by A^{-1}]

$\Rightarrow \quad (A^{-1}A)C=A^{-1} \qquad$ [By Associativity]

$\Rightarrow \quad IC=A^{-1} \Rightarrow A^{-1}=C$

$$\Rightarrow \qquad A^{-1} = \begin{bmatrix} - & & 0 & 1 \\ 9 & & 2 & -3 \\ 6 & & & \\ & & 1 & -2 \end{bmatrix}$$

Putting X, A^{-1} and B in (i) we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2+0+2 \\ 9+2-6 \\ 6+1-4 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix} \quad \Rightarrow \quad x = 0,\ y = 5 \text{ and } z = 3$$

OR

Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

For elementary row operation, we write

$$A = I A$$

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \leftrightarrow R_1$

$$\begin{bmatrix} 5 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \rightarrow R_1 - 2R_2$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} A$$

$R_1 \rightarrow R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} A$$

$R_3 \rightarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & -1 \\ 0 & 0 & 1 \\ 5 & -2 & 2 \end{bmatrix} A$$

$R_1 \rightarrow R_1 + R_3$, $R_2 \rightarrow R_2 - 3R_3$

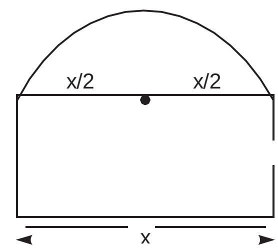
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \end{bmatrix} A$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 5 & -2 & 2 \end{bmatrix}$$

$$I = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} A$$

$$\Rightarrow \qquad A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

24. Let x and y be the length and width of rectangle part of window respectively. Let A be the opening area of window which admit Light. Obviously, for admitting the maximum light through the opening, A must be maximum.



Now A = Area of rectangle + Area of semi-circle

$$A = xy + \frac{1}{2}\pi.\frac{x^2}{4}$$

$$\Rightarrow \qquad A = xy + \frac{\pi x^2}{8}$$

$$\Rightarrow \qquad A = x\left\{5 - \frac{x(\pi+2)}{4}\right\} + \frac{\pi x^2}{8}$$

$$\Rightarrow \qquad A = 5x - \frac{(\pi+2)x^2}{4} + \frac{\pi x^2}{8}$$

$$\Rightarrow \qquad A = 5x - \left(\frac{\pi+2}{4} - \frac{\pi}{8}\right)x^2$$

$$\Rightarrow \qquad A = 5x - \frac{\pi+4}{8}x^2 \Rightarrow \frac{dA}{dx} = 5 - \left(\frac{\pi+4}{8}\right)2x$$

From question

$$\therefore \quad x + 2y + \pi \times \frac{x}{2} = 10$$

$$\Rightarrow \quad x\left(\frac{\pi}{2} + 1\right) + 2y = 10$$

$$\Rightarrow \quad 2y = 10 - x\left(\frac{\pi+2}{2}\right)$$

$$\Rightarrow \quad y = 5 - \frac{x(\pi+2)}{4} \qquad \ldots(i)$$

For maximum or minimum value of A,

$$\frac{dA}{dx} = 0$$

$$\Rightarrow \qquad 5 - \left(\frac{\pi+4}{8}\right).2x = 0 \Rightarrow \frac{\pi+4}{8}.2x = 5$$

$$\Rightarrow \qquad x = \frac{20}{\pi+4}$$

Now $$\frac{d^2A}{dx^2} = -\frac{\pi+4}{8} \times 2 = -\frac{\pi+4}{4}$$

i.e., $$\left.\frac{d^2A}{dx^2}\right]_{x=\frac{20}{\pi+4}} < 0$$

Hence for $x = \dfrac{20}{\pi+4}$, A is maximum

and thus $$y = 5 - \frac{20}{\pi+4} \times \frac{\pi+2}{4} \qquad \left[\text{Putting } x = \frac{20}{\pi+4} \text{ } in \text{ } (i)\right]$$

$$= 5 - \frac{5(\pi + 2)}{\pi + 4}$$

$$= \frac{5\pi + 20 - 5\pi - 10}{\pi + 4} = \frac{10}{\pi + 4}$$

Therefore, for maximum A *i.e.*, for admitting the maximum light

$$\text{Length of rectangle} = x = \frac{20}{\pi + 4}.$$

Breadth of rectangle $= y = \dfrac{10}{\pi + 4}$

25. Given lines are

$$2x + y = 4 \qquad \ldots (i)$$
$$3x - 2y = 6 \qquad \ldots (ii)$$
$$x - 3y + 5 = 0 \qquad \ldots (iii)$$

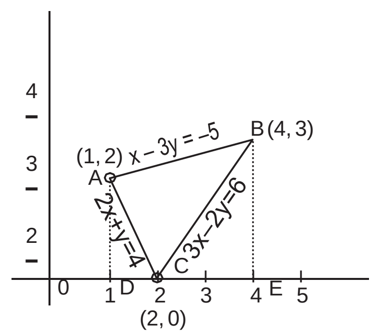
For intersection point of (i) and (ii)

Multiplying (i) by 2 and adding with (ii), we get

$$\begin{aligned} 4x + 2y &= 8 \\ \underline{3x - 2y} &\underline{= 6} \\ 7x &= 14 \quad \Rightarrow \quad x = 2 \end{aligned}$$

$\therefore \qquad y = 0$

Here, intersection point of (i) and (ii) is (2, 0).



For intersection point of (i) and (iii)

Multiplying (i) by 3 and adding with (iii), we get

$$\begin{aligned} 6x + 3y &= 12 \\ \underline{x - 3y} &\underline{= -5} \\ 7x &= 7 \quad \Rightarrow \quad x = 1 \end{aligned}$$

$\therefore \qquad y = 2$

Hence, intersection point of (i) and (iii) is (1, 2).

For intersection point of (ii) and (iii)

Multiplying (iii) by 3 and subtracting from (ii), we get

$$\begin{aligned} 3x - 2y &= 6 \\ \underline{{}_{-}3x \mp 9y} &\underline{= \mp 15} \\ 7y &= 21 \end{aligned}$$

$\Rightarrow \qquad y = 3$

$\therefore \qquad x = 4$

Hence intersection point of (ii) and (iii) is (4, 3).

With the help of intersecting points, required region ΔABC in ploted.

Shaded region is required region.

$\therefore$ Required Area $=$ Area of ΔABC

$=$ Area of trap $ABED$ – Area of ΔADC – Area of ΔCBE

$$= \int_1^4 \frac{x+5}{3}\,dx - \int_1^2 (4 - 2x)\,dx - \int_2^4 \frac{3x - 6}{2}\,dx$$

$$= \frac{1}{3}\left[\frac{x^2}{2} + 5x\right]_1^4 - \left[4x - x^2\right]_1^2 - \frac{1}{2}\left[\frac{3x^2}{2} - 6x\right]_2^4$$

$$= \frac{1}{3}\left\{\left(\frac{16}{2} + 20\right) - \left(\frac{1}{2} + 5\right)\right\} - \{(8-4) - (4-1)\} - \frac{1}{2}\left\{\left(\frac{3\times 16}{2} - 24\right) - \left(\frac{3\times 4}{2} - 12\right)\right\} = \frac{1}{3}\left[28 - \frac{11}{2}\right] - \{4-3\} - \frac{1}{2}\{0+6\}$$

$$= \frac{1}{3} \times \frac{45}{2} - 1 - 3$$

$$= \frac{7}{2} \text{ sq. unit.}$$

26. Comparing $\int_1^4 (x^2 - x)\, dx$ with $\int_a^b f(x)\, dx$, we get

$$f(x) = x^2 - x \quad \text{and} \quad a = 1,\ b = 4$$

By definition

$$\int_a^b f(x)\, dx = \lim_{h \to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f(a + (n-1)h)\right]$$

Here $h = \frac{4-1}{n} = \frac{3}{n}$ where $h = \frac{b-a}{n}$

$\Rightarrow$ $nh = 3$ Also $n \to \infty \Leftrightarrow h \to 0$

$\therefore$
$$\int_1^4 (x^2 - x)\, dx = \lim_{h \to 0} h\,[f(1) + f(1+h) + f(1+2h) + \ldots + f(1 + (n-1)h)]$$

$$= \lim_{h \to 0} h\,[0 + \{(1+h)^2 - (1+h)\} + \{(1+2h)^2 - (1+2h)\} + \ldots + \{(1+(n-1)h)^2 - (1+(n-1)h)\}]$$

$$= \lim_{h \to 0} h\,[0 + \{1 + h^2 + 2h - 1 - h\} + \{1 + 4h^2 + 4h - 1 - 2h\} + \ldots + \{1 + (n-1)^2 h^2 + 2(n-1)h - 1 - (n-1)h\}]$$

$$= \lim_{h \to 0} h\,[0 + (h^2 + h) + (4h^2 + 2h) + \ldots \{(n-1)^2 h^2 + (n-1)h\}]$$

$$= \lim_{h \to 0} h\,[h^2 \{1 + 2^2 + \ldots + (n-1)^2\} + h\{1 + 2 + \ldots + (n-1)\}]$$

$$= \lim_{h \to 0} h\left[h^2 . \frac{(n-1)\,n\,(2n-1)}{6} + h\,\frac{(n-1)n}{2}\right]$$

$$= \lim_{h \to 0}\left[\frac{h^3 . n^3 \left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)}{6} + \frac{h^2 . n^2 \left(1 - \frac{1}{n}\right)}{2}\right]$$

$$= \lim_{n \to \infty}\left[\frac{27\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)}{6} + \frac{9\left(1 - \frac{1}{n}\right)}{2}\right]$$

$$= \frac{27 \times (1-0)(2-0)}{6} + \frac{9(1-0)}{2} = 9 + \frac{9}{2} = \frac{27}{2}$$

$\because\ h = \frac{3}{}$ $\quad \left| \therefore\ h \to 0 \Rightarrow n \to \infty \right]$ $\quad \left[\quad \frac{}{n} \quad \right]$

OR

Let $\sin x - \cos x = z$ If $x = 0, z = -1$

$(\cos x + \sin x)\, dx = dz$ If $x = \frac{\pi}{4}, z = 0$

Also, $\because$ $\sin x - \cos x = z$

$\Rightarrow$ $(\sin x - \cos x)^2 = z^2 \Rightarrow \sin^2 x + \cos^2 x - 2 \sin x . \cos x = z^2$

$\Rightarrow$ $1 - \sin 2x = z^2 \Rightarrow \sin 2x = 1 - z^2$

Now $\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x}\, dx = \int_{-1}^{0} \frac{dz}{9 + 16(1 - z^2)}$

$$= \int_{-1}^{0} \frac{dz}{9 + 16 - 16z^2} = \int_{-1}^{0} \frac{dz}{25 - 16z^2}$$

$$= \frac{1}{16} \int_{-1}^{0} \frac{dz}{\left(\frac{5}{4}\right)^2 - z^2} = \frac{1}{16} \cdot \frac{1}{2 \cdot \frac{5}{4}} \left[\log \left| \frac{\frac{5}{4} + z}{\frac{5}{4} - z} \right| \right]_{-1}^{0}$$

$$= \frac{1}{40} \left[\log 1 - \log \frac{1}{9} \right] = \frac{1}{40} [\log 1 - \log 1 + \log 9]$$

$$= \frac{1}{40} \log 9$$

27. Let equation of plane passing through (–1, 3, 2) be

$$a(x + 1) + b(y - 3) + c(z - 2) = 0 \qquad \ldots (i)$$

Since (*i*) is perpendicular to plane $x + 2y + 3z = 5$

$\Rightarrow$ $a . 1 + b . 2 + c . 3 = 0$

$\Rightarrow$ $a + 2b + 3c = 0 \qquad \ldots (ii)$

Again plane (*i*) is perpendicular to plane $3x + 3y + z = 0$

$\Rightarrow$ $a . 3 + b . 3 + c . 1 = 0$

$\Rightarrow$ $3a + 3b + c = 0 \qquad \ldots (iii)$

From (*ii*) and (*iii*)

$$\frac{a}{2 - 9} = \frac{b}{9 - 1} = \frac{c}{3 - 6}$$

$$\frac{a}{-7} = \frac{b}{8} = \frac{c}{-3} = \lambda \qquad \text{(say)}$$

$\Rightarrow$ $a = -7\lambda, \quad b = 8\lambda, \quad c = -3\lambda$

Putting the value of a, b, c in (*i*), we get

$$-7\lambda(x + 1) + 8\lambda(y - 3) - 3\lambda(z - 2) = 0$$

$\Rightarrow$ $-7x - 7 + 8y - 24 - 3z + 6 = 0$

$\Rightarrow$ $-7x + 8y - 3z - 25 = 0 \Rightarrow 7x - 8y + 3z + 25 = 0$

It is required plane.

28. Let the number of padestal lamps and wooden shades manufactured by cottage industry be x and y respectively.

Here profit is the objective function Z.

$\therefore$ $Z = 5x + 3y \qquad \ldots (i)$

We have to maximise Z subject to the constrains

$$2x + y \le 12 \qquad \dots (ii)$$
$$3x + 2y \le 20 \qquad \dots (iii)$$
$$\left.\begin{array}{l} x \ge 0 \\ y \ge 0 \end{array}\right\} \qquad \dots (iv)$$

$\because$ Graph of $x = 0, y = 0$ is the y-axis and x-axis respectively.

$\therefore$ Graph of $x \ge 0, y \ge 0$ is the Ist quadrant.

Graph for $2x + y \le 12$

Graph of $2x + y = 12$

x	0	6
y	12	0

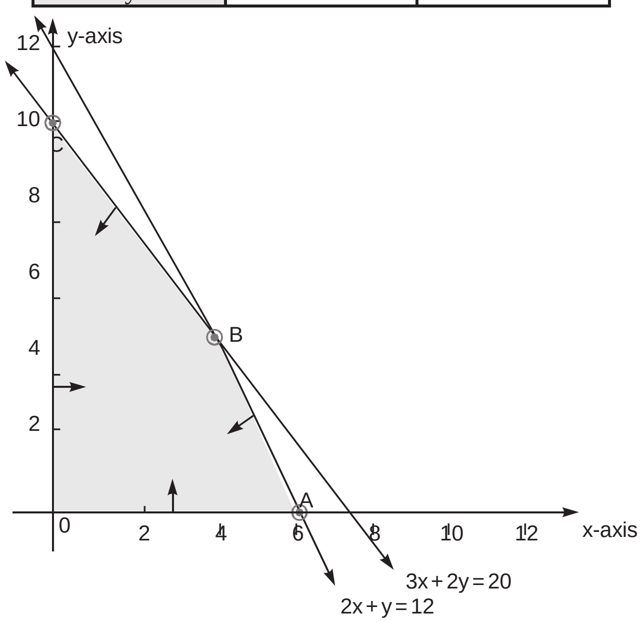
Since (0, 0) satisfy $2x + y \le 12$

$\Rightarrow$ Graph of $2x + y \le 12$ is that half plane in which origin lies.

Graph of $3x + 2y = 20$

Graph for $3x + 2y \le 20$

x	0	20/3
y	10	0



Since (0, 0) Satisfy $3x + 2y \le 20$

$\Rightarrow$ Graph of $3x + 2y \le 20$ is that half plane in which origin lies.

The shaded area $OABC$ is the feasible region whose corner points are O, A, B and C.

For coordinate B.

Equation $2x + y = 12$ and $3x + 2y = 20$ are solved as

$$3x + 2(12 - 2x) = 20$$

$\Rightarrow$ $3x + 24 - 4x = 20 \quad x = 4$

$\therefore$ $\Rightarrow \Rightarrow \quad y = 12 - 8 = 4$

Coordinate of $B = (4, 4)$

Now we evaluate objective function Z at each corner.

Corner points	$\boldsymbol{Z = 5x + 3y}$
0 (0, 0)	0
A (6, 0)	30
B (4, 4)	32 ← maximum
C (0, 10)	30

Hence maximum profit is ₹ 32 when manufacturer produces 4 lamps and 4 shades.

29. Let E_1, E_2 and A be event such that

E_1 = Production of items by machine A

E_2 = Production of items by machine B

A = Selection of defective items.

$$P(E_1) = \frac{60}{100} = \frac{3}{5}, \quad P(E_2) = \frac{40}{100} = \frac{2}{5}$$

$$P\left(\frac{A}{E_1}\right) = \frac{2}{100} = \frac{1}{50}, \quad P\left(\frac{A}{E_2}\right) = \frac{1}{100}$$

$P\left(\frac{E_2}{A}\right)$ is required

By Baye's theorem

$$P\left(\frac{E_2}{A}\right) = \frac{P(E_2) \cdot P\left(\frac{A}{E_2}\right)}{P(E_1) \cdot P\left(\frac{A}{E_1}\right) + P(E_2) \cdot P\left(\frac{A}{E_2}\right)}$$

$$P\left(\frac{E_2}{A}\right) = \frac{\frac{2}{5} \times \frac{1}{100}}{\frac{3}{5} \times \frac{1}{50} + \frac{2}{5} \times \frac{1}{100}}$$

$$= \frac{\frac{2}{500}}{\frac{3}{250} + \frac{2}{500}} = \frac{2}{500} \times \frac{500}{6 + 2} = \frac{1}{4}$$

CBSE (Foreign) Set-II

9.
$$
\begin{aligned}
fog\,(x) &= f(g(x)) \\
&= f(|5x-2|) \\
&= \big|\,|5x-2|\,\big| \\
&= |5x-2|
\end{aligned}
$$

10.
$$I = \int \frac{(e^{2x} - e^{-2x})}{e^{2x} + e^{-2x}}\,dx$$

Let $e^{2x} + e^{-2x} = z$

$$(2e^{2x} - 2e^{-2x})\,dx = dz$$

$$(e^{2x} - e^{-2x})\,dx = \frac{dz}{2}$$

$\therefore$
$$
\begin{aligned}
I &= \frac{1}{2}\int \frac{dz}{z} \\
&= \frac{1}{2}\log|z| + C \\
&= \frac{1}{2}\log\left| e^{2x} + e^{-2x} \right| + C
\end{aligned}
$$

19. L.H.S. $= \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

Applying $R_1 \rightarrow R_1 + R_2 + R_3$, we get

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Applying $C_1 \rightarrow C_1 - C_3$ and $C_2 \rightarrow C_2 - C_3$, we get

$$= (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ 0 & -b-c-a & 2b \\ c+a+b & c+a+b & c-a-b \end{vmatrix}$$

Expanding along R_1, we get

$$
\begin{aligned}
&= (a+b+c)\,[0 - 0 + 1\,\{0 - (-b-c-a).(c+a+b)\}] \\
&= (a+b+c).(a+b+c)^2 \\
&= (a+b+c)^3 = \text{RHS}
\end{aligned}
$$

20. $\lim_{x \to \pi^-} f(x) = \lim_{h \to 0} f(\pi - h)$ $\left[\begin{array}{l} \text{Let } x = \pi - h \\ x \to \pi^- \Rightarrow h \to 0 \end{array}\right.$

$= \lim_{h \to 0} K(\pi - h) + 1$ $\qquad [\because f(x) = kx + 1 \text{ for } x \le \pi$

$= K\pi + 1$

$\lim_{h \to \pi^+} f(x) = \lim_{h \to 0} f(\pi + h)$ $\left[\begin{array}{l} \text{Let } x = \pi + h \\ x \to \pi^+ \Rightarrow h \to 0 \end{array}\right]$

$= \lim_{h \to 0} \cos(\pi + h)$ $\quad [\because f(x) = \cos x \text{ for } x > \pi]$

$= \lim_{h \to 0} -\cos h \quad = -1$

Also $\quad f(\pi) = k\pi + 1$

Since $f(x)$ is continuous at $x = \pi$

$\Rightarrow \quad \lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x) = f(\pi)$

$\Rightarrow \quad k\pi + 1 = -1 = k\pi + 1$

$\Rightarrow \quad k\pi = -2$

$\Rightarrow \quad k = -\dfrac{2}{\pi}$

21. Given differential equation is

$$\frac{dy}{dx} + 2\tan x . y = \sin x$$

Comparing it with $\dfrac{dy}{dx} + Py = Q$, we get

$$P = 2\tan x,\ Q = \sin x$$

$\therefore \quad \text{I.F.} = e^{\int 2\tan x dx}$

$= e^{2\log \sec x} = e^{\log \sec^2 x}$ $\qquad [\because e^{\log z} = z]$

$= \sec^2 x$

Hence general solution is

$$y.\sec^2 x = \int \sin x . \sec^2 x \, dx + C$$

$$y.\sec^2 x = \int \sec x . \tan x \, dx + C \quad \Rightarrow y.\sec^2 x = \sec x + C$$

$\Rightarrow \quad y = \cos x + C\cos^2 x$

Putting $y = 0$ and $x = \dfrac{\pi}{3}$, we get

$$0 = \cos\frac{\pi}{3} + C.\cos^2\frac{\pi}{3}$$

$$0 = \frac{1}{2} + \frac{C}{4} \quad \Rightarrow \quad C = -2$$

$\therefore$ Required solution is $y = \cos x - 2\cos^2 x$

22. Given equation of lines are

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k}) \qquad \dots (i)$$

$$\vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu(2\hat{i} + 3\hat{j} + \hat{k}) \qquad \dots (ii)$$

Comparing (i) and (ii) with $\vec{r} = \vec{a}_1 + \lambda \vec{b}$ and $\vec{r} = \vec{a}_2 + \lambda \vec{b}$ respectively we get.

$$\vec{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k} \qquad \vec{a}_2 = 4\hat{i} + 5\hat{j} + 6\hat{k}$$

$$\vec{b}_1 = \hat{i} - 3\hat{j} + 2\hat{k} \qquad \vec{b}_2 = 2\hat{i} + 3\hat{j} + \hat{k}$$

Now $\vec{a}_2 - \vec{a}_1 = 3\hat{i} + 3\hat{j} + 3\hat{k}$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= (-3-6)\hat{i} - (1-4)\hat{j} + (3+6)\hat{k} = -9\hat{i} + 3\hat{j} + 9\hat{k}$$

$\therefore$
$$|\vec{b}_1 \times \vec{b}_1| = \sqrt{(-9)^2 + 3^2 + 9^2} = 3\sqrt{19}$$

$\therefore$
$$\text{S.D.} = \left| \frac{(\vec{a}_2 - \vec{a}_1) . (\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

$$= \left| \frac{(3\hat{i} + 3\hat{j} + 3\hat{k}) . (-9\hat{i} + 3\hat{j} + 9\hat{k})}{3\sqrt{19}} \right| = \frac{-27 + 9 + 27}{3\sqrt{19}} = \frac{3}{\sqrt{19}}.$$

28. Let equation of plane passing through A (2, 2, –1) be

$$a(x-2) + b(y-2) + c(z+1) = 0 \qquad \dots (i)$$

Since, B (3, 4, 2) lies on plane (i)

$\Rightarrow$ $$a(3-2) + b(4-2) + c(2+1) = 0$$

$\Rightarrow$ $$a + 2b + 3c = 0 \qquad \dots (ii)$$

Again C (7,0,6) lie on plane (i)

$\Rightarrow$ $$a(7-2) + b(0-2) + c(6+1) = 0$$

$\Rightarrow$ $$5a - 2b + 7c = 0 \qquad \dots (iii)$$

From (ii) and (iii)

$$\frac{a}{14+6} = \frac{b}{15-7} = \frac{c}{-2-10}$$

$$\frac{a}{20} = \frac{b}{8} = \frac{c}{-12} = \lambda \text{ (say)}$$

$$a = 20\lambda,\ b = 8\lambda,\ c$$

$= -12\lambda$ Putting the value of a, b, c in (i)

$$20\lambda(x-2) + 8\lambda(y-2) - 12\lambda(z+1) = 0$$

$\Rightarrow$ $$20x - 40 + 8y - 16 - 12z - 12 = 0$$

$\Rightarrow$ $$20x + 8y - 12z - 68 = 0$$

$\Rightarrow$ $$5x + 2y - 3z - 17 = 0$$

$\Rightarrow 5x + 2y - 3z = 17$ which is required cartesian equation of plane.

Its vector form is

$$(x\hat{i} + y\hat{j} + z\hat{k}).(5\hat{i} + 2\hat{j} - 3\hat{k}) = 17$$

$\Rightarrow \vec{r}\,.(5\hat{i} + 2\hat{j} - 3\hat{k}) = 17$

29. Let E_1, E_2 and A be event such that

E_1 = red ball is transferred from Bag I to Bag II

E_2 = black ball is transferred from Bag I to Bag II

A = drawing red ball from Bag II

Now $\quad P(E_1) = \dfrac{3}{7} \quad P(E_2) = \dfrac{4}{7}$

$$P\left(\frac{A}{E_1}\right) = \frac{5}{10},\ P\left(\frac{A}{E_2}\right) = \frac{4}{10},\ P\left(\frac{E_2}{A}\right) \text{ is required.}$$

From Baye's theorem.

$$P\left(\frac{E_2}{A}\right) = \frac{P(E_2).P\left(\frac{A}{E_2}\right)}{P(E_1).P\left(\frac{A}{E_1}\right) + P(E_2).P\left(\frac{A}{E_2}\right)}$$

$$= \frac{\frac{4}{7} \times \frac{4}{10}}{\frac{3}{7} \times \frac{5}{10} + \frac{4}{7} \times \frac{4}{10}} = \frac{16}{15+16} = \frac{16}{31}$$

CBSE (Foreign) Set-III

1. $fog(x) = f(g(x))$

$= f(x^{1/3})$

$= 8(x^{1/3})^3$

$= 8x$

2. $$I = \int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$$

Let $\quad \sqrt{x} = t$

$$\frac{1}{2\sqrt{x}}\,dx = dt \quad \Rightarrow \quad \frac{1}{\sqrt{x}}\,dx = 2dt$$

$\therefore \quad I = 2\int \cos t\,dt$

$= 2\sin t + C$

$= 2\sin\sqrt{x} + C$

11. L.H.S $= \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$

Applying $C_1 \to C_1 \to C_2 + C_3$ we get

$$= \begin{vmatrix} 2(x+y+z) & x & y \\ 2(x+y+z) & y+z+2x & y \\ 2(x+y+z) & x & z+x+2y \end{vmatrix}$$

$$= 2(x+y+z) \begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix} \qquad \text{[Taking common from } C_1]$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$= 2(x+y+z) \begin{vmatrix} 1 & x & y \\ 0 & x+y+z & 0 \\ 0 & 0 & x+y+z \end{vmatrix}$$

Expanding along C_1, we get

$$= 2(x+y+z)\,[1\{(x+y+z)^2 - 0\} - 0 + 0]$$

$$= 2(x+y+z)^3 = \text{RHS}$$

12. $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0} \lambda(x^2 - 2x) \qquad [\therefore f(x) = \lambda(x^2 - 2x) \text{ for } x \le 0]$

$$= \lambda(0 - 0) = 0$$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0} 4x + 1 \qquad [\therefore f(x) = 4x + 1 \text{ for } x > \text{o}]$

$$= 4 \times 0 + 1 = 1$$

Since $\lim\limits_{x \to o^-} f(x) \ne \lim\limits_{x \to 0^+} f(x)$ for any value of λ. Hence for no value of λ, f is continuous at $x = 0$

13. Given differential equation is

$$(1+x^2)\frac{dy}{dx} + 2xy = \frac{1}{1+x^2} \qquad \Rightarrow \quad \frac{dy}{dx} + \frac{2x}{1+x^2}\cdot y = \frac{1}{(1+x^2)^2}$$

Comparing this equation with $\frac{dy}{dx} + Py = Q$ we get

$$P = \frac{2x}{1+x^2},\ Q = \frac{1}{(1+x^2)^2}$$

$\therefore \qquad$ I.F. $= e^{\int P dx}$

$$\text{I.F.} = e^{\int \frac{2x}{1+x^2} dx}$$

$$= e^{\int \frac{dt}{t}} \qquad \left[\begin{array}{l} \text{Let } \quad t = 1 + x^2 \\ \qquad dt = 2x dx \end{array}\right.$$

$$= e^{\log t}$$

$$= t = 1 + x^2$$

Hence general solution is

$$y.(1+x^2)=\int \frac{1}{(1+x^2)^2}.(1+x^2)\,dx+C$$

$$\Rightarrow \qquad y.(1+x^2)=\int \frac{dx}{1+x^2}+C$$

$$\Rightarrow \qquad y.(1+x^2)=\tan^{-1}x+C$$

Putting $y=0$ and $x=1$ we get

$$0=\tan^{-1}(1)+C$$

$$C=-\frac{\pi}{4}$$

Hence required solution is

$$y.(1+x^2)=\tan^{-1}x-\frac{\pi}{4}$$

14. Given lines are

$$\vec{r}=(\hat{i}+2\hat{j}+\hat{k})+\lambda(\hat{i}-\hat{j}+\hat{k}) \qquad \ldots (i)$$

$$\vec{r}=(2\hat{i}-\hat{j}-\hat{k})+\mu(2\hat{i}+\hat{j}+2\hat{k}) \qquad \ldots (ii)$$

Comparing the equation (i) and (ii) with $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\lambda\vec{b}_2$.

We get

$$\vec{a}_1=\hat{i}+2\hat{j}+\hat{k} \qquad \vec{a}_2=2\hat{i}-\hat{j}-\hat{k}$$

$$\vec{b}_1=\hat{i}-\hat{j}+\hat{k} \qquad \vec{b}_2=2\hat{i}+\hat{j}+2\hat{k}$$

Now
$$\vec{a}_1-\vec{a}_1=(2\hat{i}-\hat{j}-\hat{k})-(\hat{i}+2\hat{j}+\hat{k})$$

$$=\hat{i}-3\hat{j}-2\hat{k}$$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 1\\ 2 & 1 & 2\end{vmatrix}$$

$$=(-2-1)\hat{i}-(2-2)\hat{j}+(1+2)\hat{k}=-3\hat{i}+3\hat{k}$$

$$\therefore \qquad |\vec{b}_1\times\vec{b}_2|=\sqrt{(-3)^2+(3)^2}=3\sqrt{2}$$

$$\therefore \qquad \text{Shortest distance}=\left|\frac{(\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|$$

$$=\left|\frac{(\hat{i}-3\hat{j}-2\hat{k}).(-3\hat{i}+0\hat{j}+3\hat{k})}{|\vec{b}_1\times\vec{b}_2|}\right|$$

$$= \left| \frac{-3-0-6}{3\sqrt{2}} \right|$$

$$= \frac{9}{3\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{9\sqrt{2}}{3 \times 2} = \frac{3\sqrt{2}}{2}$$

23. Let the equation of plane passing through point (1, 1, –1) be

$$a(x-1) + b(y-1) + c(z+1) = 0 \quad \ldots (i)$$

Since (i) is perpendicular to the plane $x + 2y + 3z - 7 = 0$

$\therefore \quad 1 \,.\, a + 2 \,.\, b + 3 \,.\, c = 0$

$$a + 2b + 3c = 0 \quad \ldots (ii)$$

Again plane (i) is perpendicular to the plane $2x - 3y + 4z = 0$

$\therefore \quad 2 \,.\, a - 3 \,.\, b + 4 \,.\, c = 0$

$$2a - 3b + 4c = 0 \quad \ldots (iii)$$

From (ii) and (iii), we get

$$\frac{a}{8+9} = \frac{b}{6-4} = \frac{c}{-3-4}$$

$\Rightarrow \quad \frac{a}{17} = \frac{b}{2} = \frac{c}{-7} = \lambda$

$\Rightarrow \quad a = 17\lambda,\ b = 2\lambda,\ c$

$= -7\lambda$ Puttting the value of a, b, c in (i) we get

$$17\lambda(x-1) + 2\lambda(y-1) - 7\lambda(z+1) = 0$$

$\Rightarrow \quad 17(x-1) + 2(y-1) - 7(z+1) = 0$

$$17x + 2y - 7z - 17 - 2 - 7 = 0$$

$\Rightarrow \quad 17x + 2y - 7z - 26 = 0$

$\Rightarrow$

It is required equation.

[**Note:** The equation of plane pasing through (x_1, y_1, z_1) is given by

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

where a, b, c are direction ratios of normal of plane.]

24. Let E_1, E_2, E_3 and A be events such that

E_1 = event of selecting two headed coin.

E_2 = event of selecting biased coin.

E_3 = event of selecting unbiased coin.

A = event of getting head.

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

$$P\left(\frac{A}{E_1}\right)=1,\quad P\left(\frac{A}{E_2}\right)=\frac{75}{100}=\frac{3}{4},\quad P\left(\frac{A}{E_3}\right)=\frac{1}{2}$$

$P\left(\frac{E_1}{A}\right)$ is required.

By Baye's Theorem,

$$P\left(\frac{E_1}{A}\right)=\frac{P(E_1)\,.\,P\left(\frac{A}{E_1}\right)}{P(E_1)\,.\,P\left(\frac{A}{E_1}\right)+P(E_2)\,.\,P\left(\frac{A}{E_2}\right)+P(E_3)\,.\,P\left(\frac{A}{E_3}\right)}$$

$$=\frac{\frac{1}{3}\times 1}{\frac{1}{3}\times 1+\frac{1}{3}\times\frac{3}{4}+\frac{1}{3}\times\frac{1}{2}}$$

$$P\left(\frac{E_1}{A}\right)=\frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{4}+\frac{1}{6}}$$

$$=\frac{1}{3}\times\frac{12}{9}=\frac{4}{9}$$

CBSE Examination Paper (Delhi 2012)

Time allowed: 3 hours **Maximum marks: 100**

General Instructions:

1. ***All*** *questions are compulsory.*
2. *The question paper consists of* ***29*** *questions divided into three Sections* ***A, B*** *and* ***C.*** *Section A comprises of* ***10*** *questions of* ***one*** *mark each, Section B comprises of* ***12*** *questions of* ***four*** *marks each and Section C comprises of* ***7*** *questions of* ***six*** *marks each.*
3. ***All*** *questions in Section A are to be answered in* ***one*** *word,* ***one*** *sentence or as per the exact requirement of the question.*
4. *There is no overall choice. However, internal choice has been provided in* ***4*** *questions of* ***four*** *marks each and* ***2*** *questions of* ***six*** *marks each. You have to attempt only* ***one*** *of the alternatives in all such questions.*
5. *Use of calculators is* ***not*** *permitted.*

Set–I

SECTION–A

Question numbers 1 to 10 carry 1 mark each.

1. If a line has direction ratios 2, –1, –2, then what are its direction cosines?
2. Find 'λ' when the projection of $\vec{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$ on $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$ is 4 units.
3. Find the sum of the vectors $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.
4. Evaluate: $\int_2^3 \frac{1}{x}\,dx$.
5. Evaluate: $\int (1 - x)\sqrt{x}\,dx$.
6. If $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$, write the minor of the element a_{23}.
7. If $\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ -9 & x \end{bmatrix}$, write the value of x.

8. Simplify: $\cos\theta\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}+\sin\theta\begin{bmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{bmatrix}$.

9. Write the principal value of

$$\cos^{-1}\left(\frac{1}{2}\right)-2\sin^{-1}\left(-\frac{1}{2}\right).$$

10. Let $*$ be a 'binary' operation on N given by $a*b=\text{LCM}\,(a, b)$ for all $a,b\in N$. Find $5*7$.

SECTION–B

Question numbers 11 to 22 carry 4 marks each.

11. If $(\cos x)^y=(\cos y)^x$, find $\frac{dy}{dx}$.

OR

If $\sin y = x\sin(a+y)$, prove that $\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$.

12. How many times must a man toss a fair coin, so that the probability of having at least one head is more than 80%?

13. Find the Vector and Cartesian equations of the line passing through the point (1, 2, –4) and perpendicular to the two lines $\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}$ and $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$.

14. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $|\vec{a}|=5$, $|\vec{b}|=12$ and $|\vec{c}|=13$, and $\vec{a}+\vec{b}+\vec{c}=\vec{O}$, find the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$.

15. Solve the following differential equation:

$$2x^2\frac{dy}{dx}-2xy+y^2=0.$$

16. Find the particular solution of the following differential equation:

$$\frac{dy}{dx}=1+x^2+y^2+x^2y^2,\text{ given that } y=1 \text{ when } x=0.$$

17. Evaluate: $\int \sin x \sin 2x \sin 3x\, dx$

OR

Evaluate: $\int \frac{2}{(1-x)(1+x^2)}\,dx$

18. Find the point on the curve $y=x^3-11x+5$ at which the equation of tangent is $y=x-11$.

OR

Using differentials, find the approximate value of $\sqrt{49.5}$.

19. If $y=(\tan^{-1}x)^2$, show that $(x^2+1)^2\frac{d^2y}{dx^2}+2x(x^2+1)\frac{dy}{dx}=2$.

20. Using properties of determinants, prove that

$$\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

21. Prove that $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \frac{\pi}{4} - \frac{x}{2}, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

OR

Prove that $\sin^{-1}\left(\frac{8}{17}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \cos^{-1}\left(\frac{36}{85}\right)$.

22. Let $A = R - \{3\}$ and $B = R - \{1\}$. Consider the function $f: A \to B$ defined by $f(x) = \left(\frac{x-2}{x-3}\right)$. Show that f is one-one and onto and hence find f^{-1}.

SECTION–C

Question numbers 23 to 29 carry 6 marks each.

23. Find the equation of the plane determined by the points $A(3, -1, 2)$, $B(5, 2, 4)$ and $C(-1, -1, 6)$ and hence find the distance between the plane and the point $P(6, 5, 9)$.

24. Of the students in a college, it is known that 60% reside in hostel and 40% day scholars (not residing in hostel). Previous year results report that 30% of all students who reside in hostel attain 'A' grade and 20% of day scholars attain 'A' grade in their annual examination. At the end of the year, one student is chosen at random from the college and he has an 'A' grade, what is the probability that the student is a hosteler?

25. A manufacturer produces nuts and bolts. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts. It takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of ₹17.50 per package on nuts and ₹7 per package of bolts. How many packages of each should be produced each day so as to maximize his profits if he operates his machines for at the most 12 hours a day? Form the linear programming problem and solve it graphically.

26. Prove that: $\int_0^{\frac{\pi}{4}} (\sqrt{\tan x} + \sqrt{\cot x})\,dx = \sqrt{2}.\frac{\pi}{2}$

OR

Evaluate: $\int_1^3 (2x^2 + 5x)\,dx$ as a limit of a sum.

27. Using the method of integration, find the area of the region bounded by the lines $3x - 2y + 1 = 0$, $2x + 3y - 21 = 0$ and $x - 5y + 9 = 0$.

28. Show that the height of a closed right circular cylinder of given surface and maximum volume, is equal to the diameter of its base.

29. Using matrices, solve the following system of linear equations:

$$x - y + 2z = 7$$
$$3x + 4y - 5z = -5$$
$$2x - y + 3z = 12$$

OR

Using elementary operations, find the inverse of the following matrix:

$$\begin{pmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$$

Set–II

Only those questions, not included in Set I, are given.

9. Find the sum of the following vectors:

$$\vec{a} = \hat{i} - 2\hat{j},\ \vec{b} = 2\hat{i} - 3\hat{j},\ \vec{c} = 2\hat{i} + 3\hat{k}.$$

10. If $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$, write the cofactor of the element a_{32}.

19. Using properties of determinants, prove the following:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

20. If $y = 3\cos(\log x) + 4\sin(\log x)$, show that

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

21. Find the equation of the line passing through the point (–1, 3, –2) and perpendicular to the lines

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \text{ and } \frac{x+2}{-3} = \frac{y-1}{2} = \frac{z+1}{5}$$

22. Find the particular solution of the following differential equation:

$$(x+1)\frac{dy}{dx} = 2e^{-y} - 1;\ y = 0 \text{ when } x = 0.$$

28. A girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin two times and notes the number of heads obtained. If she obtained exactly two heads, what is the probability that she threw 1, 2, 3 or 4 with the die?

29. Using the method of integration, find the area of the region bounded by the following lines:

$$3x - y - 3 = 0$$
$$2x + y - 12 = 0$$
$$x - 2y - 1 = 0$$

Set–III

Only those questions, not included in Set I and Set II, are given.

9. Find the sum of the following vectors:

$$\vec{a} = \hat{i} - 3\hat{k},\ \vec{b} = 2\hat{j} - \hat{k},\ \vec{c} = 2\hat{i} - 3\hat{j} + 2\hat{k}.$$

10. If $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 5 & 3 & 8 \end{vmatrix}$, write the minor of element a_{22}.

19. Using properties of determinants, prove the following:

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = ab + bc + ca + abc$$

20. If $y = \sin^{-1} x$, show that $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 0.$

21. Find the particular solution of the following differential equation:

$$xy\frac{dy}{dx} = (x+2)(y+2);\ y = -1 \text{ when } x = 1$$

22. Find the equation of a line passing through the point $P(2, -1, 3)$ and perpendicular to the lines

$$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k}) \text{ and } \vec{r} = (2\hat{i} - \hat{j} - 3\hat{k}) + \mu(\hat{i} + 2\hat{j} + 2\hat{k}).$$

28. Bag I contains 3 red and 4 black balls and Bag II contains 4 red and 5 black balls. Two balls are transferred at random from Bag I to Bag II and then a ball is drawn from Bag II. The ball so drawn is found to be red in colour. Find the probability that the transferred balls were both black.

29. Using the method of integration, find the area of the region bounded by the following lines:

$$5x - 2y - 10 = 0$$
$$x + y - 9 = 0$$
$$2x - 5y - 4 = 0$$

Solutions

Set–I

SECTION–A

1. Here direction ratios of line are 2, –1, –2

$\therefore$ Direction cosines of line are $\dfrac{2}{\sqrt{2^2+(-1)^2+(-2)^2}}, \dfrac{-1}{\sqrt{2^2+(-1)^2+(-2)^2}}, \dfrac{-2}{\sqrt{2^2+(-1)^2+(-2)^2}}$

i.e., $\dfrac{2}{3}, \dfrac{-1}{3}, \dfrac{-2}{3}$

[**Note:** If *a, b, c* are the direction ratios of a line, the direction cosines are $\dfrac{a}{\sqrt{a^2+b^2+c^2}}$, $\dfrac{b}{\sqrt{a^2+b^2+c^2}}, \dfrac{c}{\sqrt{a^2+b^2+c^2}}$]

2. We know that projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

$$\Rightarrow \qquad 4 = \frac{\vec{a}.\vec{b}}{|\vec{b}|} \qquad ...(i)$$

Now, $\vec{a}.\vec{b} = 2\lambda + 6 + 12 = 2\lambda + 18$

Also $|\vec{b}| = \sqrt{2^2 + 6^2 + 3^2} = \sqrt{4 + 36 + 9} = 7$

Putting in (*i*) we get

$$4 = \frac{2\lambda + 18}{7}$$

$$\Rightarrow \qquad 2\lambda = 28 - 18 \qquad \Rightarrow \qquad \lambda = \frac{10}{2} = 5$$

3. $\vec{a} + \vec{b} + \vec{c} = (1-2+1)\hat{i} + (-2+4-6)\hat{j} + (1+5-7)\hat{k}$

$= -4\hat{j} - \hat{k}$

4. $\displaystyle\int_2^3 \frac{1}{x}\,dx = [\log x]_2^3 = \log 3 - \log 2$

5. $\displaystyle\int (1-x)\sqrt{x}\,dx = \int \sqrt{x}\,dx - \int x^{1+\frac{1}{2}}\,dx$

$\displaystyle = \int x^{\frac{1}{2}}\,dx - \int x^{\frac{3}{2}}\,dx$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + c = \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} + c$$

6. Minor of $a_{23} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 10 - 3 = 7.$

7. Given $\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \cdot \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ -9 & x \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 2 \times 1 + 3 \times (-2) & 2 \times (-3) + 3 \times 4 \\ 5 \times 1 + 7 \times (-2) & 5 \times (-3) + 7 \times 4 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ -9 & x \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -4 & 6 \\ -9 & 13 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ -9 & x \end{bmatrix}$$

Equating the corresponding elements, we get

$x = 13$

8. $\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta & \sin\theta.\cos\theta \\ -\sin\theta.\cos\theta & \cos^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & -\sin\theta.\cos\theta \\ \sin\theta.\cos\theta & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \sin^2\theta + \cos^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

9. We have, $\cos^{-1}\left(\frac{1}{2}\right) = \cos^{-1}\left(\cos\frac{\pi}{3}\right)$

$$= \frac{\pi}{3} \qquad \left[\because \frac{\pi}{3} \in [0, \pi]\right]$$

Also $\sin^{-1}\left(-\frac{1}{2}\right) = \sin^{-1}\left(-\sin\frac{\pi}{6}\right) = \sin^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right)$

$$= -\frac{\pi}{6} \qquad \left[\because -\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right]$$

$$\therefore \cos^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{3} - 2\left(-\frac{\pi}{6}\right)$$

$$= \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$$

[**Note:** Principal value branches of sin x and cos x are $\left[-\frac{}{2}, \frac{}{2}\right]$ and [0, π] respectively.]

10. $5 * 7$ = LCM of 5 and 7 = 35

SECTION–B

11. Given,

$$(\cos x)^y = (\cos y)^x$$

Taking logrithm of both sides, we get

$$\log(\cos x)^y = \log(\cos y)^x$$

$\Rightarrow \qquad y \,.\, \log(\cos x) = x \,.\, \log(\cos y) \qquad\qquad [\text{Q } \log m^n = n\log m]$

Differentiating both sides we get

$$\Rightarrow \qquad y \,.\, \frac{1}{\cos x}(-\sin x) + \log(\cos x)\,.\,\frac{dy}{dx} = x\,.\,\frac{1}{\cos y}\,.\,(-\sin y)\,.\,\frac{dy}{dx} + \log(\cos y)$$

$$\Rightarrow \qquad -\frac{y\sin x}{\cos x} + \log(\cos x)\,.\,\frac{dy}{dx} = -\frac{x\sin y}{\cos y}\,.\,\frac{dy}{dx} + \log(\cos y)$$

$$\Rightarrow \qquad \log(\cos x)\,.\,\frac{dy}{dx} + \frac{x\sin y}{\cos y}\,.\,\frac{dy}{dx} = \log(\cos y) + \frac{y\sin x}{\cos x}$$

$$\Rightarrow \qquad \frac{dy}{dx}\left[\log(\cos x) + \frac{x\sin y}{\cos y}\right] = \log(\cos y) + \frac{y\sin x}{\cos x}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{\log(\cos y) + \dfrac{y\sin x}{\cos x}}{\log(\cos x) + \dfrac{x\sin y}{\cos y}} = \frac{\log(\cos y) + y\tan x}{\log(\cos x) + x\tan y}$$

OR

Here $\qquad \sin y = x\sin(a+y)$

$$\Rightarrow \qquad \frac{\sin y}{\sin(a+y)} = x$$

$$\Rightarrow \qquad \frac{\sin(a+y)\,.\cos y\,.\dfrac{dy}{dx} - \sin y\,.\cos(a+y).\,\dfrac{dy}{dx}}{\sin^2(a+y)} = 1$$

$$\Rightarrow \qquad \frac{dy}{dx}\{\sin(a+y)\,.\cos y - \sin y\,.\cos(a+y)\} = \sin^2(a+y)$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin(a+y-y)}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

12. Let no. of times of tossing a coin be n.

Here, Probability of getting a head in a chance $= p = \dfrac{1}{2}$

Probability of getting no head in a chance $= q = 1 - \dfrac{1}{2} = \dfrac{1}{2}$

Now, P (having at least one head) $= P\,(X \geq 1)$

$$= 1 - P\,(X = 0)$$
$$= 1 - {}^nC_0 p^0 . q^{n-0}$$
$$= 1 - 1.1.\left(\frac{1}{2}\right)^n = 1 - \left(\frac{1}{2}\right)^n$$

From question

$$1 - \left(\frac{1}{2}\right)^n > \frac{80}{100}$$

$$\Rightarrow \quad 1 - \left(\frac{1}{2}\right)^n > \frac{8}{10} \quad \Rightarrow \quad 1 - \frac{8}{10} > \frac{1}{2^n}$$

$$\Rightarrow \quad \frac{1}{5} > \frac{1}{2^n} \quad \Rightarrow \quad 2^n > 5$$

$$\Rightarrow \quad n \geq 3$$

A man must have to toss a fair coin 3 times.

13. Let the cartesian equation of line passing through (1, 2, –4) be

$$\frac{x-1}{a} = \frac{y-2}{b} = \frac{z+4}{c} \qquad ...(i)$$

Given lines are

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \qquad ...(ii)$$

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \qquad ...(iii)$$

Obviously parallel vectors $\vec{b}_1$, $\vec{b}_2$ and $\vec{b}_3$ of (*i*), (*ii*) and (*iii*) respectively are given as

$$\vec{b}_1 = a\hat{i} + b\hat{j} + c\hat{k}$$

$$\vec{b}_2 = 3\hat{i} - 16\hat{j} + 7\hat{k}$$

$$\vec{b}_3 = 3\hat{i} + 8\hat{j} - 5\hat{k}$$

From question

$$(i) \perp (ii) \quad \Rightarrow \quad \vec{b}_1 \perp \vec{b}_2 \quad \Rightarrow \quad \vec{b}_1 . \vec{b}_2 = 0$$

$$(i) \perp (iii) \quad \Rightarrow \quad \vec{b}_1 \perp \vec{b}_3 \quad \Rightarrow \quad \vec{b}_1 . \vec{b}_3 = 0$$

Hence, $3a - 16b + 7c = 0$ ...(*iv*)

and $3a + 8b - 5c = 0$ ...(*v*)

From equation (*iv*) and (*v*)

$$\frac{a}{80-56} = \frac{b}{21+15} = \frac{c}{24+48}$$

$$\Rightarrow \quad \frac{a}{24} = \frac{b}{36} = \frac{c}{72}$$

$$\Rightarrow \qquad \frac{a}{2} = \frac{b}{3} = \frac{c}{6} = \lambda \text{ (say)}$$

$$\Rightarrow \qquad a = 2\lambda, \;\; b = 3\lambda, \;\; c = 6\lambda$$

Putting the value of a, b, c in (i) we get required cartesian equation of line as

$$\frac{x-1}{2\lambda} = \frac{y-2}{3\lambda} = \frac{z+4}{6\lambda}$$

$$\Rightarrow \qquad \frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$$

Hence vector equation is

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

14. $\because \qquad \vec{a} + \vec{b} + \vec{c} = \vec{O} \qquad \ldots(i)$

$$\Rightarrow \qquad \vec{a}\cdot(\vec{a} + \vec{b} + \vec{c}) = \vec{a}\cdot\vec{O}$$

$$\Rightarrow \qquad \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} = 0$$

$$\Rightarrow \qquad \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} = -|\vec{a}|^2 \qquad \left[\because \vec{a}\cdot\vec{a} = |\vec{a}|^2\right]$$

$$\Rightarrow \qquad \vec{a}\cdot\vec{b} + \vec{c}\cdot\vec{a} = -25 \qquad \ldots(ii) \qquad \left[\because \vec{a}\cdot\vec{c} = \vec{c}\cdot\vec{a}\right]$$

Similarly taking dot product of both sides of (i) by $\vec{b}$ and $\vec{c}$ respectively we get

$$\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} = -|\vec{b}|^2 = -144 \qquad \ldots(iii)$$

and $\qquad \vec{c}\cdot\vec{a} + \vec{b}\cdot\vec{c} = -|\vec{c}|^2 = -169 \qquad \ldots(iv)$

Adding (ii), (iii) and (iv) we get

$$\vec{a}\cdot\vec{b} + \vec{c}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} + \vec{b}\cdot\vec{c} = -25 - 144 - 169$$

$$\Rightarrow \qquad 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = -338$$

$$\Rightarrow \qquad \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} = -\frac{338}{2} = -169$$

15. Given $\qquad 2x^2\frac{dy}{dx} - 2xy + y^2 = 0$

$$\Rightarrow \qquad 2x^2\frac{dy}{dx} = 2xy - y^2$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{2xy - y^2}{2x^2} \qquad \ldots(i)$$

It is homogeneous differential equation.

Let $\qquad y = vx \qquad \Rightarrow \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}$

Equation (i) becomes

$$v + x\frac{dv}{dx} = \frac{2x.vx - v^2x^2}{2x^2}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{2x^2\left(v - \frac{v^2}{2}\right)}{2x^2} \quad \Rightarrow \quad x\frac{dv}{dx} = v - \frac{v^2}{2} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{v^2}{2} \quad \Rightarrow \quad \frac{dx}{x} = -\frac{2dv}{v^2}$$

Integrating both sides we get

$$\Rightarrow \quad \int\frac{dx}{x} = -2\int\frac{dv}{v^2}$$

$$\Rightarrow \quad \log|x| + c = -2\frac{v^{-2+1}}{-2+1} \quad \Rightarrow \quad \log|x| + c = 2.\frac{1}{v}$$

Putting $v = \frac{y}{x}$, we get

$$\log|x| + c = \frac{2x}{y}$$

16. Given: $\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$

$$\Rightarrow \quad \frac{dy}{dx} = (1 + x^2) + y^2(1 + x^2) \quad \Rightarrow \quad \frac{dy}{dx} = (1 + x^2)(1 + y^2)$$

$$\Rightarrow \quad (1 + x^2)\,dx = \frac{dy}{(1 + y^2)}$$

Integrating both sides we get

$$\int(1 + x^2)\,dx = \int\frac{dy}{(1 + y^2)}$$

$$\Rightarrow \quad \int dx + \int x^2\,dx = \int\frac{dy}{(1+y^2)} \quad \Rightarrow \quad x + \frac{x^3}{3} + c = \tan^{-1} y$$

$$\Rightarrow \quad \tan^{-1} y = x + \frac{x^3}{3} + c$$

Putting $y = 1$ and $x = 0$, we get

$$\Rightarrow \quad \tan^{-1}(1) = 0 + 0 + c$$

$$\Rightarrow \quad c = \tan^{-1}(1) = \frac{\pi}{4}$$

Therefore required particular solution is

$$\tan^{-1} y = x + \frac{x^3}{3} + \frac{\pi}{4}$$

17. Let $I = \int \sin x \,.\, \sin 2x \,.\, \sin 3x \, dx.$

$$= \frac{1}{2} \int 2 \sin x \,.\, \sin 2x \,.\, \sin 3x \, dx$$

$$= \frac{1}{2} \int \sin x \,.\, (2 \sin 2x \,.\, \sin 3x) \, dx$$

$$= \frac{1}{2} \int \sin x \,.\, (\cos x - \cos 5x) \, dx \qquad [\because 2 \sin A \sin B = \cos (A - B) - \cos (A + B)]$$

$$= \frac{1}{2 \times 2} \int 2 \sin x \,.\, \cos x \, dx - \frac{1}{2 \times 2} \int 2 \sin x \,.\, \cos 5x \, dx$$

$$= \frac{1}{4} \int \sin 2x \, dx - \frac{1}{4} \int (\sin 6x - \sin 4x) \, dx$$

$$= -\frac{\cos 2x}{8} + \frac{\cos 6x}{24} - \frac{\cos 4x}{16} + C$$

OR

Here $\int \frac{2}{(1-x)(1+x^2)} dx$

Now, $\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx + C}{1 + x^2}$

$\Rightarrow \quad \frac{2}{(1-x)(1+x^2)} = \frac{A(1 + x^2) + (Bx + C)(1 - x)}{(1-x)(1+x^2)}$

$\Rightarrow \quad 2 = A(1 + x^2) + (Bx + C)(1 - x)$

$\Rightarrow \quad 2 = A + Ax^2 + Bx - Bx^2 + C - Cx$

$\Rightarrow \quad 2 = (A + C) + (A - B)x^2 + (B - C)x$

Equating co-efficient both sides, we get

$A + C = 2 \qquad \text{...}(i)$

$A - B = 0 \qquad \text{...}(ii)$

$B - C = 0 \qquad \text{...}(iii)$

From (*ii*) and (*iii*) $A = B = C$

Putting $C = A$ in (*i*), we get

$A + A = 2$

$\Rightarrow i \quad 2A = 2 \quad \Rightarrow \quad A = 1$

.e., $\quad A = B = C = 1$

$\therefore \quad \frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2}$

$\therefore \quad \int \frac{2}{(1-x)(1+x^2)} = \int \frac{1}{1-x} dx + \int \frac{x+1}{1+x^2} dx$

$$= -\log|1-x| + \int \frac{x}{1+x^2}\,dx + \int \frac{1}{1+x^2}\,dx$$

$$= -\log|1-x| + \frac{1}{2}\log|1+x^2| + \tan^{-1}x + c$$

18. Let the required point of contact be (x_1, y_1).

Given curve is

$$y = x^3 - 11x + 5 \qquad ...(i)$$

$\therefore$ $\dfrac{dy}{dx} = 3x^2 - 11$

$\Rightarrow$ $\left[\dfrac{dy}{dx}\right]_{(x_1, y_1)} = 3x_1^2 - 11$

i.e., Slope of tangent at (x_1, y_1) to give curve $(i) = 3x_1^2 - 11$

From question

$3x_1^2 - 11 =$ Slope of line $y = x - 11$, which is also tangent

$3x_1^2 - 11 = 1$

$\Rightarrow$ $x_1^2 = 4 \quad \Rightarrow \quad x_1 = \pm 2$

Since (x_1, y_1) lie on curve (i)

$\therefore$ $y_1 = x_1^3 - 11x_1 + 5$

When $x_1 = 2,\ y_1 = 2^3 - 11 \times 2 + 5 = -9$

$x_1 = -2,\ y_1 = (-2)^3 - 11 \times (-2) + 5 = 19$

But $(-2, 19)$ does not satisfy the line $y = x - 11$

Therefore $(2, -9)$ is required point of curve at which tangent is $y = x - 11$

OR

Let $f(x) = \sqrt{x}$, where $x = 49$

let $\delta x = 0.5$

$\therefore$ $f(x + \delta x) = \sqrt{x + \delta x} = \sqrt{49.5}$

Now by definition, approximately we can write

$$f'(x) = \frac{f(x+\delta x) - f(x)}{\delta x} \qquad ...(i)$$

Here $f(x) = \sqrt{x} = \sqrt{49} = 7$

$\delta x = 0.5$

$$f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{49}} = \frac{1}{14}$$

Putting these values in (i), we get

$$\frac{1}{14} = \frac{\sqrt{49.5} - 7}{0.5}$$

$$\Rightarrow \qquad \sqrt{49.5} = \frac{0.5}{14} + 7$$

$$= \frac{0.5 + 98}{14} = \frac{98.5}{14} = 7.036$$

19. We have $\quad y = (\tan^{-1} x)^2 \qquad \ldots (i)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2\tan^{-1} x \cdot \frac{1}{1+x^2} \qquad \ldots (ii)$$

or $\qquad (1+x^2)y_1 = 2\tan^{-1} x$

Again differentiating w.r.t. x, we get

$$(1+x^2).\frac{dy_1}{dx} + y_1 \frac{d}{dx}(1+x^2) = 2.\frac{1}{1+x^2}$$

$$\Rightarrow \qquad (1+x^2).y_2 + y_1.2x = \frac{2}{1+x^2}$$

or $\qquad (1+x^2)^2 y_2 + 2x(1+x^2)y_1 = 2$

20. LHS $\qquad \Delta = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$

Applying, $R_1 \leftrightarrow R_3$ and $R_3 \leftrightarrow R_2$, we get

$$= \begin{vmatrix} a+b & p+q & x+y \\ b+c & q+r & y+z \\ c+a & r+p & z+x \end{vmatrix}$$

Applying, $R_1 \to R_1 + R_2 + R_3$, we get

$$\Delta = \begin{vmatrix} 2(a+b+c) & 2(p+q+r) & 2(x+y+z) \\ b+c & q+r & y+z \\ c+a & r+p & z+x \end{vmatrix}$$

$$= 2\begin{vmatrix} a+b+c & p+q+r & x+y+z \\ b+c & q+r & y+z \\ c+a & r+p & z+x \end{vmatrix}$$

$$= 2\begin{vmatrix} a & p & x \\ b+c & q+r & y+z \\ c+a & r+p & z+x \end{vmatrix} \quad [\text{Applying } R_1 \to R_1 - R_2]$$

$$= 2\begin{vmatrix} a & p & x \\ b+c & q+r & y+z \\ c & r & z \end{vmatrix} \quad [\text{Applying } R_3 \to R_3 - R_1]$$

Again applying $R_2 \to R_2 - R_3$, we get

$$\Delta = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = \text{RHS}$$

21. Now, $\tan^{-1}\left(\dfrac{\cos x}{1+\sin x}\right) = \tan^{-1}\left(\dfrac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\cos\frac{x}{2}.\sin\frac{x}{2}}\right)$

$$= \tan^{-1}\left[\frac{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}\right]$$

$$= \tan^{-1}\left(\frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}}\right) = \tan^{-1}\left(\frac{\frac{\cos\frac{x}{2}}{\cos\frac{x}{2}} - \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{\frac{\cos\frac{x}{2}}{\cos\frac{x}{2}} + \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}\right)$$

$$= \tan^{-1}\left(\frac{1 - \tan\frac{x}{2}}{1 + \tan\frac{x}{2}}\right) = \tan^{-1}\left(\frac{\tan\frac{\pi}{4} - \tan\frac{x}{2}}{1 + \tan\frac{\pi}{4}\tan\frac{x}{2}}\right)$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right]$$

$$= \frac{\pi}{4} - \frac{x}{2}$$

$\because x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow -\frac{\pi}{2} < x < \frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{4}$

$\Rightarrow \frac{\pi}{4} > -\frac{x}{2} > -\frac{\pi}{4}$

$\Rightarrow \frac{\pi}{4} + \frac{\pi}{4} > \frac{\pi}{4} - \frac{x}{2} > -\frac{\pi}{4} + \frac{\pi}{4}$

$\Rightarrow \frac{\pi}{2} > \frac{\pi}{4} - \frac{x}{2} > 0$

$\Rightarrow \left(\frac{\pi}{4} - \frac{x}{2}\right) \in \left(0, \frac{\pi}{2}\right) \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

OR

Let $\sin^{-1}\left(\frac{8}{17}\right) = \alpha$ and $\sin^{-1}\left(\frac{3}{5}\right) = \beta$

$\Rightarrow \quad \sin\alpha = \frac{8}{17}$ and $\sin\beta = \frac{3}{5}$

$\Rightarrow \quad \cos\alpha = \sqrt{1-\sin^2\alpha}$ and $\cos\beta = \sqrt{1-\sin^2\beta}$

$\Rightarrow \quad \cos\alpha = \sqrt{1-\frac{64}{289}}$ and $\cos\beta = \sqrt{1-\frac{9}{25}}$

$\Rightarrow \quad \cos\alpha = \sqrt{\frac{289-64}{289}}$ and $\cos\beta = \sqrt{\frac{25-9}{25}}$

$\Rightarrow \quad \cos\alpha = \sqrt{\frac{225}{289}}$ and $\cos\beta = \sqrt{\frac{16}{25}}$

$\Rightarrow \quad \cos\alpha = \frac{15}{17}$ and $\cos\beta = \frac{4}{5}$

Now, $\quad \cos(\alpha+\beta) = \cos\alpha.\cos\beta - \sin\alpha.\sin\beta$

$\Rightarrow \quad \cos(\alpha+\beta) = \frac{15}{17}\times\frac{4}{5} - \frac{8}{17}\times\frac{3}{5}$

$\Rightarrow \quad \cos(\alpha+\beta) = \frac{60}{85} - \frac{24}{85} \quad \Rightarrow \quad \cos(\alpha+\beta) = \frac{36}{85}$

$\Rightarrow \quad \alpha+\beta = \cos^{-1}\left(\frac{36}{85}\right)$

$\Rightarrow \quad \sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \cos^{-1}\left(\frac{36}{85}\right)$ [Putting the value of α, β]

22. Let $x_1, x_2 \in A$.

Now, $\quad f(x_1) = f(x_2) \quad \Rightarrow \quad \frac{x_1-2}{x_1-3} = \frac{x_2-2}{x_2-3}$

$\Rightarrow \quad (x_1-2)(x_2-3) = (x_1-3)(x_2-2)$

$x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 2x_1 - 3x_2 + 6$

$\Rightarrow \quad -3x_1 - 2x_2 = -2x_1 - 3x_2$

$-x_1 = -x_2 \quad \Rightarrow \quad x_1 = x_2$

$\Rightarrow$

$x - 2 \quad \Rightarrow$

Hence f is one-one function.

For Onto

Let $\quad y = \frac{}{x-3}$

$x =$

$\Rightarrow \quad xy - 3y = x - 2 \quad \Rightarrow \quad xy - x = 3y - 2$

$\Rightarrow \quad x(y-1) = 3y - 2$

$\Rightarrow \quad \frac{3y-2}{y-1}$ ...(i)

From above it is obvious that $\forall\ y$ except 1, *i.e.*, $\forall\ y \in B = R - \{1\}\ \exists\, x \in A$

Hence f is onto function.

Thus f is one-one onto function.

It f^{-1} is inverse function of f then

$$f^{-1}(y) = \frac{3y-2}{y-1} \qquad \text{[from } (i)\text{]}$$

SECTION–C

23. The equation of the plane through three non-collinear points A(3, –1, 2), B(5, 2, 4) and C (–1, –1, 6) can be expressed as

$$\begin{vmatrix} x-3 & y+1 & z-2 \\ 5-3 & 2+1 & 4-2 \\ -1-3 & -1+1 & 6-2 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x-3 & y+1 & z-2 \\ 2 & 3 & 2 \\ -4 & 0 & 4 \end{vmatrix} = 0$$

$$\Rightarrow \quad 12(x-3) - 16(y+1) + 12(z-2) = 0$$

$\Rightarrow \quad 12x - 16y + 12z - 76 = 0 \Rightarrow 3x - 4y + 3z - 19 = 0$ is the required equation.

Now, distance of $P(6, 5, 9)$ from the plane is given by

$$= \left|\frac{3\times 6 - 4(5) + 3(9) - 19}{\sqrt{9+16+9}}\right| = \left|\frac{6}{\sqrt{34}}\right| = \frac{6}{\sqrt{34}} \text{ units.}$$

24. Let E_1, E_2 and A be events such that

E_1 = student is a hosteler

E_2 = student is a day scholar

A = getting A grade.

Now from question

$$P(E_1) = \frac{60}{100} = \frac{6}{10}, \qquad P(E_2) = \frac{40}{100} = \frac{4}{10}$$

$$P\left(\frac{A}{E_1}\right) = \frac{30}{100} = \frac{3}{10}, \qquad P\left(\frac{A}{E_2}\right) = \frac{20}{100} = \frac{2}{10}$$

We have to find $P\left(\frac{E_1}{A}\right)$.

Now, $$P\left(\frac{E_1}{A}\right) = \frac{P(E_1).P\left(\frac{A}{E_1}\right)}{P(E_1).P\left(\frac{A}{E_1}\right) + P(E_2).P\left(\frac{A}{E_2}\right)}$$

$$= \frac{\frac{6}{10}.\frac{3}{10}}{\frac{6}{10}.\frac{3}{10} + \frac{4}{10}.\frac{2}{10}} = \frac{\frac{18}{100}}{\frac{18}{100} + \frac{8}{100}} = \frac{18}{100} \times \frac{100}{26} = \frac{18}{26} = \frac{9}{13}$$

25. Let x package nuts and y package bolts are produced

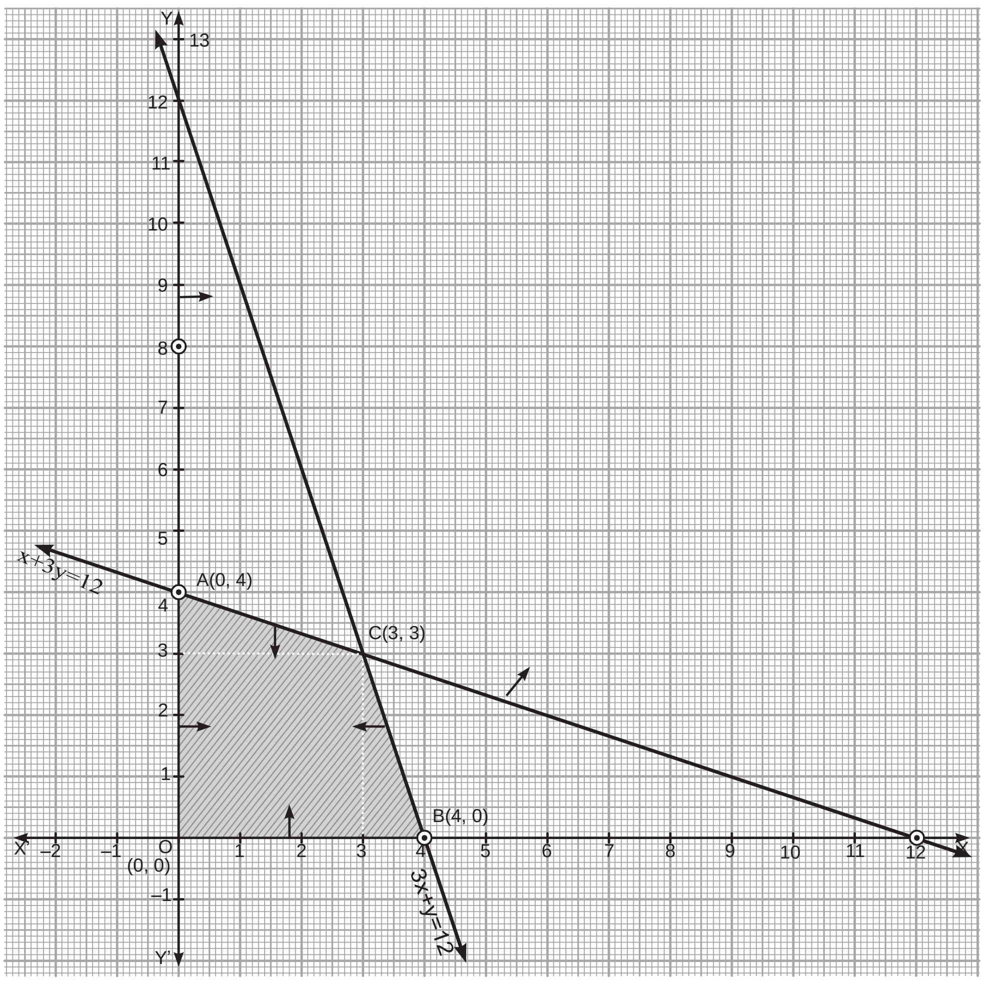
Let Z be the profit function, which we have to maximize.

Here $Z = 17.50x + 7y$ ... (i) is objective function.

And constraints are

$$x + 3y \le 12 \quad \ldots(ii)$$
$$3x + y \le 12 \quad \ldots(iii)$$
$$x \ge 0 \quad \ldots(iv)$$
$$y \ge 0 \quad \ldots(v)$$

On plotting graph of above constraints or inequalities (ii), (iii), (iv) and (v) we get shaded region as feasible region having corner points A, O, B and C.



For coordinate of 'C' two equations

$$x + 3y = 12 \quad \ldots(vi)$$
$$3x + y = 12 \quad \ldots(vii) \text{ are solved}$$

Applying (*vi*) × 3 – (*vii*), we get

$$3x + 9y - 3x - y = 36 - 12$$

$\Rightarrow \quad 8y = 24 \quad \Rightarrow \quad y = 3 \quad$ and $\quad x = 3$

Hence coordinate of C are (3, 3).

Now the value of Z is evaluated at corner point as

Corner point	$Z = 17.5x + 7y$
(0, 4)	28
(0, 0)	0
(4, 0)	70
(3, 3)	73.5 ← Maximum

Therefore maximum profit is ₹73.5 when 3 package nuts and 3 package bolt are produced.

26. LHS $= \int_0^{\frac{\pi}{4}} (\sqrt{\tan x} + \sqrt{\cot x})dx$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right)dx = \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{\sqrt{\sin x . \cos x}}dx$$

$$= \sqrt{2}\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{\sqrt{2\sin x . \cos x}}dx = \sqrt{2}\int_0^{\frac{\pi}{4}} \frac{(\sin x + \cos x)}{\sqrt{1 - (\sin x - \cos x)^2}}dx$$

Let $\sin x - \cos x = z$

$\Rightarrow \quad (\cos x + \sin x)dx = dz$

Also if $x = 0, \ z = -1$

and $\quad x = \frac{\pi}{4}, z = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$

$\therefore$ LHS $= \sqrt{2}\int_{-1}^{0} \frac{dz}{\sqrt{1 - z^2}}$

$$= \sqrt{2}\,[\sin^{-1} z]_{-1}^{0} = \sqrt{2}\,[\sin^{-1} 0 - \sin^{-1}(-1)]$$

$$= \sqrt{2}\left[0 - \left(-\frac{\pi}{2}\right)\right] = \sqrt{2} . \frac{\pi}{2} = \text{RHS}$$

OR

Let $\quad f(x) = 2x^2 + 5x$

Here $a = 1, \ b = 3 \qquad \therefore \quad h = \frac{b - a}{n} = \frac{3 - 1}{n} = \frac{2}{n}$

$\Rightarrow \quad nh = 2$

Also, $n \to \infty \Leftrightarrow h \to 0.$

Q $\quad \int_a^b f(x)dx = \lim_{h\to 0} h[f(a) + f(a+h) + \ldots + f\{a+(n-1)h\}]$

$\therefore \quad \int_1^3 (2x^2 + 5x)dx = \lim_{h\to 0} h[f(1) + f(1+h) + \ldots + f\{1+(n-1)h\}]$

$$= \lim_{h\to 0} h[\{2\times 1^2 + 5\times 1\} + \{2(1+h)^2 + 5(1+h)\} + \ldots + \{2(1-(n-1)h)^2 + 5((1+(n-1)h\}]$$

$$= \lim_{h\to 0} h[(2+5) + \{2+4h+2h^2 + 5 + 5h\} + \ldots + \{2 + 4(n-1)h + 2(n-1)^2 h^2 + 5 + 5(n-1)h\}]$$

$$= \lim_{h\to 0} h[7 + \{7 + 9h + 2h^2\} + \ldots + \{7 + 9(n-1)h + 2(n-1)^2 h^2\}]$$

$$= \lim_{h\to 0} h[7n + 9h\{1 + 2 + \ldots + (n-1)\} + 2h^2\{1^2 + 2^2 + \ldots + (n-1)^2\}]$$

$$= \lim_{h\to 0}\left[7nh + 9h^2\,\frac{(n-1).n}{2} + 2h^3\,\frac{(n-1).n(2n-1)}{6}\right]$$

$$= \lim_{h\to 0}\left[7(nh) + \frac{9(nh)^2.\left(1-\frac{1}{n}\right)}{2} + \frac{2(nh)^3.\left(1-\frac{1}{n}\right).\left(2-\frac{1}{n}\right)}{6}\right]$$

$$= \lim_{n\to\infty}\left[14 + \frac{36\left(1-\frac{1}{n}\right)}{2} + \frac{16\left(1-\frac{1}{n}\right).\left(2-\frac{1}{n}\right)}{6}\right] \qquad [\text{Q } nh = 2]$$

$$= \lim_{n\to\infty}\left[14 + 18\left(1-\frac{1}{n}\right) + \frac{8}{3}\left(1-\frac{1}{n}\right).\left(2-\frac{1}{n}\right)\right]$$

$$= 14 + 18 + \frac{8}{3}\times 1\times 2$$

$$= 32 + \frac{16}{3} = \frac{96+16}{3} = \frac{112}{3}$$

27. Given lines are

$3x - 2y + 1 = 0 \qquad \ldots(i)$

$2x + 3y - 21 = 0 \qquad \ldots(ii)$

$x - 5y + 9 = 0 \qquad \ldots(iii)$

For intersection of (*i*) and (*ii*)

Applying (*i*) × 3 + (*ii*) × 2, we get

$9x - 6y + 3 + 4x + 6y - 42 = 0$

$\Rightarrow \quad 13x - 39 = 0$

$\Rightarrow \quad x = 3$

Putting it in (*i*), we get

$9 - 2y + 1 = 0$

$\Rightarrow \quad 2y = 10 \quad \Rightarrow \quad y = 5$

Intersection point of (*i*) and (*ii*) is (3, 5)

For intersection of (*ii*) and (*iii*)

Applying (*ii*) – (*iii*) × 2, we get

$2x + 3y - 21 - 2x + 10y - 18 = 0$

$\Rightarrow \quad 13y - 39 = 0$

$\Rightarrow \quad y = 3$

Putting $y = 3$ in (*ii*), we get

$2x + 9 - 21 = 0$

$\Rightarrow \quad 2x - 12 = 0$

$\Rightarrow \quad x = 6$

Intersection point of (*ii*) and (*iii*) is (6, 3)

For intersection of (*i*) and (*iii*)

Applying (*i*) – (*iii*) × 3, we get

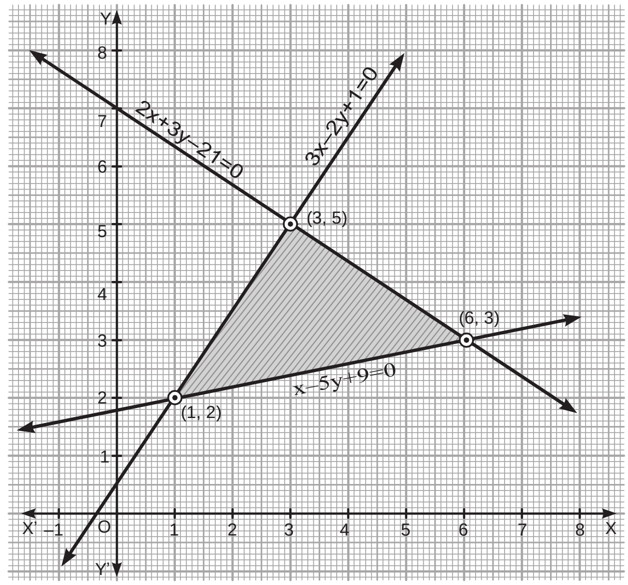
$3x - 2y + 1 - 3x + 15y - 27 = 0$

$\Rightarrow \quad 13y - 26 = 0 \quad \Rightarrow \quad y = 2$

Putting $y = 2$ in (*i*), we get

$3x - 4 + 1 = 0$

$\Rightarrow \quad x = 1$



Intersection point of (*i*) and (*iii*) is (1, 2)

With the help of point of intersection we draw the graph of lines (*i*), (*ii*) and (*iii*)

Shaded region is required region.

$$\therefore \text{ Area of Required region} = \int_1^3 \frac{3x+1}{2}\,dx + \int_3^6 \frac{-2x+21}{3}\,dx - \int_1^6 \frac{x+9}{5}\,dx$$

$$= \frac{3}{2}\int_1^3 x\,dx + \frac{1}{2}\int_1^3 dx - \frac{2}{3}\int_3^6 x\,dx + 7\int_3^6 dx - \frac{1}{5}\int_1^6 x\,dx - \frac{9}{5}\int_1^6 dx$$

$$= \frac{3}{2}\left[\frac{x^2}{2}\right]_1^3 + \frac{1}{2}[x]_1^3 - \frac{2}{3}\left[\frac{x^2}{2}\right]_3^6 + 7[x]_3^6 - \frac{1}{5}\left[\frac{x^2}{2}\right]_1^6 - \frac{9}{5}[x]_1^6$$

$$= \frac{3}{4}(9-1) + \frac{1}{2}(3-1) - \frac{2}{6}(36-9) + 7(6-3) - \frac{1}{10}(36-1) - \frac{9}{5}(6-1)$$

$$= 6 + 1 - 9 + 21 - \frac{7}{2} - 9$$

$$= 10 - \frac{7}{2} = \frac{20-7}{2} = \frac{13}{2}$$

28. Let r and h be radius and height of given cylinder of surface area S.

If V be the volume of cylinder then

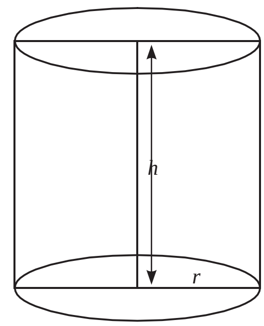
$$V = \pi r^2 h$$

$$V = \frac{\pi r^2.(S-2\pi r^2)}{2\pi r} \qquad \left[\because\ S = 2\pi r^2 + 2\pi rh \ \Rightarrow\ \frac{S-2\pi r^2}{2\pi r} = h\right]$$

$$\Rightarrow \quad V = \frac{Sr - 2\pi r^3}{2}$$

$$\Rightarrow \quad \frac{dV}{dr} = \frac{1}{2}(S - 6\pi r^2)$$

For maximum or minimum value of V



$$\frac{dV}{dr} = 0$$

$$\Rightarrow \quad \frac{1}{2}(S - 6\pi r^2) = 0 \Rightarrow \quad S - 6\pi r^2 = 0$$

$$\Rightarrow \quad r^2 = \frac{S}{6\pi} \qquad \Rightarrow \quad r = \sqrt{\frac{S}{}}$$

$$\frac{2}{d^2}\ 6\pi \quad \text{Now}\ \frac{d\ \ V}{2\quad r} = -\frac{1}{} \times 12\pi r$$

$$\Rightarrow \quad \frac{d^2V}{dr^2} = -6\pi r$$

$$\Rightarrow \quad \left[\frac{d^2V}{dr^2}\right]_{r=\sqrt{\frac{S}{6\pi}}} = -\text{ve}$$

Hence for $r = \sqrt{\frac{S}{6\pi}}$. Volume V is maximum.

$$\Rightarrow \quad h = \frac{S - 2\pi.\frac{S}{6\pi}}{2\pi\sqrt{\frac{S}{6\pi}}} \quad \Rightarrow \quad h = \frac{3S - S}{3 \times 2\pi} \times \sqrt{\frac{6\pi}{S}}$$

$$\Rightarrow \quad h = \frac{2S}{6\pi}.\frac{\sqrt{6\pi}}{\sqrt{S}} = 2\sqrt{\frac{S}{6\pi}}$$

$$\Rightarrow \quad h = 2r \text{ (diameter)} \qquad \left[\because\ r = \sqrt{\frac{S}{6\pi}}\right]$$

Therefore, for maximum volume height of cylinder in equal to diameter of its base.

29. The given system of equation can be written in matric form as $AX = B$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 7 \\ -5 \\ \ \end{bmatrix}$$

$$\text{Now, } |A| = \begin{vmatrix} \lfloor 12 \rfloor 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{vmatrix} = 1(12-5) + 1(9+10) + 2(-3-8)$$

$$= 7 + 19 - 22 = 4 \neq 0$$

Hence A^{-1} exist and system have unique solution.

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 4 & -5 \\ -1 & 3 \end{vmatrix} = 12 - 5 = 7$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} 3 & -5 \\ 2 & 3 \end{vmatrix} = -(9 + 10) = -19$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} = +(-3 - 8) = -11$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 2 \\ -1 & 3 \end{vmatrix} = -(-3+2) = 1$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = +(3 - 4) = -1$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -(-1 + 2) = -1$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 2 \\ 4 & -5 \end{vmatrix} = +(5 - 8) = -3$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ 3 & -5 \end{vmatrix} = -(-5 - 6) = 11$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix} = +(4 + 3) = 7$$

$$\therefore \quad adj A = \begin{bmatrix} 7 & -19 & -11 \\ 1 & -1 & -1 \\ -3 & 11 & 7 \end{bmatrix}^T = \begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|} adj A = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

Q $\quad AX = B$

$\Rightarrow \quad X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}\begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 49 - 5 - 36 \\ -133 + 5 + 132 \\ -77 + 5 + 84 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix} \qquad \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Equating the corresponding elements, we get

$x = 2,\ y = 1,\ z = 3$

OR

Let $A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

For applying elementary row operation we write,

$$A = IA$$

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \leftrightarrow R_2$, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 - 3R_1$, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 - \frac{2}{3}R_2$, we get

$$\begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 3 & 5 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2/3 & 1/3 & 0 \\ 1 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

Applying $R_2 \to \frac{1}{3}R_2$, we get

$$\begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 5/3 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2/3 & 1/3 & 0 \\ 1/3 & 1/3 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

Applying $R_3 \to R_3 + 5R_2$, we get

$$\begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 5/3 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} -2/3 & 1/3 & 0 \\ 1/3 & 1/3 & 0 \\ 5/3 & -4/3 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 + R_3$ and $R_2 \to R_2 - 5R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5/3 & -4/3 & 1 \end{bmatrix} A$$

Applying $R_3 \to 3R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{bmatrix} A$$

Hence $\quad A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{bmatrix}$

Set–II

9. $\vec{a} + \vec{b} + \vec{c} = (\hat{i} - 2\hat{j}) + (2\hat{i} - 3\hat{j}) + (2\hat{i} + 3\hat{k})$

$$= 5\hat{i} - 5\hat{j} + 3\hat{k}$$

10. Co-factor of $a_{32} = (-1)^{3+2} \begin{vmatrix} 5 & 8 \\ 2 & 1 \end{vmatrix} = -(5 - 16) = 11$

19. LHS $= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b - a & c - a \\ a^3 & b^3 - a^3 & c^3 - a^3 \end{vmatrix}$$

Taking out $(b - a)$, $(c - a)$ common from C_2 and C_3 respectively, we get

$$= (b - a)(c - a) \begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^3 & b^2 + ab + a^2 & c^2 + ac + a^2 \end{vmatrix}$$

Expanding along R_1, we get

$$= -(a - b)(c - a)[1(c^2 + ac + a^2 - b^2 - ab - a^2) - 0 + 0]$$

$$= -(a - b)(c - a)(c^2 + ac - b^2 - ab)$$

$$= -(a - b)(c - a)\{-(b^2 - c^2) - a(b - c)\}$$

$$= -(a - b)(c - a)\{(b - c)(-b - c - a)\}$$

$$= (a - b)(b - c)(c - a)(a + b + c)$$

20. Given, $y = 3\cos(\log x) + 4\sin(\log x)$

Differentiating w.r.t. x, we have

$$\frac{dy}{dx} = -\frac{3\sin(\log x)}{x} + \frac{4\cos(\log x)}{x}$$

$$\Rightarrow \qquad y_1 = \frac{1}{x}\left[-3\sin(\log x) + 4\cos(\log x)\right]$$

Again differentiating w.r.t. x, we have

$$\frac{d^2y}{dx^2} = \frac{x\left[\frac{-3\cos(\log x)}{x} - \frac{4\sin(\log x)}{x}\right] - [-3\sin(\log x) + 4\cos(\log x)]}{x^2}$$

$$= \frac{-3\cos(\log x) - 4\sin(\log x) + 3\sin(\log x) + 4\cos(\log x)}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{-\sin(\log x) - 7\cos(\log x)}{x^2}$$

$$\Rightarrow \qquad y_2 = \frac{-\sin(\log x) - 7\cos(\log x)}{x^2}$$

Now, L.H.S. $= x^2 y_2 + x y_1 + y$

$$= x^2\left(\frac{-\sin(\log x) - 7\cos(\log x)}{x^2}\right) + x \times \frac{1}{x}[-3\sin(\log x) + 4\cos(\log x)]$$
$$+ 3\cos(\log x) + 4\sin(\log x)$$

$$= -\sin(\log x) - 7\cos(\log x) - 3\sin(\log x) + 4\cos(\log x)$$
$$+ 3\cos(\log x) + 4\sin(\log x)$$

$= 0 =$ RHS

21. Let the direction ratios of the required line be a, b, c. Since the required line is perpendicular to the given lines, therefore,

$$a + 2b + 3c = 0 \qquad \ldots(i)$$

and $\qquad -3a + 2b + 5c = 0 \qquad \ldots(ii)$

Solving (i) and (ii), by cross multiplication, we get

$$\frac{a}{10-6} = \frac{b}{-9-5} = \frac{c}{2+6} = k \text{ (let)}$$

$$\Rightarrow \qquad a = 4k,\ b = -14k,\ c = 8k$$

Thus, the required line passing through $P(-1, 3, -2)$ and having the direction ratios $a = 4k,\ b = -14k,\ c = 8k$ is $\frac{x+1}{4k} = \frac{y-3}{-14k} = \frac{z+2}{8k}$.

Removing k, we get $\frac{x+1}{4} = \frac{y-3}{-14} = \frac{z+2}{8}$ or $\frac{x+1}{2} = \frac{y-3}{-7} = \frac{z+2}{4}$ which is the required equation of the line.

22. Given $(x+1)\dfrac{dy}{dx} = 2e^{-y} - 1$

$\Rightarrow \quad \dfrac{dy}{2e^{-y}-1} = \dfrac{dx}{x+1}$

Integrating both sides we get

$$\int \frac{dy}{2e^{-y}-1} = \int \frac{dx}{x+1}$$

$\Rightarrow \quad \displaystyle\int \frac{e^y\,dy}{2-e^y} = \log|x+1| + c$

$\Rightarrow \quad \displaystyle -\int \frac{dz}{z} = \log|x+1| + c \qquad$ [Let $2 - e^y = z \Rightarrow -e^y\,dy = dz \Rightarrow e^y\,dy = -dz$]

$\Rightarrow \quad -\log z = \log|x+1| + c$

$\Rightarrow \quad -\log|2-e^y| = \log|x+1| + c$

$\Rightarrow \quad \log|x+1| + \log|2-e^y| = \log k$

$\Rightarrow \quad \log|(x+1).(2-e^y)| = \log k$

$\Rightarrow \quad (x+1)(2-e^y) = k$

Putting $x = 0, y = 0$, we get

$$1.(2-e^0) = k \quad \Rightarrow \quad k = 1$$

Therefore, required particular solution is

$$(x+1)(2-e^y) = 1$$

28. Let E_1, E_2, A be events such that

E_1 = getting 5 or 6 in a single throw of die

E_2 = getting 1, 2, 3 or 4 in a single throw of a die

A = getting exactly two heads

$P\left(\dfrac{E_2}{A}\right)$ is required.

Now, $P(E_1) = \dfrac{2}{6} = \dfrac{1}{3}$ and $P(E_2) = \dfrac{4}{6} = \dfrac{2}{3}$

$P\left(\dfrac{A}{E_1}\right) = \dfrac{3}{8} \qquad$ [∵ {HHH, <u>HHT</u>, HTT, TTT, TTH, <u>THH</u>, THT, <u>HTH</u>}]

$P\left(\dfrac{A}{E_2}\right) = \dfrac{1}{4} \qquad$ [{<u>HH</u>, HT, TH, TT}]

$$|\therefore \quad P\left(\frac{E_2}{A}\right) = \frac{P(E_2).P\left(\frac{A}{E_2}\right)}{P(E_1).P\left(\frac{A}{E_1}\right) + P(E_2).P\left(\frac{A}{E_2}\right)}$$

$$= \frac{\frac{2}{3}\times\frac{1}{4}}{\frac{1}{3}\times\frac{3}{8}+\frac{2}{3}\times\frac{1}{4}} = \frac{\frac{1}{6}}{\frac{1}{8}+\frac{1}{6}} = \frac{\frac{1}{6}}{\frac{3+4}{24}} = \frac{1}{6}\times\frac{24}{7} = \frac{4}{7}$$

29. Given lines are

$$3x - y - 3 = 0 \qquad ...(i)$$
$$2x + y - 12 = 0 \qquad ...(ii)$$
$$x - 2y - 1 = 0 \qquad ...(iii)$$

For intersecting point of (*i*) and (*ii*)

(*i*) + (*ii*) $\Rightarrow 3x - y - 3 + 2x + y - 12 = 0$

$\Rightarrow 5x - 15 = 0$

$\Rightarrow x = 3$

Putting $x = 3$ in (*i*), we get

$9 - y - 3 = 0$

$\Rightarrow y = 6$

Intersecting point of (*i*) and (*ii*) is (3, 6)

For intersecting point of (*ii*) and (*iii*)

(*ii*) – 2 × (*iii*) $\Rightarrow 2x + y - 12 - 2x + 4y + 2 = 0$

$\Rightarrow 5y - 10 = 0$

$\Rightarrow y = 2$

Putting $y = 2$ in (*ii*) we get

$2x + 2 - 12 = 0$

$\Rightarrow x = 5$

Intersecting point of (*ii*) and (*iii*) is (5, 2).

For Intersecting point of (*i*) and (*iii*)

(*i*) – 3 × (*iii*) $\Rightarrow 3x - y - 3 - 3x + 6y + 3 = 0$

$\Rightarrow 5y = 0$

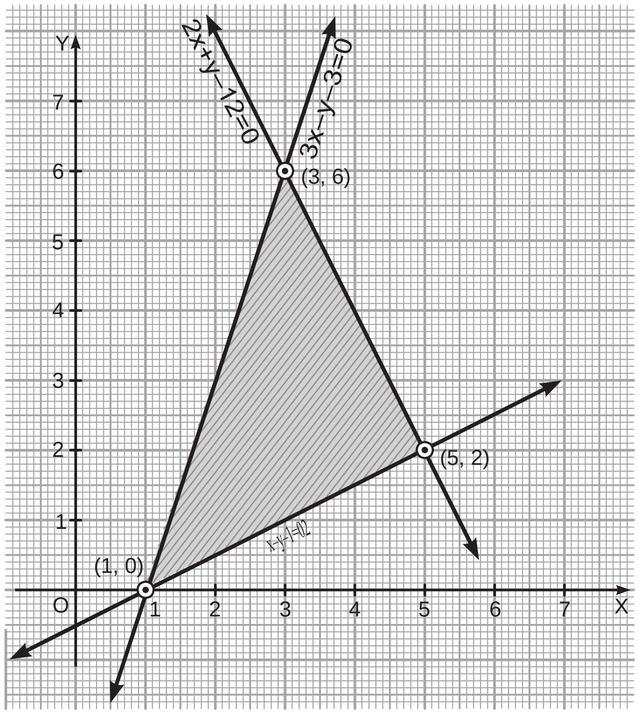
$\Rightarrow y = 0$

Putting $y = 0$ in (*i*), we get

$3x - 3 = 0$

$\Rightarrow x = 1$

Intersecting point (*i*) and (*iii*) is (1, 0).



Shaded region is required region.

$$\therefore \text{ Required Area} = \int_1^3 (3x-3)\,dx + \int_3^5 (-2x+12)\,dx - \int_1^5 \frac{x-1}{2}\,dx$$

$$= 3\int_1^3 x\,dx - 3\int_1^3 dx - 2\int_3^5 x\,dx + 12\int_3^5 dx - \frac{1}{2}\int_1^5 x\,dx + \frac{1}{2}\int_1^5 dx$$

$$= 3\left[\frac{x^2}{2}\right]_1^3 - 3[x]_1^3 - 2\left[\frac{x^2}{2}\right]_3^5 + 12[x]_3^5 - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^5 + \frac{1}{2}[x]_1^5$$

$$= \frac{3}{2}(9-1) - 3(3-1) - (25-9) + 12(5-3) - \frac{1}{4}(25-1) + \frac{1}{2}(5-1)$$

$$= 12 - 6 - 16 + 24 - 6 + 2$$

$$= 10 \text{ sq. unit}$$

Set–III

9. $\vec{a} + \vec{b} + \vec{c} = \hat{i} - 3\hat{k} + 2\hat{j} - \hat{k} + 2\hat{i} - 3\hat{j} + 2\hat{k}$

$$= 3\hat{i} - \hat{j} - 2\hat{k}$$

10. Minor of $a_{22} = \begin{vmatrix} 1 & 3 \\ 5 & 8 \end{vmatrix} = 8 - 15 = -7$

19. LHS $= \Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

Taking out *a, b, c* common from I, II, and III row respectively, we get

$$\Delta = abc \begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$

$$\Delta = abc \begin{vmatrix} \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1 & \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1 & \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$$= abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$\Delta = abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1\right)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$$

$$= abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1\right) \times (1 \times 1 \times 1)$$

($\because$ the determinant of a triangular matrix is the product of its diagonal elements.)

$$= abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1\right) = abc\left(\frac{bc+ac+ab+abc}{abc}\right) = ab+bc+ca+abc = \text{R.H.S.}$$

20. $\because \quad y = \sin^{-1} x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \quad \Rightarrow \quad \sqrt{1-x^2}\,\frac{dy}{dx} = 1$$

Again differentiating w.r.t. x, we get

$$\sqrt{1-x^2}\,\frac{d^2y}{dx^2} + \frac{dy}{dx}.\frac{1 \times (-2x)}{2\sqrt{1-x^2}} = 0$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2}-\frac{xdy}{dx}=0$$

21. Given differential equation is

$$xy\frac{dy}{dx}=(x+2)(y+2)$$

$$\Rightarrow \quad \frac{y}{y+2}\,dy=\frac{x+2}{x}\,dx$$

Integrating both sides

$$\int\frac{y}{y+2}\,dy=\int\left(1+\frac{2}{x}\right)dx$$

$$\Rightarrow \quad \int\left(1-\frac{2}{y+2}\right)dy=\int\left(1+\frac{2}{x}\right)dx$$

$$\Rightarrow \quad y-2\log|y+2|=x+2\log|x|+c \qquad \ldots(i)$$

Given that $y=-1$ when $x=1$

$$\therefore \quad -1-2\log 1=1+2\log|1|+C$$

$$\Rightarrow \quad C=-2$$

$\therefore$ The required particular solution is

$$y-2\log|y+2|=x+2\log|x|-2$$

22. Let the equation of line passing through the point (2, –1, 3) be

$$\frac{x-2}{a}=\frac{y+1}{b}=\frac{z-3}{c} \qquad \ldots(i)$$

Given lines are

$$\vec{r}=(\hat{i}+\hat{j}-\hat{k})+\lambda(2\hat{i}-2\hat{j}+\hat{k}) \ldots(ii)$$

$$\vec{r}=(2\hat{i}-\hat{j}-3\hat{k})+\mu(\hat{i}+2\hat{j}+2\hat{k}) \qquad \ldots(iii)$$

Since (*i*), (*ii*) and (*i*), (*iii*) are perpendicular to each other

$$\Rightarrow \quad 2a-2b+c=0$$

$$a+2b+2c=0$$

$$\Rightarrow \quad \frac{a}{-4-2}=\frac{b}{1-4}=\frac{c}{4+2}$$

$$\Rightarrow \quad \frac{a}{-6}=\frac{b}{-3}=\frac{c}{6}=l \quad \text{(say)}$$

$$\Rightarrow \quad a=-6\,l,\ b=-3\,l,\ c=6\,l$$

Putting it in (*i*) we get required equation of line as

$$\frac{x-2}{-6\,l}=\frac{y+1}{-3\,l}=\frac{z-3}{6\,l}$$

$$\Rightarrow \quad \frac{x-2}{2}=y+1=\frac{z-3}{-2}$$

28. Let E_1, E_2, E_3 and A be events such that

E_1 = Both transfered ball from Bag I to Bag II are red.

E_2 = Both transfered ball from Bag I to Bag II are black.

E_3 = Out of two transfered ball one is red and other is black.

A = Drawing a red ball from Bag II.

Here, $P\left(\frac{E_2}{A}\right)$ is required.

Now, $P(E_1) = \dfrac{{}^3C_2}{{}^7C_2} = \dfrac{3!}{2!\,1!} \times \dfrac{2!\,5!}{7!} = \dfrac{1}{7}$

$P(E_2) = \dfrac{{}^4C_2}{{}^7C_2} = \dfrac{4!}{2!\,2!} \times \dfrac{2!\,5!}{7!} = \dfrac{2}{7}$

$P(E_3) = \dfrac{{}^3C_1 \times {}^4C_1}{{}^7C_2} = \dfrac{3 \times 4}{7!} \times \dfrac{2!5!}{1} = \dfrac{4}{7}$

$P\left(\dfrac{A}{E_1}\right) = \dfrac{6}{11}$, $P\left(\dfrac{A}{E_2}\right) = \dfrac{4}{11}$, $P\left(\dfrac{A}{E_3}\right) = \dfrac{5}{11}$

$$\therefore \quad P\left(\frac{E_2}{A}\right) = \frac{P(E_2).P\left(\frac{A}{E_2}\right)}{P(E_1).P\left(\frac{A}{E_1}\right) + P(E_2).P\left(\frac{A}{E_2}\right) + P(E_3).P\left(\frac{A}{E_3}\right)}$$

$$= \frac{\frac{2}{7} \times \frac{4}{11}}{\frac{1}{7} \times \frac{6}{11} + \frac{2}{7} \times \frac{4}{11} + \frac{4}{7} \times \frac{5}{11}} = \frac{\frac{8}{77}}{\frac{6}{77} + \frac{8}{77} + \frac{20}{77}} = \frac{8}{77} \times \frac{77}{34} = \frac{4}{17}$$

29. Given lines are

$5x - 2y - 10 = 0$ ...(*i*)

$x + y - 9 = 0$ ...(*ii*)

$2x - 5y - 4 = 0$ ...(*iii*)

For intersecting point of (*i*) and (*ii*)

(*i*) + 2 × (*ii*) $\Rightarrow 5x - 2y - 10 + 2x + 2y - 18 = 0$

$\Rightarrow 7x - 28 = 0 \Rightarrow x = 4$

Putting $x = 4$ in(*i*), we get

$20 - 2y - 10 = 0 \Rightarrow y = 5$

Intersecting point of (*i*) and (*ii*) is (4, 5).

For intersecting point of (*i*) and (*iii*)

(*i*) × 5 – (*iii*) × 2 $\Rightarrow 25x - 10y - 50 - 4x + 10y + 8 = 0$

$\Rightarrow 21x - 42 = 0 \Rightarrow x = 2$

Putting $x = 2$ in (i) we get

$$10 - 2y - 10 = 0 \qquad \Rightarrow \qquad y = 0$$

i.e., Intersecting points of (i) and (iii) is $(2, 0)$

For intersecting point of (ii) and (iii)

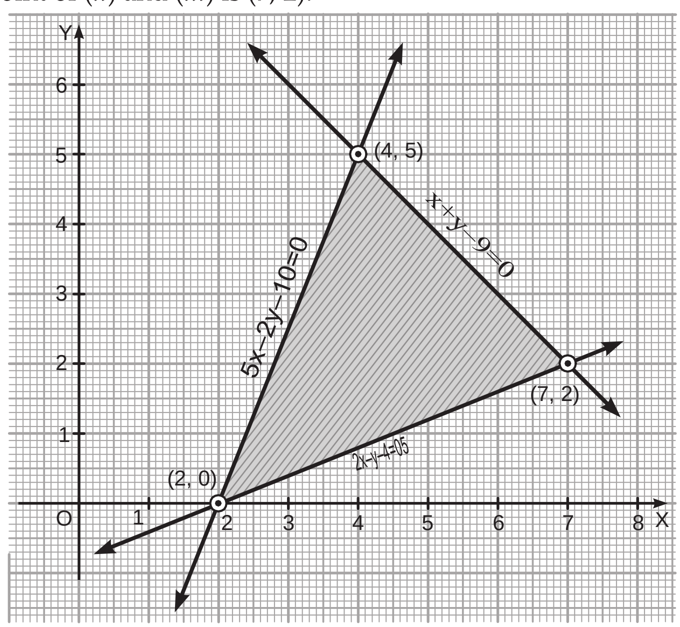
$$2 \times (ii) \times (iii) \qquad \Rightarrow \quad 2x + 2y - 18 - 2x + 5y + 4 = 0$$

$$\Rightarrow \quad 7y - 14 = 0 \quad \Rightarrow \quad y = 2$$

Putting $y = 2$ in (ii) we get

$$x + 2 - 9 = 0 \;\Rightarrow\; x = 7$$

Intersecting point of (ii) and (iii) is $(7, 2)$.



Shaded region is required region.

$$\therefore \quad \text{Required Area} = \int_2^4 \left(\frac{5x - 10}{2}\right) dx + \int_4^7 (-x + 9)\,dx - \int_2^7 \frac{(2x - 4)}{5}\,dx$$

$$= \frac{5}{2}\int_2^4 x\,dx - 5\int_2^4 dx - \int_4^7 x\,dx + 9\int_4^7 dx - \frac{2}{5}\int_2^7 x\,dx + \frac{4}{5}\int_2^7 dx$$

$$= \frac{5}{2}\left[\frac{x^2}{2}\right]_2^4 - 5\,[x]_2^4 - \left[\frac{x^2}{2}\right]_4^7 + 9\,[x]_4^7 - \frac{2}{5}\left[\frac{x^2}{2}\right]_2^7 + \frac{4}{5}[x]_2^7$$

$$= \frac{5}{4}(16 - 4) - 5(4 - 2) - \frac{1}{2}(49 - 16) + 9(7 - 4) - \frac{1}{5}(49 - 4) + \frac{4}{5}(7 - 2)$$

$$= 15 - 10 - \frac{33}{2} + 27 - 9 + 4 = 27 - \frac{33}{2} = \frac{54 - 33}{2} = \frac{21}{2} \text{ sq. unit}$$

CBSE Examination Paper (All India 2012)

Time allowed: 3 hours **Maximum marks: 100**

General Instructions:

1. ***All*** *questions are compulsory.*
2. *The question paper consists of* ***29*** *questions divided into three Sections* ***A, B*** *and* ***C.*** *Section A comprises of* ***10*** *questions of* ***one*** *mark each, Section B comprises of* ***12*** *questions of* ***four*** *marks each and Section C comprises of* ***7*** *questions of* ***six*** *marks each.*
3. ***All*** *questions in Section A are to be answered in* ***one*** *word,* ***one*** *sentence or as per the exact requirement of the question.*
4. *There is no overall choice. However, internal choice has been provided in* ***4*** *questions of* ***four*** *marks each and* ***2*** *questions of* ***six*** *marks each. You have to attempt only* ***one*** *of the alternatives in all such questions.*
5. *Use of calculators is* ***not*** *permitted.*

Set–I

SECTION–A

Question numbers 1 to 10 carry 1 mark each.

1. The binary operation $* : R \times R \to R$ is defined as $a * b = 2a + b$. Find $(2 * 3) * 4$
2. Find the principal value of $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$.
3. Find the value of $x + y$ from the following equation:

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

4. If $A^{T} = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, then find $A^{T} - B^{T}$.
5. Let A be a square matric of order 3×3. Write the value of $|2A|$, where $|A| = 4$.
6. Evaluate: $\int_0^2 \sqrt{4 - x^2}\, dx$
7. Given $\int e^x (\tan x + 1) \sec x \, dx = e^x f(x) + c$.

   Write $f(x)$ satisfying the above.
8. Write the value of $(\hat{i} \times \hat{j}) \cdot \hat{k} + \hat{i} \cdot \hat{j}$.

9. Find the scalar components of the vector $\overrightarrow{AB}$ with initial point A (2,1) and terminal point B (–5, 7).

10. Find the distance of the plane $3x - 4y + 12z = 3$ from the origin.

SECTION–B

Question numbers 11 to 22 carry 4 marks each.

11. Prove the following: $\cos\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right) = \frac{6}{5\sqrt{13}}$

12. Using properties of determinants, show that

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$$

13. Show that f : N → N, given by

$$f(x) = \begin{cases} x+1, & \text{if } x \text{ is odd} \\ x-1, & \text{if } x \text{ is even} \end{cases}$$

is both one-one and onto.

OR

Consider the binary operations $*$: $R \times R \to$ R and o : $R \times R \to R$ defined as $a * b = |a - b|$ and $aob = a$ for all $a, b \in R$. Show that '*' is commutative but not associative, 'o' is associative but not commutative.

14. If $x = \sqrt{a^{\sin^{-1} t}}$, $y = \sqrt{a^{\cos^{-1} t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$

OR

Differentiate $\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$ with respect to x.

15. If $x = a$ (cos t + t sin t) and $y = a$ (sin t – t cos t), 0 < t < $\frac{\pi}{2}$, find $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$.

16. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall?

17. Evaluate: $\int_{-1}^{2} |x^3 - x|\, dx$

OR

Evaluate: $\int_{0}^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx$

18. Form the differential equation of the family of circles in the second quadrant and touching the coordinate axes.

OR

Find the particular solution of the differential equation

$$x(x^2 - 1)\frac{dy}{dx} = 1; \qquad y = 0 \text{ when } x = 2$$

19. Solve the following differential equation:
$(1 + x^2)\, dy + 2xy\, dx = \cot x\, dx;\ \ x \neq 0$

20. Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k},\ \vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$.

Find a vector $\vec{p}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{p}\cdot\vec{c} = 18$.

21. Find the coordinates of the point where the line through the points A (3, 4, 1) and B (5, 1, 6) crosses the XY-plane.

22. Two cards are drawn simultaneously (without replacement) from a well-shuffled pack of 52 cards. Find the mean and variance of the number of red cards.

SECTION–C

Question numbers 23 to 29 carry 6 marks each.

23. Using matrices, solve the following system of equations:
$2x + 3y + 3z = 5,\ x - 2y + z = -4,\ 3x - y - 2z = 3$

24. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

OR

An open box with a square base is to be made out of a given quantity of cardboard of area c^2 square units. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ cubic units.

25. Evaluate: $\int \frac{x \sin^{-1} x}{\sqrt{1 - x^2}}\, dx$

OR

Evaluate: $\int \frac{x^2 + 1}{(x-1)^2 (x+3)}\, dx$

26. Find the area of the region $\{(x, y) : x^2 + y^2 \leq 4,\ x + y \geq 2\}$.

27. If the lines $\frac{x-1}{-3} = \frac{y-2}{-2k} = \frac{z-3}{2}$ and $\frac{x-1}{k} = \frac{y-2}{1} = \frac{z-3}{5}$ are perpendicular, find the value of k and hence find the equation of plane containing these lines.

28. Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin 3 times and notes the number of heads. If she gets 1,2,3 or 4 she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1,2,3, or 4 with the die?

29. A dietician wishes to mix two types of foods in such a way that the vitamin contents of the mixture contains at least 8 units of vitamin A and 10 units of vitamin C. Food I contains 2 units/kg of vitamin A and 1 units/kg of vitamin C while Food II contains 1 unit/kg of vitamin A and 2 units/kg of vitamin C. It costs ₹5 per kg to purchase Food I and ₹7 per kg to purchase Food II. Determine the minimum cost of such a mixture. Formulate the above as a LPP and solve it graphically.

Set–II

Only those questions, not included in Set I, are given

10. Write the value of $(\hat{k} \times \hat{j}) \cdot \hat{i} + \hat{j} \cdot \hat{k}$

19. Prove that: $\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$

20. If $y = (\tan^{-1} x)^2$, show that $(x^2+1)^2 \frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2.$

21. Find the particular solution of the differential equation
$\frac{dy}{dx} + y \cot x = 4x \operatorname{cosec} x$, $(x \neq 0)$ given that $y = 0$ when $x = \frac{\pi}{2}$.

22. Find the coordinates of the point where the line through the points (3,–4, –5) and (2,–3, 1) crosses the plane $2x + y + z = 7$.

28. Using matrices, solve the following system of equations:
$$x + y - z = 3;\ 2x + 3y + z = 10;\ 3x - y - 7z = 1$$

29. Find the length and the foot of the perpendicular from the point P (7, 14, 5) to the plane $2x + 4y - z = 2$. Also find the image of point P in the plane.

Set–III

Only those questions, not included in Set I and Set II are given

10. Find the value of $x + y$ from the following equation:
$$2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

19. If $x = a\left(\cos t + \log \tan \frac{t}{2}\right)$, $y = a \sin t$, find $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$.

20. Find the co-ordinates of the point where the line through the points (3, –4, –5) and (2, –3, 1) crosses the plane $3x + 2y + z + 14 = 0$.

21. Find the particular solution of the following differential equation.
$x\frac{dy}{dx} - y + x \sin\left(\frac{y}{x}\right) = 0$, given that when $x = 2$, $y = \pi$

22. Prove that: $\cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right)$

28. Find the coordinates of the foot of perpendicular and the length of the perpendicular drawn from the point P (5, 4, 2) to the line $\vec{r} = -\hat{i} + 3\hat{j} + \hat{k} + \lambda(2\hat{i} + 3\hat{j} - \hat{k})$. Also find the image of P in this line.

29. Using matrices, solve the following system of equations.
$$3x + 4y + 7z = 4$$
$$2x - y + 3z = -3$$
$$x + 2y - 3z = 8$$

Solutions

Set–I

SECTION–A

1. $(2 * 3) * 4 = (2 \times 2 + 3) * 4$
$$= 7 * 4$$
$$= 2 \times 7 + 4 = 18$$

2. $\tan^{-1}(\sqrt{3}) - \sec^{-1}(-2)$

$$= \tan^{-1}\left(\tan\frac{\pi}{3}\right) - \sec^{-1}\left(-\sec\frac{\pi}{3}\right)$$

$$= \frac{\pi}{3} - \sec^{-1}\left[\sec\left(\pi - \frac{\pi}{3}\right)\right] = \frac{\pi}{3} - \sec^{-1}\left(\sec\frac{2\pi}{3}\right)$$

$$= \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}.$$

$\left[\tan^{-1}(\tan x) = x \text{ if } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right.$ Here $\frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

also, $\sec^{-1}(\sec x) = x$ if $x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

Here $\left.\frac{2\pi}{3} \in [0, \pi] - \left\{\frac{\pi}{2}\right\}\right]$

3. Given: $2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2x & 10 \\ 14 & 2y-6 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2x+3 & 6 \\ 15 & 2y-4 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$

Equating the corresponding element we get

$2x + 3 = 7$ and $2y - 4 = 14$

$\Rightarrow \quad x = \dfrac{7-3}{2}$ and $y = \dfrac{14+4}{2}$

$\Rightarrow \quad x = 2$ and $y = 9$

$\therefore \quad x + y = 2 + 9 = 11$

4. Given: $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$\therefore \quad B^T = \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$

Now $A^T - B^T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -3 & 0 \\ -1 & -2 \end{bmatrix}$

5. $\because \quad |2A| = 2^n |A|$ Where n is order of matrix A.

Here $\quad |A| = 4$ and $n = 3$

$\therefore \quad |2A| = 2^3 \times 4 = 32$

6. Let $I = \int_0^2 \sqrt{4 - x^2}\, dx = \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{2^2}{2} \sin^{-1} \frac{x}{2} \right]_0^2$

$= (0 + 2 \sin^{-1} 1) - (0 + 0)$ $\quad \left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c \right]$

$= 2 \times \frac{\pi}{2} = \pi$

7. Given $\int e^x (\tan x + 1) \sec x\, dx = e^x f(x) + c$

$\Rightarrow \quad \int e^x (\tan x \sec x + \sec x)\, dx = e^x f(x) + c$

$\Rightarrow \quad \int e^x (\sec x + \tan x \sec x)\, dx = e^x f(x) + c$

$\Rightarrow \quad \int e^x \sec x + c = e^x f(x) + c$

$\Rightarrow \quad f(x) = \sec x$

[**Note:** $\int e^x [f(x) + f'(x)]\, dx = e^x . f(x) + c$, Here $f(x) = \sec x$]

8. $(\hat{i} \times \hat{j}).\hat{k} + \hat{i}.\hat{j} = \hat{k}.\hat{k} + 0$

$= 1 + 0 = 1$

[**Note:** $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0,\ \hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1,\ \hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i}$ and $\hat{k} \times \hat{i} = \hat{j}$]

9. Let $\quad AB = (-5 - 2)\hat{i} + (7 - 1)\hat{j}$

$= -7\hat{i} + 6\hat{j}$

Hence scalar components are –7, 6

[**Note:** If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then x, y, z are called scalar component and $x\hat{i}, y\hat{j}, z\hat{k}$ are called vector component.]

10. Given plane is $3x - 4y + 12z - 3 = 0$

$\therefore$ Distance from origin $= \left| \frac{3 \times 0 + (-4) \times 0 + 12 \times 0 - 3}{\sqrt{3^2 + (-4)^2 + (12)^2}} \right|$

$= \left| \frac{-3}{\sqrt{9 + 16 + 144}} \right|$

$= \left| \frac{-3}{\sqrt{169}} \right|$

$= \frac{3}{13}$ units

SECTION–B

11. Here

$\text{LHS} = \cos\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$

Let $\sin^{-1}\frac{3}{5} = \theta$ and $\cot^{-1}\frac{3}{2}$

$= \phi \Rightarrow \sin\theta = \frac{3}{5}$ and $\cot\phi = \frac{3}{2}$

$\Rightarrow \quad \cos\theta = \frac{4}{5}$ and $\sin\phi = \frac{2}{\sqrt{13}}, \cos\phi = \frac{3}{\sqrt{13}}$

$$\begin{aligned}\therefore \quad \text{LHS} &= \cos(\theta + \phi)\\ &= \cos\theta . \cos\phi - \sin\theta \times \sin\phi\\ &= \frac{4}{5}.\frac{3}{\sqrt{13}} - \frac{3}{5}.\frac{2}{\sqrt{13}} = \frac{12}{5\sqrt{13}} - \frac{6}{5\sqrt{13}} = \frac{6}{5\sqrt{13}}\end{aligned}$$

12. $\text{LHS} = \begin{vmatrix} \boldsymbol{b+c} & \boldsymbol{a} & \boldsymbol{a} \\ \boldsymbol{b} & \boldsymbol{c+a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{c} & \boldsymbol{a+b} \end{vmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3$ we get

$$= \begin{vmatrix} 2(b+c) & 2(c+a) & 2(a+b) \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Taking 2 common from R_1 we get

$$= 2\begin{vmatrix} (b+c) & (c+a) & (a+b) \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ we get

$$= 2\begin{vmatrix} (b+c) & (c+a) & (a+b) \\ -c & 0 & -a \\ -b & -a & 0 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$ we get

$$= 2\begin{vmatrix} 0 & c & b \\ -c & 0 & -a \\ -b & -a & 0 \end{vmatrix}$$

Expanding along R_1 we get

$$\begin{aligned}&= 2\,[0 - c\,(0 - ab) + b\,(ac - 0)]\\ &= 2\,[abc + abc]\\ &= 4\,abc\end{aligned}$$

13. For one-one

Case I : When x_1, x_2 are odd natural number.

$\therefore \quad f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 + 1 \qquad \forall\, x_1, x_2 \in N$

$\Rightarrow x_1 = x_2$

i.e., f is one-one.

Case II : When x_1, x_2 are even natural number

$\therefore \quad f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 - 1$

$\Rightarrow x_1 = x_2$

i.e., f is one-one.

Case III : When x_1 is odd and x_2 is even natural number

$f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 - 1$

$\Rightarrow x_2 - x_1 = 2$ which is never possible as the difference of odd and even number is always odd number.

Hence in this case $f(x_1) \neq f(x_2)$

i.e., f is one-one.

Case IV: When x_1 is even and x_2 is odd natural number

Similar as case III, We can prove f is one-one

For onto:

$\therefore \quad f(x) = x + 1$ if x is odd

$= x - 1$ if x is even

$\Rightarrow$ For every even number 'y' of codomain $\exists$ odd number $y - 1$ in domain and for every odd number y of codomain $\exists$ even number $y + 1$ in Domain.

i.e. f is onto function.

Hence f is one-one onto function.

OR

For operation '$*$'

'$*$' : $R \times R \to R$ s.t.

$a * b = |a - b| \qquad \forall a, b \in R$

Commutativity

$a * b = |a - b|$

$= |b - a| = b * a$

i.e., '$*$' is commutative

Associativity

$\forall\, a, b, c \in R \ (a * b) * c = |a - b| * c$

$= \big||a - b| - c\big|$

$a * (b * c) = a * |b - c|$

$= \big|a - |b - c|\big|$

But $\big||a - b| - c\big| \neq \big|a - |b - c|\big|$

$\Rightarrow \quad (a * b) * c \neq a * (b * c) \ \forall\, a, b, c \in R$

$\Rightarrow$ $*$ is not associative.

Hence, '*' is commutative but not associative.

For Operation 'o'

$$o : R \times R \to R \text{ s.t.}$$
$$aob = a$$

Commutativity $\forall\, a, b \in R$

$$aob = a \quad \text{and } boa = b$$

$\because$ $a \neq b \;\Rightarrow\; aob \neq boa$

$\Rightarrow$ 'o' is not commutative.

Associativity: $\forall\, a, b, c \in R$

$$(aob)\, oc = aoc = a$$
$$ao(boc) = aob = a$$

$\Rightarrow$ $(aob)\, oc = ao\,(boc)$

$\Rightarrow$ 'o' is associative

Hence 'o' is not commutative but associative.

14. Given $x = \sqrt{a^{\sin^{-1} t}}$

Taking log on both sides, we have

$$\log x = \log\,(a^{\sin^{-1} t})^{1/2}$$
$$= \frac{1}{2}\log\,(a^{\sin^{-1} t}) = \frac{1}{2} \times \sin^{-1} t \,.\log a$$
$$\log x = \frac{1}{2}\sin^{-1} t \,.\log a$$

Differentiating both sides w.r.t. t, we have

$$\frac{1}{x}\frac{dx}{dt} = \frac{1}{2}\log a \times \frac{1}{\sqrt{1-t^2}}$$

$$\therefore \quad \frac{dx}{dt} = x\left(\frac{1}{2}\log a \times \frac{1}{\sqrt{1-t^2}}\right)$$

Again, $y = \sqrt{a^{\cos^{-1} t}}$

Taking log on both sides, we have

$$\log y = \frac{1}{2}\log a^{\cos^{-1} t}$$

$$\Rightarrow \quad \log y = \frac{1}{2} \times \cos^{-1} t \log a$$

Differentiating both sides w.r.t. t, we have

$$\frac{1}{y}\frac{dy}{dt} = \frac{1}{2}\log a \times \frac{-1}{\sqrt{1-t^2}}$$

$$\frac{dy}{dt} = y \times \frac{1}{2}\log a \times \frac{-1}{\sqrt{1-t^2}}$$

$$\therefore \qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y \times \frac{1}{2}\log a \times -\frac{1}{\sqrt{1-t^2}}}{x \times \frac{1}{2}\log a \times \frac{1}{\sqrt{1-t^2}}} \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

OR

Let $y = \tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$

Let $x = \tan\theta \quad \Rightarrow \quad \theta = \tan^{-1} x$

Now, $\quad y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$

$$= \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}\right)$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$$

$$\left[\begin{array}{l} \because \quad -\infty < x < \infty \\ \Rightarrow \tan\left(-\frac{\pi}{2}\right) < \tan\theta < \tan\left(\frac{\pi}{2}\right) \\ \Rightarrow -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ \Rightarrow -\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{4} \\ \Rightarrow \frac{\theta}{2} \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{array}\right]$$

$$\Rightarrow \qquad y = \frac{1}{2}\tan^{-1} x$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{1}{2(1+x^2)}$$

15. Given $x = a(\cos t + t \sin t)$

Differentiating both sides w.r.t. x we get

$$\frac{dx}{dt} = a(-\sin t + t\cos t + \sin t)$$

$$\Rightarrow \qquad \frac{dx}{dt} = a\, t \cos t \qquad \qquad ...(i)$$

Differentiating again w.r.t. t we get

$$\frac{d\quad x}{dt^2} = a(-t\sin t + \cos t) = a(\cos t - t\sin t).$$

Again $y = a(\sin t - t\cos t)$

Differentiating w.r.t. t we get

$$\frac{dy}{dt} = a(\cos t + t\sin t - \cos t)$$

$$\Rightarrow \quad \frac{dy}{dt} = at\sin t \qquad \text{...}(ii)$$

Differentiating again w.r.t. t we get

$$\frac{d^2y}{dt^2} = a(t\cos t + \sin t)$$

Now, $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$ [from (i) and (ii)]

$$\Rightarrow \quad \frac{dy}{dx} = \frac{at\sin t}{at\cos t}$$

$$\Rightarrow \quad \frac{dy}{dx} = \tan t$$

Differentiating w.r.t. x we get

$$\frac{d^2y}{dx^2} = \sec^2 t \cdot \frac{dt}{dx}$$

$$= \sec^2 t \cdot \frac{1}{dx/dt} = \frac{\sec^2 t}{at\cos t} \qquad \text{[from } (i)]$$

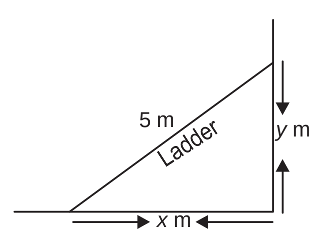
$$= \frac{\sec^3 t}{at}.$$

Hence $\dfrac{d^2x}{dt^2} = a(\cos t - t\sin t)$, $\dfrac{d^2y}{dt^2} = a(t\cos t + \sin t)$ and $\dfrac{d^2y}{dx^2} = \dfrac{\sec^3 t}{at}$.

16. Let x, y be the distance of the bottom and top of the ladder respectively from the edge of the wall.

Here, $\dfrac{dx}{dt} = 2$ cm/s

$$x^2 + y^2 = 25$$



When $x = 4$ m,

$$(4)^2 + y^2 = 25 \quad \Rightarrow \quad y^2 = 25 - 16 = 9$$

$$y = 3 \text{ m}$$

Now, $x^2 + y^2 = 25$

Differentiating w.r.t. t, we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad \Rightarrow \quad x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

$\Rightarrow \quad 4 \times 2 + 3 \times \dfrac{dy}{dt} = 0$

$\dfrac{dy}{dt} = -\dfrac{8}{3}$

Hence, the rate of decrease of its height $= \dfrac{8}{3}$ cm/s

17. If $\quad x^3 - x = 0$

$\Rightarrow \quad x(x^2 - 1) = 0$

$\Rightarrow \quad x = 0 \text{ or } x^2 = 1$

$\Rightarrow \quad x = 0 \text{ or } x = \pm 1$

$\Rightarrow \quad x = 0, -1, 1$

Hence [–1, 2] divided into three sub intervals [–1, 0], [0, 1] and [1, 2] such that

$x^3 - x \ge 0 \quad$ on $\quad [-1, 0]$

$x^3 - x \le 0 \quad$ on $\quad [0, 1]$

and $\quad x^3 - x \ge 0 \quad$ on $\quad [1, 2]$

Now $\quad \displaystyle\int_{-1}^{2} |x^3 - x|\, dx = \int_{-1}^{0} |x^3 - x|\, dx + \int_{0}^{1} |x^3 - x|\, dx + \int_{1}^{2} |x^3 - x|\, dx$

$$= \int_{-1}^{0} (x^3 - x)dx + \int_{0}^{1} -(x^3 - x)dx + \int_{1}^{2} (x^3 - x)dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} - \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{0}^{1} + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{1}^{2}$$

$$= \left\{0 - \left(\frac{1}{4} - \frac{1}{2}\right)\right\} - \left\{\left(\frac{1}{4} - \frac{1}{2}\right) - 0\right\} + \left\{(4 - 2) - \left(\frac{1}{4} - \frac{1}{2}\right)\right\}$$

$$= -\frac{1}{4} + \frac{1}{2} - \frac{1}{4} + \frac{1}{2} + 2 - \frac{1}{4} + \frac{1}{2}$$

$$= \frac{3}{2} - \frac{3}{4} + 2 = \frac{11}{4}$$

OR

Let $\quad I = \displaystyle\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx.$

$$I = \int_0^{\pi} \frac{(\pi - x) \sin (\pi - x)\, dx}{1 + \cos^2 (\pi - x)} = \int_0^{\pi} \frac{(\pi - x) \sin x\, dx}{1 + \cos^2 x} = \pi \int_0^{\pi} \frac{\sin x\, dx}{1 + \cos^2 x} - I$$

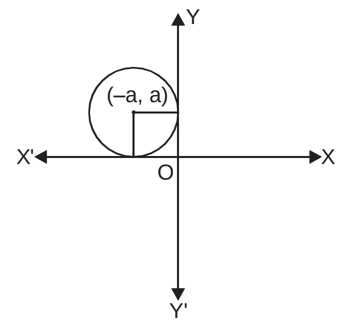
or $\quad 2I = \pi \displaystyle\int_0^{\pi} \frac{\sin x\, dx}{1 + \cos^2 x} \quad$ or $\quad I = \dfrac{\pi}{2} \displaystyle\int_0^{\pi} \frac{\sin x\, dx}{1 + \cos^2 x}$

Put $\cos x = t$ so that $-\sin x\, dx = dt$. When $x = 0$, $t = 1$ and when $x = \pi$, $t = -1$. Therefore, we get

$$I = \frac{-\pi}{2}\int_{1}^{-1}\frac{dt}{1+t^2} = \pi\int_{0}^{1}\frac{dt}{1+t^2} \qquad \left[\text{Q}\int_{a}^{-a} f(x)\,dx = -\int_{-a}^{a} f(x)\,dx \text{ and } \int_{0}^{2a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx\right]$$

$$= \pi\,[\tan^{-1} t]_0^1 = \pi\,[\tan^{-1} 1 - \tan^{-1} 0] = \pi\left[\frac{\pi}{4} - 0\right] = \frac{\pi^2}{4}$$

18. Let C denotes the family of circles in the second quadrant and touching the coordinate axes. Let $(-a, a)$ be the coordinate of the centre of any member of this family (see figure).



Equation representing the family C is

$$(x+a)^2 + (y-a)^2 = a^2 \qquad ...(i)$$

or $$x^2 + y^2 + 2ax - 2ay + a^2 = 0 \qquad ...(ii)$$

Differentiating equation (ii) w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} + 2a - 2a\frac{dy}{dx} = 0$$

or $$x + y\frac{dy}{dx} = a\left(\frac{dy}{dx} - 1\right)$$

or $$a = \frac{x}{+yy'\,y} \qquad \left(y' = \frac{dy}{dx}\right)$$
$$'-1$$

Substituting the value of a in equation (i), we get

$$\left[x + \frac{x+yy'}{y'-1}\right]^2 + \left[y - \frac{x+yy'}{y'-1}\right]^2 = \left[\frac{x+yy'}{y'-1}\right]^2$$

or $$[xy' - x + x + yy']^2 + [yy' - y - x - yy']^2 = [x + yy']^2$$

or $$(x+y)^2\,y'^2 + (x+y)^2 = (x+yy')^2$$

or $$(x+y)^2\,[(y')^2 + 1] = [x + yy']^2,$$ is the required differential equation representing the given family of circles.

OR

Given differential equation is

$$x(x^2-1)\frac{dy}{dx} = 1,$$

$$dy = \frac{dx}{x(x^2-1)}$$

$\Rightarrow$ $$dy = \frac{dx}{x(x-1)(x+1)}$$

Integrating both sides we get

$$\int dy = \int\frac{dx}{x(x-1)(x+1)}$$

$\Rightarrow$ $$y = \int\frac{dx}{x(x-1)(x+1)} \qquad ...(i)$$

$$\frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$\Rightarrow$ $$\frac{1}{x(x-1)(x+1)} = \frac{A(x-1)(x+1) + Bx(x+1) + Cx(x-1)}{x(x-1)(x+1)}$$

$\Rightarrow$ Putti $1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$

ng $x = 1$ we get $1 = 0 + B.\,1.\,2 + 0 \quad \Rightarrow B = \frac{1}{2}$

Putting $x = -1$ we get $1 = 0 + 0 + C\,.(-1).(-2) \Rightarrow C = \frac{1}{2}$

Putting $x = 0$ we get $1 = A\,(-1).1 \Rightarrow A = -1$

Hence $$\frac{1}{x(x-1)(x+1)} = \frac{-1}{x} + \frac{1}{2(x-1)} + \frac{1}{2(x+1)}$$

From (*i*)

$$y = \int\left(-\frac{1}{x} + \frac{1}{2(x-1)} + \frac{1}{2(x+1)}\right)dx$$

$\Rightarrow$ $$y = -\int\frac{dx}{x} + \frac{1}{2}\int\frac{dx}{x-1} + \frac{1}{2}\int\frac{dx}{x+1}$$

$\Rightarrow$ $$y = -\log x + \frac{1}{2}\log|x-1| + \frac{1}{2}\log|x+1| + \log c$$

$\Rightarrow$ $$2y = 2\log\frac{1}{x} + \log\left|x^2 - 1\right| + 2\log c$$

$\Rightarrow$ $$2y = \log\left|\frac{x^2-1}{x^2}\right| + \log c^2 \qquad ...(ii)$$

When $x = 2, y = 0$

$\Rightarrow$ $$0 = \log\left|\frac{4-1}{4}\right| + \log c^2$$

$\Rightarrow$ $$\log c^2 = -\log\frac{3}{4}$$

Putting $\log c^2 = -\log\frac{3}{4}$ in (*ii*) we get

$$2y = \log\left|\frac{x^2-1}{x^2}\right| - \log\frac{3}{4}$$

$\Rightarrow$ $$y = \frac{1}{2}\log\left|\frac{x^2-1}{x^2}\right| - \frac{1}{2}\log\frac{3}{4}$$

19. Given differential equation is

$$(1 + x^2)\,dy + 2xy\,dx = \cot x.\,dx$$

$\Rightarrow$ $$\frac{dy}{dx} + \frac{2x}{1+x^2}\cdot y = \frac{\cot x}{1+x^2}$$

It is in the form of $\frac{dy}{dx} + Py = Q$. Where

$$P = \frac{2x}{1+x^2}, \; Q = \frac{\cot x}{1+x^2}$$

$$\therefore \quad I.F. = e^{\int P dx}$$

$$= e^{\int \frac{2x}{1+x^2} dx}$$

$$= e^{\int \frac{dz}{z}} \quad [\text{Let } 1 + x^2 = z \Rightarrow 2x\, dx = dz]$$

$$= e^{\log z} = e^{\log (1+x^2)}$$

$$= 1 + x^2 \qquad [\because e^{\log z} = z]$$

Hence the solution is

$$y \times I.F = \int Q \times I.F \; dx + c$$

$$\Rightarrow \quad y(1+x^2) = \int \frac{\cot x}{1+x^2} . (1+x^2) dx + c$$

$$\Rightarrow \quad y(1+x^2) = \int \cot x \, dx + c$$

$$\Rightarrow \quad y(1+x^2) = \int \frac{\cos x \, dx}{\sin x} + c$$

$$\Rightarrow \quad y(1+x^2) = \log|\sin x| + c$$

$$\Rightarrow \quad y = \frac{\log|\sin x|}{1+x^2} + \frac{c}{1+x^2}$$

20. Given,

$$\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}, \quad \vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}, \quad \vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$$

Vector $\vec{p}$ is perpendicular to both $\vec{a}$ and $\vec{b}$ *i.e.,* $\vec{p}$ is parallel to vector $\vec{a} \times \vec{b}$.

$$\therefore \quad \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 2 \\ 3 & -2 & 7 \end{vmatrix} = \hat{i} \begin{vmatrix} 4 & 2 \\ -2 & 7 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 4 \\ 3 & -2 \end{vmatrix} = 32\hat{i} - \hat{j} - 14\hat{k}$$

Since $\vec{p}$ is parallel to $\vec{a} \times \vec{b}$

$$\therefore \quad \vec{p} = \mu(32\hat{i} - \hat{j} - 14\hat{k})$$

Also, $\quad \vec{p} . \vec{c} = 18$

$$\Rightarrow \quad \mu(32\hat{i} - \hat{j} - 14\hat{k}).(2\hat{i} - \hat{j} + 4\hat{k}) = 18$$

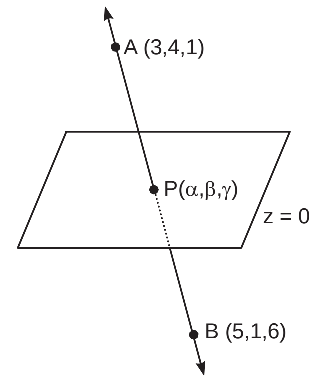
$$\Rightarrow \quad \mu(64 + 1 - 56) = 18 \Rightarrow \quad 9\mu = 18 \quad \text{or} \quad \mu = 2$$

$$\therefore \quad \vec{p} = 2(32\hat{i} - \hat{j} - 14\hat{k}) = 64\hat{i} - 2\hat{j} - 28\hat{k}$$

21. Let $P(\alpha, \beta, \gamma)$ be the point at which the given line crosses the xy plane.

Now the equation of given line is

$$\frac{x-3}{2} = \frac{y-4}{-3} = \frac{z-1}{5} \qquad \ldots(i)$$



Since P (α, β, γ) lie on line (i)

$$\therefore \quad \frac{\alpha-3}{2} = \frac{\beta-4}{-3} = \frac{\gamma-1}{5} = \lambda \text{ (say)}$$

$\Rightarrow \quad \alpha = 2\lambda + 3; \quad \beta = -3\lambda + 4$

and $\quad \gamma = 5\lambda + 1$

Also $P(\alpha, \beta, \gamma)$ lie on given xy plane, *i.e.*, $z = 0$

$\therefore \quad 0.\alpha + 0.\beta + \gamma = 0$

$\Rightarrow \quad 5\lambda + 1 = 0 \quad \Rightarrow \quad \lambda = -1/5$.

Hence the coordinates of required points are

$$\alpha = 2 \times \left(-\frac{1}{5}\right) + 3 = \frac{13}{5}$$

$$\beta = -3 \times \left(-\frac{1}{5}\right) + 4 = \frac{23}{5}$$

$$\gamma = 5 \times \left(-\frac{1}{5}\right) + 1 = 0$$

i.e., required point in $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$.

22. Total no. of cards in the deck = 52

Number of red cards = 26

No. of cards drawn = 2 simultaneously

$\therefore \quad X$ = value of random variable = 0, 1, 2

X or x_i	$P(X)$	$x_i P(X)$	$x_i^2 P(X)$
0	$\frac{^{26}C_0 \times {}^{26}C_2}{^{52}C_2} = \frac{25}{102}$	0	0
1	$\frac{^{26}C_1 \times {}^{26}C_1}{^{52}C_2} = \frac{52}{102}$	$\frac{52}{102}$	$\frac{52}{102}$
2	$\frac{^{26}C_0 \times {}^{26}C_2}{^{52}C_2} = \frac{25}{102}$	$\frac{50}{102}$	$\frac{100}{102}$
		$\Sigma x_i P(X) = 1$	$\Sigma x_i^2 P(X) = \frac{152}{102}$

Mean $= \mu = \Sigma x_i P(X) = 1$

Variance $= \sigma^2 = \Sigma x_i^2 P(X) - \mu^2$

$$= \frac{152}{102} - 1 = \frac{50}{102} = \frac{25}{51} = 0.49$$

SECTION–C

23. The given system of equation can be represented in matrix form as $AX = B$, where

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \; B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

Now $|A| = \begin{vmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{vmatrix} = 2\,(4 + 1) - 3\,(-2-3) + 3\,(-1 + 6)$

$$= 10 + 15 + 15 = 40 \neq 0$$

$$C_{11} = (-1)^{1+1} \begin{vmatrix} -2 & 1 \\ -1 & -2 \end{vmatrix} = 4 + 1 = 5$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = -(-2 - 3) = 5$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} = (-1 + 6) = 5$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 3 & 3 \\ -1 & -2 \end{vmatrix} = -(-6 + 3) = 3$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} = (-4 - 9) = -13$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = -(-2 - 9) = 11$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 3 & 3 \\ -2 & 1 \end{vmatrix} = (3 + 6) = 9$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -(2 - 3) = 1$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -4 - 3 = -7$$

$$\text{Adj } A = \begin{bmatrix} 5 & 5 & 5 \\ 3 & -13 & 11 \\ 9 & 1 & -7 \end{bmatrix}^T = \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\, adj\, A = \frac{1}{40} \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$\therefore \quad AX = B \Rightarrow X = A^{-1} B$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 25 - 12 + 27 \\ 25 + 52 + 3 \\ 25 - 44 - 21 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Equating the corresponding elements we get

$x = 1, y = 2, z = -1$

24. Let r and h be the radius and height of right circular cylinder inscribed in a given cone of radius R and height H. If S be the curved surface area of cylinder then

$S = 2\pi rh$

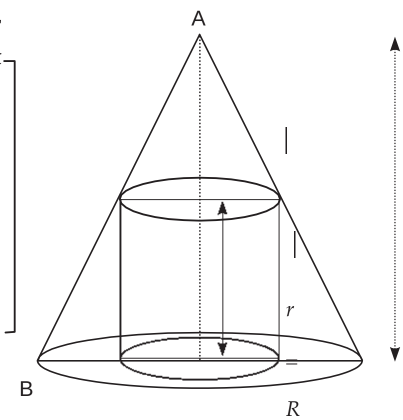
$\because \Delta AOC \sim \Delta FEC$

$\Rightarrow \dfrac{OC}{EC} = \dfrac{AO}{FE} \Rightarrow \dfrac{R}{R-r} = \dfrac{H}{h}$

$\Rightarrow h = \dfrac{(R-r).H}{R}$

$\Rightarrow \quad S = 2\pi r . \dfrac{(R-r)}{R} . H$

$\Rightarrow \quad S = \dfrac{2\pi H}{R}(rR - r^2)$



Differentiating both sides w.r.t. r, we get

$$\frac{dS}{dr} = \frac{2\pi H}{R}(R - 2r)$$

For maxima and minima

$$\frac{dS}{dr} = 0$$

$\Rightarrow \quad \dfrac{2\pi H}{R}(R - 2r) = 0$

$\Rightarrow \quad R - 2r = 0 \quad \Rightarrow \quad r = \dfrac{R}{2}$

Now, $\quad \dfrac{d^2 S}{dr^2} = \dfrac{2\pi H}{R}(0 - 2)$

$\Rightarrow \quad \left[\dfrac{d^2 S}{dr^2}\right]_{r = R/2} = -\dfrac{4\pi H}{R} = -\text{ve}$

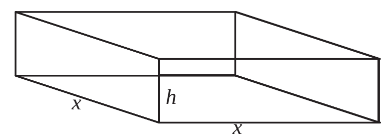

G
r
F
H
h
D
O
E
C
R

Hence for $r = \frac{R}{2}$ S is maximum.

i.e., radius of cylinder is half of that of cone.

OR

Let the length, breadth and height of open box with square base be x, x and h unit respectively. If V be the volume of box then



$$V = x.x.\,h$$

$\Rightarrow$ A $\qquad V = x^2 h \qquad \text{....}(i)$

lso $\qquad c^2 = x^2 + 4xh$

$$\Rightarrow \qquad h = \frac{c^2 - x^2}{4x}$$

Putting it in (i) we get

$$V = \frac{x^2 (c^2 - x^2)}{4x} \quad \Rightarrow \quad V = \frac{c^2 x}{4} - \frac{x^3}{4}$$

Differentiating w.r.t. x we get

$$\frac{dV}{dx} = \frac{c^2}{4} - \frac{3x^2}{4}$$

Now for maxima or minima

$$\frac{dV}{dx} = 0$$

$$\Rightarrow \qquad \frac{c^2}{4} - \frac{3x^2}{4} = 0 \quad \Rightarrow \quad \frac{3x^2}{4} = \frac{c^2}{4}$$

$$\Rightarrow \qquad x^2 = \frac{c^2}{3} \quad \Rightarrow \quad x = \frac{c}{\sqrt{3}}$$

Now, $\frac{d^2V}{dx^2} = -\frac{6x}{4} = -\frac{3x}{2}$

$$\therefore \qquad \left[\frac{d^2V}{dx^2}\right]_{x = c/\sqrt{3}} = -\frac{3c}{2\sqrt{3}} = -\text{ve}.$$

Hence, for $x = \frac{c}{\sqrt{3}}$ volume of box is maximum.

$$\therefore \qquad h = \frac{c^2 - x^2}{4x}$$

$$= \frac{c^2 - \frac{c^2}{3}}{4\frac{c}{\sqrt{3}}} = \frac{2c^2}{3} \times \frac{\sqrt{3}}{4c} = \frac{c}{2\sqrt{3}}$$

Therefore maximum volume $= x^2 . h$

$$= \frac{c^2}{3} . \frac{c}{2\sqrt{3}} = \frac{c^3}{6\sqrt{3}}$$

25. Let $\sin^{-1} x = z \Rightarrow x = \sin z$

$\therefore \quad \Rightarrow \quad \frac{1}{\sqrt{1-x^2}} dx = dz$

$\therefore \quad \int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx = \int z . \sin z \; dz$

$$= -z \cos z + \int \cos z \, dz$$

$$= -z \cos z + \sin z + c$$

$$= -\sin^{-1} x . \sqrt{1-x^2} + x + c$$

$$= x - \sqrt{1-x^2} \sin^{-1} x + c \qquad [\because \cos z = \sqrt{1-\sin^2 z} = \sqrt{1-x^2}]$$

OR

Now let $\frac{x^2+1}{(x-1)^2 (x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$

$\Rightarrow \quad \frac{x^2+1}{(x-1)^2 (x+3)} = \frac{A(x-1)(x+3) + B(x+3) + C(x-1)^2}{(x-1)^2 (x+3)}$

$\Rightarrow \quad x^2 + 1 = A(x-1)(x+3) + B(x+3) + C(x-1)^2 \qquad \ldots(i)$

Putting $x = 1$ in (i) we get

$\Rightarrow \quad 2 = 4B \Rightarrow B = \frac{1}{2}$

Putting $x = -3$ in (i) we get

$10 = 16C$

$\Rightarrow \quad C = \frac{10}{16} = \frac{5}{8}$

Putting $x = 0$, $B = \frac{1}{2}$, $C = \frac{5}{8}$ in (i) we get

$1 = A(-1).(3) + \frac{1}{2} \times 3 + \frac{5}{8}(-1)^2$

$1 = -3A + \frac{3}{2} + \frac{5}{8}$

$\Rightarrow \quad 3A = \frac{12+5}{8} - 1 = \frac{17}{8} - 1 = \frac{9}{8}$

$\Rightarrow \quad A = \frac{3}{8}$

$\therefore \quad \dfrac{x^2+1}{(x-1)^2(x+3)} = \dfrac{3}{8(x-1)} + \dfrac{1}{2(x-1)^2} + \dfrac{5}{8(x+3)}$

$\therefore \quad \displaystyle\int \frac{x^2+1}{(x-1)^2(x+3)}\,dx = \int\left(\frac{3}{8(x-1)} + \frac{1}{2(x-1)^2} + \frac{5}{8(x+3)}\right)dx$

$$= \frac{3}{8}\int\frac{dx}{x-1} + \frac{1}{2}\int (x-1)^{-2}\,dx + \frac{5}{8}\int\frac{dx}{x+3}$$

$$= \frac{3}{8}\log|x-1| - \frac{1}{2(x-1)} + \frac{5}{8}\log|x+3| + c$$

26. Let $R = \{(x, y) : x^2 + y^2 \le 4,\ x + y \ge 2\}$

$\Rightarrow \quad R = \{(x, y) : x^2 + y^2 \le 4\} \cap \{(x, y) : x + y \ge 2\}$

i.e., $\quad R = R_1 \cap R_2$ where

$R_1 = \{(x, y) : x^2 + y^2 \le 4\}$ and $R_2 = \{(x, y) : x + y \ge 2\}$

For region R_1

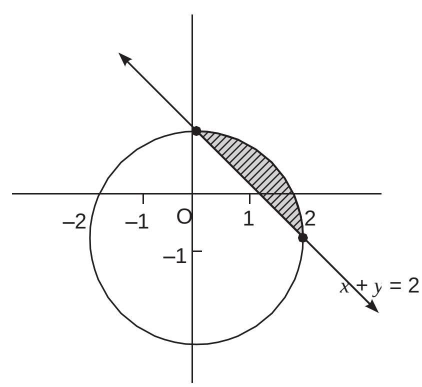
Obviously $x^2 + y^2 = 4$ is a circle having centre at (0,0) and radius 2.

Since (0,0) satisfy $x^2 + y^2 \le 4$. Therefore region R_1 is the region lying interior of circle $x^2 + y^2 = 4$

For region R_2

x	0	2
y	2	0

$x + y = 2$ is a straight line passing through (0, 2) and (2, 0). Since (0, 0) does not satisfy $x + y \ge 2$ therefore R_2 is that region which does not contain origin (0, 0) *i.e.,* above the line $x + y = 2$



Hence, shaded region is required region R.

Now area of required region

$$= \int_0^2 \sqrt{4-x^2}\,dx - \int_0^2 (2-x)\,dx$$

$$= \left[\frac{1}{2}x\sqrt{4-x^2} + \frac{1}{2}4\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2 - 2[x]_0^2 + \left[\frac{x^2}{2}\right]_0^2$$

$$= [2\sin^{-1}1 - 0] - 2[2-0] + \left[\frac{4}{2} - 0\right] = 2\times\frac{\pi}{2} - 4 + 2 = \pi - 2$$

27. Given lines are

$$\frac{x-1}{-3} = \frac{y-2}{-2k} = \frac{z-3}{2} \qquad ...(i)$$

$$\frac{x-1}{k} = \frac{y-2}{1} = \frac{z-3}{5} \qquad ...(ii)$$

Obviously, parallel vectors $\vec{b_1}$ and $\vec{b_2}$ of line (*i*) and (*ii*) respectively are:

$$\vec{b_1} = -3\hat{i} - 2k\hat{j} + 2\hat{k}$$

$$\Rightarrow \qquad \vec{b_2} = k\hat{i} + \hat{j} + 5\hat{k}$$

Line (*i*) $\perp$ (*ii*) $\Rightarrow \vec{b_1} \perp \vec{b_2}$

$$\Rightarrow \qquad \vec{b_1} \cdot \vec{b_2} = 0 \qquad \Rightarrow \qquad -3k - 2k + 10 = 0$$

$$\Rightarrow \qquad -5k + 10 = 0 \qquad \Rightarrow \qquad k = \frac{-10}{-5} = 2$$

Putting k = 2 in (*i*) and (*ii*) we get

$$\frac{x-1}{-3} = \frac{y-2}{-4} = \frac{z-3}{2}$$

$$\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{5}$$

Now the equation of plane containing above two lines is

$$\begin{bmatrix} x-1 & y-2 & z-3 \\ -3 & -4 & 2 \\ 2 & 1 & 5 \end{bmatrix} = 0$$

$\Rightarrow \quad (x-1)(-20-2) - (y-2)(-15-4) + (z-3)(-3+8) = 0$

$\Rightarrow \quad -22(x-1) + 19(y-2) + 5(z-3) = 0$

$\Rightarrow \quad -22x + 22 + 19y - 38 + 5z - 15 = 0$

$\Rightarrow \quad -22x + 19y + 5z - 31 = 0$

$\Rightarrow \quad 22x - 19y - 5z + 31 = 0$

[**Note:** Equation of plane containing lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$]

28. Consider the following events:

E_1 = Getting 5 or 6 in a single throw of a die.

E_2 = Getting 1, 2, 3, or 4 in a single throw of a die.

A = Getting exactly one head.

We have, $P(E_1) = \frac{2}{6} = \frac{1}{3}$, $P(E_2) = \frac{4}{6} = \frac{2}{3}$

$P(A / E_1)$ = Probability of getting exactly one head when a coin is tossed three times

$$= {}^3C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^2 = \frac{3}{8}$$

$P(A / E_2)$ = Probability of getting exactly one head when a coin is tossed once only $= \frac{1}{2}$

Now,

Required probability $= P(E_2 / A)$

$$= \frac{P(E_2)\ P(A / E_2)}{P(E_1)\ P(A / E_1) + P(E_2)\ P(A / E_2)} = \frac{\frac{2}{3} \times \frac{1}{2}}{\frac{1}{3} \times \frac{3}{8} + \frac{1}{2} \times \frac{2}{3}}$$

$$= \frac{\frac{1}{3}}{\frac{1}{8} + \frac{1}{3}} = \frac{1}{3} \times \frac{24}{11} = \frac{8}{11}$$

29. Let the mixture contain x kg of Food I and y kg of Food II.

According to question we have following constraints:

$2x + y \geq 8$ ...(*i*)

$x + 2y \geq 10$ ...(*ii*)

$x \geq 0$ ...(*iii*)

$y \geq 0$ ...(*iv*)

It z be the total cost of purchasing x kg of Food I and y kg of Food II then

$Z = 5x + 7y$ ...(*v*)

Here we have to minimise Z subject to the constraints (*i*) to (*iv*)

On plotting inequalities (*i*) to (*iv*) we get shaded region having corner points *A, B, C* which is required feasible region.

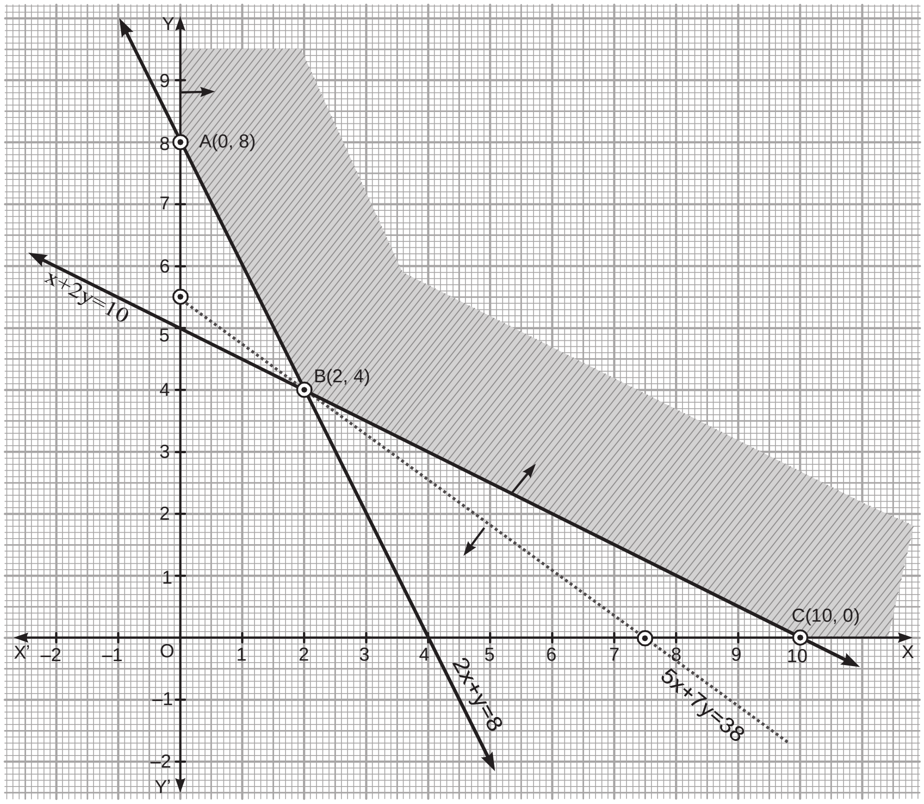
Now we evaluate Z at the corner points A (0, 8), B (2, 4) and C (10, 0)

Corner Point	$Z = 5x + 7y$	
A (0, 8)	56	
B (2, 4)	38	Minimum
C (10, 0)	50	

Since feasible region is unbounded. Therefore we have to draw the graph of the inequality.

$5x + 7y < 38$ ...(*vi*)

Since the graph of inequality (*vi*) is that open half plane which does not have any point common with the feasible region.



So the minimum value of Z is 38 at (2, 4).

i.e., the minimum cost of food mixture is `38 when 2 kg of Food I and 4 kg of Food II are mixed.

Set–II

10. $(\hat{k}\times\hat{j}).\hat{i}+\hat{j}.\hat{k} = -\hat{i}.\hat{i}+0 = -1+0 = -1$

19. Let $\cos^{-1}\dfrac{4}{5} = x,\ \cos^{-1}\dfrac{12}{13} = y$ $\qquad [x, y \in [o, \pi]]$

$\Rightarrow \quad \cos x = \dfrac{4}{5},\ \cos y = \dfrac{12}{13}$

$\therefore \quad \sin x = \sqrt{1-\left(\dfrac{4}{5}\right)^2},\ \sin y = \sqrt{1-\left(\dfrac{12}{13}\right)^2}$ $\qquad [\because x,\ y \in [0, \pi] \Rightarrow \sin x$ and $\sin y$ are +ve]

$\Rightarrow \quad \sin x = \dfrac{3}{5},\ \sin y = \dfrac{5}{13}$

Now $\quad \cos(x+y) = \cos x . \cos y - \sin x . \sin y$

$$= \frac{4}{5}\times\frac{12}{13}-\frac{3}{5}\times\frac{5}{13}$$

$\Rightarrow \quad \cos(x+y) = \dfrac{33}{65}$

$\Rightarrow \quad x + y = \cos^{-1}\left(\dfrac{33}{65}\right) \qquad \left[\because \dfrac{33}{65} \in [-1, 1]\right]$

Putting the value of x and y we get

$$\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\left(\frac{33}{65}\right) \quad \text{Proved.}$$

20. Refer to CBSE Delhi Set-I Q.No. 19.

21. Given differential equation is $\dfrac{dy}{dx} + y\cot x = 4x \operatorname{cosec} x$ and is of the type $\dfrac{dy}{dx} + Py = Q$ where

$P = \cot x$, $Q = 4x \operatorname{cosec} x$

$\therefore \quad \text{I.F.} = e^{\int P dx}$

$\therefore \quad \text{I.F.} = e^{\int \cot x dx} = e^{\log|\sin x|} = \sin x$

Its solution is given by

$\Rightarrow \quad \sin x . y = \int 4x \operatorname{cosec} x . \sin x \, dx$

$\Rightarrow \quad y \sin x = \int 4x \, dx = \dfrac{4x^2}{2} + C \qquad \Rightarrow \qquad y \sin x = 2x^2 + C$

Now $y = 0$ when $x = \dfrac{\pi}{2}$

$\therefore \quad 0 = 2 \times \dfrac{\pi^2}{4} + C \Rightarrow C = -\dfrac{\pi^2}{2}$

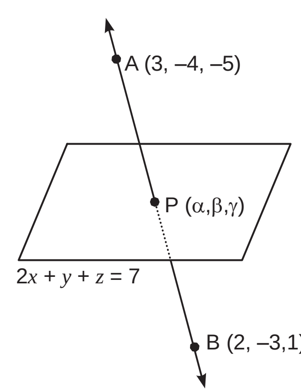
Hence, the particular solution of given differential equation is

$$y \sin x = 2x^2 - \frac{\pi^2}{2}$$

22. The equation of line passing through the point (3, – 4, –5) and (2, –3, 1) is

$$\frac{x-3}{2-3} = \frac{y+4}{-3+4} = \frac{z+5}{1+5}$$

$\Rightarrow \quad \dfrac{x-3}{-1} = \dfrac{y+4}{1} = \dfrac{z+5}{6} \qquad \ldots(i)$



Let the line (i) crosses the plane $2x + y + z = 7$ ...(ii) at point $P(\alpha, \beta, \gamma)$

$\because$ P lies on line (i), therefore (α, β, γ) satisfy equation (i)

$\therefore \quad \dfrac{\alpha - 3}{-1} = \dfrac{\beta + 4}{1} = \dfrac{\gamma + 5}{6} = \lambda$ (say)

$\Rightarrow \quad \alpha = -\lambda + 3$

$\beta = \lambda - 4$

$\gamma = 6\lambda - 5$

Also P (α, β, γ) lie on plane (ii)

$\therefore \quad 2\alpha + \beta + \gamma = 7$

$$\Rightarrow \quad 2(-\lambda+3)+(\lambda-4)+(6\lambda-5)=7$$
$$\Rightarrow \quad -2\lambda+6+\lambda-4+6\lambda-5=7$$
$$\Rightarrow \quad 5\lambda=10 \quad \Rightarrow \quad \lambda=2$$

Hence the co-ordinate of required point P is $(-2+3, 2-4, 6\times 2-5)$ *i.e.,* $(1, -2, 7)$

28. The given system of linear equations may be written in matrix form as:

$$AX = B$$

i.e.,
$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 1 \end{bmatrix}$$

Now,
$$|A| = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -7 \end{vmatrix}$$
$$= 1(-21+1) - 1(-14-3) - 1(-2-9)$$
$$= -20+17+11 = 8 \neq 0$$

$C_{11} = -20 \quad C_{12} = 17 \quad C_{13} = -11$

$C_{21} = +8 \quad C_{22} = -4 \quad C_{23} = 4$

$C_{31} = 4 \quad C_{32} = -3 \quad C_{33} = 1$

$$\therefore \quad \text{Adj } A = \begin{bmatrix} -20 & 17 & -11 \\ +8 & -4 & 4 \\ 4 & -3 & 1 \end{bmatrix}' = \begin{bmatrix} -20 & 8 & 4 \\ 17 & -4 & -3 \\ -11 & 4 & 1 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{|A|}\text{Adj } A = \frac{1}{8}\begin{bmatrix} -20 & 8 & 4 \\ +17 & -4 & -3 \\ -11 & 4 & 1 \end{bmatrix}$$

Now, $AX = B \Rightarrow X = A^{-1}B$

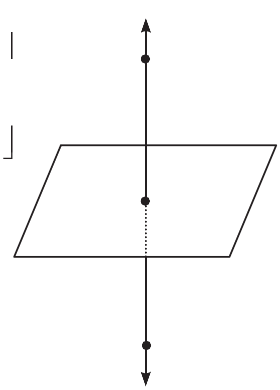
$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -20 & 8 & 4 \\ +17 & -4 & -3 \\ -11 & 4 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 10 \\ 1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -60+80+4 \\ 51-40-3 \\ -33+40+1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 24 \\ 8 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

On equating, we get

$$x = 3, y = 1, z = 1$$

29. Let $Q(\alpha, \beta, \gamma)$ be the foot of perpendicular from P to the given plane

$$2x + 4y - z = 2 \qquad ...(i)$$



Let $P'(x_1, y_1, z_1)$ be the image of P in the plane (i)

Now $\quad \vec{PQ} = (\alpha-7)\hat{i} + (\beta-14)\hat{j} + (\gamma-5)\hat{k}$

Also, Normal vector of plane (i) is

$$\vec{N} = 2\hat{i} + 4\hat{j} - \hat{k}$$

P (7, 14, 5)

Q (α, β, γ)

$2x + 4y - z = 2$

P′ (x_1, y_1, z_1)

Since $\vec{PQ} \parallel \vec{N}$

$\therefore \quad \dfrac{\alpha-7}{2} = \dfrac{\beta-14}{4} = \dfrac{\gamma-5}{-1} = \lambda$ (say)

$\Rightarrow \quad \alpha = 2\lambda + 7$

$\beta = 4\lambda + 14$

$\gamma = -\lambda + 5$

Again $\because$ $Q(\alpha, \beta, \gamma)$ lie on plane (i)

$\therefore \quad 2\alpha + 4\beta - \gamma = 2$

$2(2\lambda+7) + 4(4\lambda+14) - (-\lambda+5) = 2$

$\Rightarrow \quad 4\lambda + 14 + 16\lambda + 56 + \lambda - 5 - 2 = 0$

$\Rightarrow \quad 21\lambda + 63 = 0$

$\Rightarrow \quad 21\lambda = -63 \qquad \Rightarrow \quad \lambda = -3$

$\Rightarrow$ the coordinates of Q are $(2\times(-3)+7,\ 4\times(-3)+14,\ -(-3)+5)$ *i.e.*, (1, 2, 8)

$\therefore$ Length of perpendicular $= \sqrt{(7-1)^2 + (14-2)^2 + (5-8)^2}$

$= \sqrt{36+144+9}$

$= \sqrt{189} = 3\sqrt{21}$

Also Q (1, 2, 8) in mid point of PP'

$\therefore \quad 1 = \dfrac{7+x_1}{2} \quad \Rightarrow \quad x_1 = -5$

$2 = \dfrac{14+y_1}{2} \quad \Rightarrow \quad y_1 = -10$

$8 = \dfrac{5+z_1}{2} \quad \Rightarrow \quad z_1 = 11$

Hence the required image is (–5, –10, 11).

Set–III

10. Given:

$$2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2 & 6 \\ 0 & 2x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2+y & 6 \\ 1 & 2x+2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

Equating the corresponding elements we get

$2 + y = 5 \qquad$ and $\qquad 2x + 2 = 8$

$\Rightarrow \quad y = 3 \qquad$ and $\qquad x = 3$

$$\therefore \qquad x + y = 3 + 3 = 6.$$

19. Q $x = a\left(\cos t + \log \tan \frac{t}{2}\right)$

Differentiating w.r.t. t, we get

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan \frac{t}{2}} \cdot \sec^2 \frac{t}{2} \cdot \frac{1}{2}\right)$$

$$= a\left\{-\sin t + \frac{1}{2\sin \frac{t}{2} \cdot \cos \frac{t}{2}}\right\} = a\left\{-\sin t + \frac{1}{\sin t}\right\}$$

$$\frac{dx}{dt} = a\left(\frac{1 - \sin^2 t}{\sin t}\right) = a\frac{\cos^2 t}{\sin t}$$

Q $y = a \sin t$

Differentiating w.r.t t, we get

$$\frac{dy}{dt} = a \cdot \cos t \quad \Rightarrow \quad \frac{d^2y}{dt^2} = -a \sin t$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t \cdot \sin t}{a\cos^2 t} = \tan t$$

$$\therefore \quad \frac{d^2y}{dx^2} = \sec^2 t \cdot \frac{dt}{dx} = \sec^2 t \cdot \frac{1 \times \sin t}{a\cos^2 t} = \frac{1}{a}\sec^4 t \cdot \sin t$$

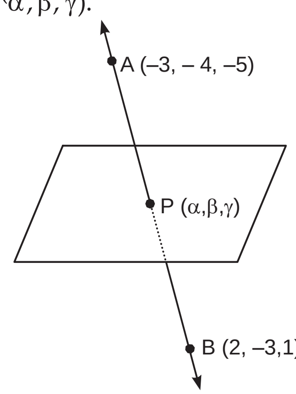
Hence, $\frac{d^2y}{dt^2} = -a\sin t$ and $\frac{d^2y}{dx^2} = \frac{\sec^4 t \sin t}{a}$

20. The equation of the line passing through the point (3, –4, –5) and (2, –3, 1) is

$$\frac{x-3}{2-3} = \frac{y+4}{-3+4} = \frac{z+5}{1+5}$$

$$\Rightarrow \quad \frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} \quad \ldots(i)$$

Let the line (*i*) crosses the plane $3x + 2y + z + 14 = 0$ ...(*ii*) at point $P(\alpha, \beta, \gamma)$.



Q P lie on line (*i*) therefore (α, β, γ) satisfy equation (*i*)

$$\therefore \quad \frac{\alpha-3}{-1} = \frac{\beta+4}{1} = \frac{\gamma+5}{6} = \lambda \text{ (say)}$$

$$\Rightarrow \quad \alpha = -\lambda + 3; \ \beta = \lambda - 4 \ \text{ and } \ \gamma = 6\lambda - 5$$

Also $P(\alpha, \beta, \gamma)$ lie on plane (ii)

$$\therefore \quad 3\alpha + 2\beta + \gamma + 14 = 0$$

$$\Rightarrow \quad 3(-\lambda + 3) + 2(\lambda - 4) + (6\lambda - 5) + 14 = 0$$

$$\Rightarrow \quad -3\lambda + 9 + 2\lambda - 8 + 6\lambda - 5 + 14 = 0$$

$$\Rightarrow \quad 5\lambda + 10 = 0 \quad \Rightarrow \quad \lambda = -2$$

Hence the coordinate of required point P is given as

$$(2+3,\ -2-4,\ 6\times-2-5)\equiv(5,-6,-17)$$

21. Given differential equation is

$$x\frac{dy}{dx}-y+x\sin\left(\frac{y}{x}\right)=0$$

$$\Rightarrow \quad \frac{dy}{dx}-\frac{y}{x}+\sin\left(\frac{y}{x}\right)=0 \qquad ...(i)$$

It is homogeneous differential equation.

Let $\quad \dfrac{y}{x}=v\Rightarrow y=vx$

$$\Rightarrow \quad \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Putting these values in (*i*) we get

$$v+x\frac{dv}{dx}-v+\sin v=0$$

$$\Rightarrow \quad x\frac{dv}{dx}+\sin v=0 \qquad \Rightarrow \qquad x\frac{dv}{dx}=-\sin v$$

$$\Rightarrow \quad \frac{dv}{\sin v}=\frac{-dx}{x}$$

$$\Rightarrow \quad \operatorname{cosec} v\,dv=-\frac{dx}{x}$$

Integrating both sides we get

$$\Rightarrow \quad \int\operatorname{cosec} v\,dv=-\int\frac{dx}{x}$$

$$\Rightarrow \quad \log\left|\operatorname{cosec} v-\cot v\right|=-\log|x|+c$$

$$\Rightarrow \quad \log\left|\operatorname{cosec}\frac{y}{x}-\cot\frac{y}{x}\right|+\log|x|=c$$

Putting $x=2$, $y=\pi$ we get

$$\Rightarrow \quad \log\left|\operatorname{cosec}\pi/2-\cot\pi/2\right|+\log 2=c$$

$$\Rightarrow \quad \log 1+\log 2=c \qquad [\because\ \log 1=0]$$

$$\Rightarrow \quad c=\log 2$$

Hence particular solution is

$$\log\left|\operatorname{cosec}\frac{y}{x}-\cot\frac{y}{x}\right|+\log|x|=\log 2$$

$$\Rightarrow \quad \log\left|x.(\operatorname{cosec} y/x-\cot y/x)\right|=\log 2$$

$$\Rightarrow \quad x\left(\operatorname{cosec}\frac{y}{x}-\cot\frac{y}{x}\right)=2$$

22. LHS $= \cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5}$

$$= \sin^{-1}\sqrt{1-\left(\frac{12}{13}\right)^2} + \sin^{-1}\frac{3}{5}$$

$$= \sin^{-1}\sqrt{1-\frac{144}{169}} + \sin^{-1}\frac{3}{5}$$

$$= \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{3}{5}$$

$$= \sin^{-1}\left[\frac{5}{13}\sqrt{1-\left(\frac{3}{5}\right)^2} + \frac{3}{5}\sqrt{1-\left(\frac{5}{13}\right)^2}\right] \qquad \left[\left(\frac{5}{13}\right)^2 + \left(\frac{3}{5}\right)^2 \le 1\right]$$

$$= \sin^{-1}\left[\frac{5}{13}\sqrt{1-\frac{9}{25}} + \frac{3}{5}\sqrt{1-\frac{25}{169}}\right]$$

$$= \sin^{-1}\left[\frac{5}{13}\times\frac{4}{5} + \frac{3}{5}\times\frac{12}{13}\right] = \sin^{-1}\left[\frac{20}{65} + \frac{36}{65}\right]$$

$$= \sin^{-1}\left[\frac{56}{65}\right] = \text{RHS}$$

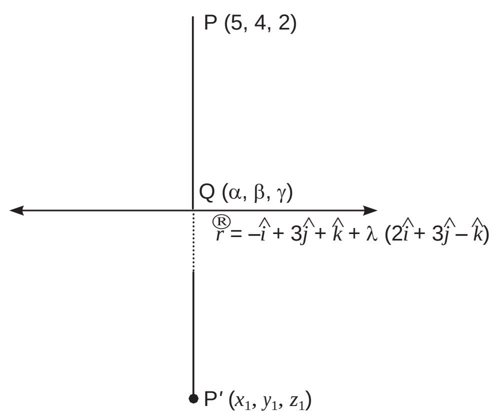
28. Given line is

$$\vec{r} = -\hat{i} + 3\hat{j} + \hat{k} + \lambda(2\hat{i} + 3\hat{j} - \hat{k})$$

It can be written in cartesian form as

$$\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1} \qquad \text{...}(i)$$

Let $Q(\alpha, \beta, \gamma)$ be the foot of perpendicular drawn from $P(5, 4, 2)$ to the line (i) and $P'(x_1, y_1, z_1)$ be the image of P on the line (i)



Q $\quad Q(\alpha, \beta, \gamma)$ lie on line (i)

$\therefore \quad \frac{\alpha+1}{2} = \frac{\beta-3}{3} = \frac{\gamma-1}{-1} = \lambda$ (say)

$\Rightarrow \quad \alpha = 2\lambda - 1;\ \beta = 3\lambda + 3$ and $\gamma = -\lambda + 1$

Now $\vec{PQ} = (\alpha - 5)\hat{i} + (\beta - 4)\hat{j} + (\gamma - 2)\hat{k}$

Parallel vector of line (i) $\vec{b} = 2\hat{i} + 3\hat{j} - \hat{k}$.

Obviously $\vec{PQ} \perp \vec{b}$

$\therefore \quad \vec{PQ}\cdot\vec{b} = 0$

$\quad 2(\alpha - 5) + 3(\beta - 4) + (-1)(\gamma - 2) = 0$

$\Rightarrow \quad 2\alpha - 10 + 3\beta - 12 - \gamma + 2 = 0$

$\Rightarrow \quad 2\alpha + 3\beta - \gamma - 20 = 0$

$\Rightarrow \quad 2(2\lambda - 1) + 3(3\lambda + 3) - (-\lambda + 1) - 20 = 0 \qquad$ [Putting α, β, γ]

$\Rightarrow \quad 4\lambda - 2 + 9\lambda + 9 + \lambda - 1 - 20 = 0$

$\Rightarrow \quad 14\lambda - 14 = 0 \quad \Rightarrow \quad \lambda = 1$

Hence the coordinates of foot of perpendicular Q are $(2 \times 1 - 1,\ 3 \times 1 + 3, -1 + 1)$, *i.e.*, $(1, 6, 0)$

$$\therefore \text{ Length of perpendicular } = \sqrt{(5-1)^2 + (4-6)^2 + (2-0)^2}$$
$$= \sqrt{16 + 4 + 4}$$
$$= \sqrt{24} = 2\sqrt{6} \text{ unit.}$$

Also since Q is mid-point of PP′

$$\therefore \quad 1 = \frac{x_1 + 5}{2} \quad \Rightarrow \quad x_1 = -3$$
$$6 = \frac{y_1 + 4}{2} \quad \Rightarrow \quad y_1 = 8$$
$$0 = \frac{z_1 + 2}{2} \quad \Rightarrow \quad z_1 = -2$$

Therefore required image is (–3, 8, –2).

29. The given system of linear equations may be written in matrix form as

$AX = B$ Where

$$A = \begin{bmatrix} 3 & 4 & 7 \\ 2 & -1 & 3 \\ 1 & 2 & -3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$$

$$\text{Now, } |A| = \begin{vmatrix} 3 & 4 & 7 \\ 2 & -1 & 3 \\ 1 & 2 & -3 \end{vmatrix}$$

$$= 3(3-6) - 4(-6-3) + 7(4+1)$$
$$= -9 + 36 + 35 = 62 \neq 0$$

$$C_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 3 \\ 2 & -3 \end{vmatrix} = 3 - 6 = -3$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 3 \\ 1 & -3 \end{vmatrix} = -\{-6 - 3\} = 9$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 4 + 1 = 5$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} 4 & 7 \\ 2 & -3 \end{vmatrix} = -(-12 - 14) = 26$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 3 & 7 \\ 1 & -3 \end{vmatrix} = -9 - 7 = -16$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = -(6 - 4) = -2$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 4 & 7 \\ -1 & 3 \end{vmatrix} = 12 + 7 = 19$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 3 & 7 \\ 2 & 3 \end{vmatrix} = -(9 - 14) = 5$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} = (-3 - 8) = -11$$

$$\therefore \quad Adj.\ A = \begin{bmatrix} -3 & 9 & 5 \\ 26 & -16 & -2 \\ 19 & 5 & -11 \end{bmatrix}'$$

$$= \begin{bmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} Adj\ A$$

$$= \frac{1}{62}\begin{bmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{bmatrix}$$

$\therefore \qquad AX = B$

$\Rightarrow \qquad X = A^{-1}B$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{62}\begin{bmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{bmatrix}\begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$$

$$= \frac{1}{62}\begin{bmatrix} -12 - 78 + 152 \\ 36 + 48 + 40 \\ 20 + 6 - 88 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{62}\begin{bmatrix} 62 \\ 124 \\ -62 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Equating the corresponding elements we get

$x = 1, \ y = 2, \ z = -1$

CBSE Examination Paper (Foreign 2012)

Time allowed: 3 hours **Maximum marks: 100**

General Instructions:

1. ***All*** *questions are compulsory.*
2. *The question paper consists of* ***29*** *questions divided into three Sections* ***A, B*** *and* ***C.*** *Section A comprises of* ***10*** *questions of* ***one*** *mark each, Section B comprises of* ***12*** *questions of* ***four*** *marks each and Section C comprises of* ***7*** *questions of* ***six*** *marks each.*
3. ***All*** *questions in Section A are to be answered in* ***one*** *word,* ***one*** *sentence or as per the exact requirement of the question.*
4. *There is no overall choice. However, internal choice has been provided in* ***4*** *questions of* ***four*** *marks each and* ***2*** *questions of* ***six*** *marks each. You have to attempt only* ***one*** *of the alternatives in all such questions.*
5. *Use of calculators is* ***not*** *permitted.*

Set–I

SECTION–A

Question number 1 to 10 carry 1 mark each.

1. If the binary operation $*$ on the set Z of integers is defined by $a * b = a + b - 5$, then write the identity element for the operation $*$ in Z.
2. Write the value of $\cot(\tan^{-1} a + \cot^{-1} a)$.
3. If A is a square matrix such that $A^2 = A$, then write the value of $(\mathrm{I} + A)^2 - 3A$.
4. If $x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$, write the value of x.
5. Write the value of the following determinant:
$$\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$$
6. If $\int \left(\frac{x-1}{x^2}\right) e^x \, dx = f(x)\, e^x + c$, then write the value of $f(x)$.
7. If $\int_0^a 3x^2 dx = 8$, write the value of 'a'.
8. Write the value of $(\hat{i} \times \hat{j}) \cdot \hat{k} + (\hat{j} \times \hat{k}) \cdot \hat{i}$
9. Write the value of the area of the parallelogram determined by the vectors $2\hat{i}$ and $3\hat{j}$.

10. Write the direction cosines of a line parallel to z-axis.

SECTION–B

Question numbers 11 to 22 carry 4 marks each.

11. If $f(x) = \dfrac{4x+3}{6x-4}, x \neq \dfrac{2}{3}$, show that $fof(x) = x$ for all $x \neq \dfrac{2}{3}$. What is the inverse of f?

12. Prove that: $\sin^{-1}\left(\dfrac{63}{65}\right) = \sin^{-1}\left(\dfrac{5}{13}\right) + \cos^{-1}\left(\dfrac{3}{5}\right)$

OR

Solve for x:

$$2\tan^{-1}(\sin x) = \tan^{-1}(2\sec x), x \neq \frac{\pi}{2}$$

13. Using properties of determinants, prove that

$$\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$$

14. If $x^m y^n = (x+y)^{m+n}$, prove that $\dfrac{dy}{dx} = \dfrac{y}{x}$.

15. If $y = e^{a\cos^{-1}x}, -1 \le x \le 1$, show that

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0.$$

OR

If $x\sqrt{1+y} + y\sqrt{1+x} = 0, -1 < x < 1, x \neq y$, then prove that $\dfrac{dy}{dx} = -\dfrac{1}{(1+x)^2}$.

16. Show that $y = \log(1+x) - \dfrac{2x}{2+x}, x > -1$ is an increasing function of x throughout its domain.

OR

Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$.

17. Evaluate: $\int x^2 \tan^{-1} x\, dx$

OR

Evaluate: $\int \dfrac{3x-1}{(x+2)^2}\, dx$

18. Solve the following differential equation:

$$\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right]\frac{dx}{dy} = 1, x \neq 0$$

19. Solve the following differential equation:

$3e^x \tan y\, dx + (2 - e^x)\sec^2 y\, dy = 0$, given that when $x = 0$, $y = \dfrac{\pi}{4}$.

20. If $\vec{\alpha} = 3\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{\beta} = 2\hat{i} + \hat{j} - 4\hat{k}$, then express $\vec{\beta}$ in the form $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

21. Find the vector and cartesian equations of the line passing through the point $P\,(1, 2, 3)$ and parallel to the planes $\vec{r}\cdot(\hat{i} - \hat{j} + 2\hat{k}) = 5$ and $\vec{r}\cdot(3\hat{i} + \hat{j} + \hat{k}) = 6$.

22. A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability distribution of the number of successes and hence find its mean.

SECTION–C

Question numbers 23 to 29 carry 6 marks each.

23. Using matrices, solve the following system of equations:

$$x - y + z = 4; \quad 2x + y - 3z = 0; \; x + y + z = 2$$

OR

If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, find $(AB)^{-1}$.

24. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius R is $\dfrac{4R}{3}$.

25. Find the area of the region in the first quadrant enclosed by x-axis, the line $x = \sqrt{3}\,y$ and the circle $x^2 + y^2 = 4$.

26. Evaluate: $\displaystyle\int_1^3 (x^2 + x)\,dx$

OR

Evaluate: $\displaystyle\int_0^{\pi/4} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x}\,dx$

27. Find the vector equation of the plane passing through the points $(2, 1, -1)$ and $(-1, 3, 4)$ and perpendicular to the plane $x - 2y + 4z = 10$. Also show that the plane thus obtained contains the line $\vec{r} = -\hat{i} + 3\hat{j} + 4\hat{k} + \lambda(3\hat{i} - 2\hat{j} - 5\hat{k})$.

28. A company produces soft drinks that has a contract which requires that a minimum of 80 units of the chemical A and 60 units of the chemical B go into each bottle of the drink. The chemicals are available in prepared mix packets from two different suppliers. Supplier S had a packet of mix of 4 units of A and 2 units of B that costs ₹10. The supplier T has a packet of mix of 1 unit of A and 1 unit of B costs ₹4. How many packets of mixed from S and T should the company purchase to honour the contract requirement and yet minimize cost? Make a LPP and solve graphically.

29. In a certain college, 4% of boys and 1% of girls are taller than 1.75 metres. Furthermore, 60% of the students in the college are girls. A student is selected at random from the college and is found to be taller than 1.75 metres. Find the probability that the selected student is a girl.

Set–II

Only those questions, not included in Set I, are given

9. If the binary operation $*$ on set R of real numbers is defined as $a*b=\dfrac{3ab}{7}$, write the identity element in R for $*$.

10. Evaluate: $\displaystyle\int \frac{2}{1+\cos 2x}\,dx$

19. If $x^{13}y^{7}=(x+y)^{20}$, prove that $\dfrac{dy}{dx}=\dfrac{y}{x}$.

20. Find the particular solution of the following differential equation:

$$e^{x}\sqrt{1-y^{2}}\,dx+\frac{y}{x}\,dy=0, \qquad x=0,\ y=1$$

21. If $\vec{\alpha}=3\vec{i}-\vec{j}$ and $\vec{\beta}=2\hat{i}+\hat{j}-3\hat{k}$, then express $\vec{\beta}$ in the form $\vec{\beta}=\vec{\beta}_1+\vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

22. Find the vector and cartesian equations of the line passing through the point P (3, 0, 1) and parallel to the planes $\vec{r}\cdot(\hat{i}+2\hat{j})=0$ and $\vec{r}\cdot(3\hat{j}-\hat{k})=0$.

28. Find the area of the region in the first quadrant enclosed by $x-axis$, the line $y=\sqrt{3}x$ and the circle $x^{2}+y^{2}=16$.

29. Find the vector equation of the plane passing through the points (3, 4, 2) and (7, 0, 6) and perpendicular to the plane $2x-5y-15=0$. Also show that the plane thus obtained contains the line $\vec{r}=\hat{i}+3\hat{j}-2\hat{k}+\lambda(\hat{i}-\hat{j}+\hat{k})$.

Set–III

Only those questions, not included in Set I and Set II are given

9. If the binary operation $*$ on the set Z of integers is defined by $a*b=a+b+2$, then write the identity element for the operation $*$ in Z.

19. If $x^{16}y^{9}=(x^{2}+y)^{17}$, prove that $\dfrac{dy}{dx}=\dfrac{2y}{x}$.

20. Find the particular solution of the following differential equation:

$$(x^{2}-yx^{2})dy+(y^{2}+x^{2}y^{2})dx=0;\ \ y=1,\ x=1$$

21. Find the distance between the point P (6, 5, 9) and the plane determined by the points $A(3,-1,2)$, B (5, 2, 4) and C (–1, –1, 6).

22. The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}+5\hat{k}$ and $\hat{i}-2\hat{j}-3\hat{k}$. Find the unit vector parallel to one of its diagonals. Also, find its area.

28. Using the method of integration, find the area of the ΔABC, coordinates of whose vertices are A (2, 0), B(4, 5) and C(6, 3).

29. Find the equation of the plane passing through the points (2, 2, 1) and (9, 3, 6) and perpendicular to the plane $2x+6y+6z=1$. Also, show that the plane thus obtained contains the line $\vec{r}=4\hat{i}+3\hat{j}+3\hat{k}+\lambda(7\hat{i}+\hat{j}+5\hat{k})$.

Solutions

Set–I

SECTION–A

1. Let $e \in Z$ be required identity

$\therefore \quad a * e = a \qquad \forall\, a \in Z$

$\Rightarrow \quad a + e - 5 = a$

$\Rightarrow \quad e = a - a + 5$

$\Rightarrow \quad e = 5$

2. $\cot(\tan^{-1} a + \cot^{-1} a) = \cot\left(\frac{\pi}{2} - \cot^{-1} a + \cot^{-1} a\right)$

$$= \cot\frac{\pi}{2} = 0$$

[**Note:** $\tan^{-1} x + \cot^{-1} x = \pi/2 \quad \forall\ x \in R$]

3. $(I + A)^2 - 3A = I^2 + A^2 + 2A - 3A$

$= I^2 + A^2 - A$

$= I^2 + A - A \qquad [\because A^2 = A]$

$= I^2 = I\,.\,I = I$

4. Given

$$x\begin{bmatrix}2\\3\end{bmatrix} + y\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}10\\5\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}2x\\3x\end{bmatrix} + \begin{bmatrix}-y\\y\end{bmatrix} = \begin{bmatrix}10\\5\end{bmatrix}$$

$$\begin{bmatrix}2x - y\\3x + y\end{bmatrix} = \begin{bmatrix}10\\5\end{bmatrix}$$

Equating the corresponding elements we get.

$2x - y = 10 \qquad ...(i)$

$3x + y = 5 \qquad ...(ii)$

(i) and $(ii) \Rightarrow 2x - y + 3x + y = 10 + 5$

$\Rightarrow 5x = 15 \quad \Rightarrow x = 3.$

5. Let $\Delta = \begin{vmatrix}102 & 18 & 36\\ 1 & 3 & 4\\ 17 & 3 & 6\end{vmatrix}$

Applying $R_1 \to R_1 - 6R_3$

$$\Delta = \begin{vmatrix}0 & 0 & 0\\ 1 & 3 & 4\end{vmatrix} = 0 \qquad [\because R_1 \text{ is zero}]$$

17 3 6

6. Given $$\int \left(\frac{x-1}{x^2}\right) e^x \, dx = f(x).e^x + c$$

$$\Rightarrow \quad \int \left(\frac{1}{x} - \frac{1}{x^2}\right) e^x \, dx = f(x).e^x + c$$

$$\Rightarrow \quad \frac{1}{x}.e^x + c = f(x).e^x + c$$

Equating we get

$$f(x) = \frac{1}{x}$$

[Note: $\int [f(x) + f'(x)]\, e^x = f(x)\, e^x + c$]

7. Given $$\int_0^a 3x^2 dx = 8$$

$$\Rightarrow \quad 3\left[\frac{x^3}{3}\right]_0^a = 8$$

$$\Rightarrow \quad a^3 = 8 \Rightarrow \quad a = 2$$

8. $(\hat{i} \times \hat{j}).\hat{k} + (\hat{j} \times \hat{k}).\hat{i} = \hat{k}.\hat{k} + \hat{i}.\hat{i}$

$$= 1 + 1 = 2$$

[Note $\vec{a}.\vec{b} = |\vec{a}|.|\vec{b}| \cos\theta$. Also $|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$]

9. Required area of parallelogram $= \left|2\hat{i} \times 3\hat{j}\right|$

$$= 6|\hat{i} \times \hat{j}| = 6|\hat{k}|$$

$= 6$ square unit.

[**Note:** Area of parallelogram whose sides are represented by $\vec{a}$ and $\vec{b}$ is $|\vec{a} \times \vec{b}|$]

10. The angle made by a line parallel to z axis with x, y and z axis are 90°, 90° and 0° respectively.

$\therefore$ The direction cosines of the line are $\cos 90°$, $\cos 90°$, $\cos 0°$ *i.e,* 0, 0, 1.

SECTION–B

11. Given $f(x) = \dfrac{4x+3}{6x-4}$, $x \neq \dfrac{2}{3}$

$$\therefore \quad fof(x) = f(f(x)) = f\left(\frac{4x+3}{6x-4}\right)$$

$$= \frac{4\left(\frac{4x+3}{6x-4}\right) + 3}{6\left(\frac{4x+3}{6x-4}\right) - 4} = \frac{16x + 12 + 18x - 12}{24x + 18 - 24x + 16} = \frac{34x}{34} = x$$

Now for inverse of f,

Let $\quad y = \dfrac{4x+3}{6x-4}$

$\therefore \quad 6xy - 4y = 4x + 3 \qquad 6xy - 4x = 3 + 4y$

$\Rightarrow x(6y-4) = 3 + 4y \qquad \Rightarrow \qquad x = \dfrac{4y+3}{6y-4}$

$\therefore$ Inverse of f is given by

$$f^{-1}(x) = \frac{4x+3}{6x-4}$$

12. Let $\sin^{-1}\left(\dfrac{5}{13}\right) = \alpha$, $\cos^{-1}\left(\dfrac{3}{5}\right) = \beta$

$\Rightarrow \quad \sin\alpha = \dfrac{5}{13}, \cos\beta = \dfrac{3}{5}$

$\Rightarrow \quad \cos\alpha = \sqrt{1 - \left(\dfrac{5}{13}\right)^2}, \sin\beta = \sqrt{1 - \left(\dfrac{3}{5}\right)^2}$

$\Rightarrow \quad \cos\alpha = \dfrac{12}{13}, \qquad \sin\beta = \dfrac{4}{5}$

Now $\quad \sin(\alpha+\beta) =$

$$= \sin\alpha . \cos\beta + \cos\alpha . \sin\beta$$

$$= \frac{5}{13} \cdot \frac{3}{5} + \frac{12}{13} \cdot \frac{4}{5}$$

$$= \frac{15}{65} + \frac{48}{65} = \frac{63}{65}$$

$\Rightarrow \qquad \alpha + \beta = \sin^{-1}\left(\dfrac{63}{65}\right)$

Putting the value of α and β we get

$$\sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5} = \sin^{-1}\left(\frac{63}{65}\right)$$

OR

Given, $\quad 2\tan^{-1}(\sin x) = \tan^{-1}(2\sec x)$

$\Rightarrow \qquad \tan^{-1}\left(\dfrac{2\sin x}{1-\sin^2 x}\right) = \tan^{-1}(2\sec x)$

$\Rightarrow \qquad \dfrac{2\sin x}{1-\sin^2 x} = 2\sec x \qquad \left[\because x \neq \dfrac{\pi}{2} \Rightarrow 1 - \sin^2 x \neq 0\right]$

$\Rightarrow \qquad \dfrac{2\sin x}{\cos^2 x} = 2\sec x \qquad \Rightarrow \qquad \sin x = \sec x . \cos^2 x$

$\Rightarrow \qquad \sin x = \dfrac{1}{\cos x} . \cos^2 x \qquad \Rightarrow \qquad \sin x = \cos x$

$\Rightarrow \qquad \tan x = 1 \qquad \Rightarrow \qquad x = \dfrac{\pi}{4}$

13. L.H.S. $= \begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$

$$= \begin{vmatrix} a & a & a+b+c \\ 2a & 3a & 4a+3b+2c \\ 3a & 6a & 10a+6b+3c \end{vmatrix} + \begin{vmatrix} a & b & a+b+c \\ 2a & 2b & 4a+3b+2c \\ 3a & 3b & 10a+6b+3c \end{vmatrix}$$

$$= a^2 \begin{vmatrix} 1 & 1 & a+b+c \\ 2 & 3 & 4a+3b+2c \\ 3 & 6 & 10a+6b+3c \end{vmatrix} + ab \begin{vmatrix} 1 & 1 & a+b+c \\ 2 & 2 & 4a+3b+2c \\ 3 & 3 & 10a+6b+3c \end{vmatrix}$$

$$= a^2 \begin{vmatrix} 1 & 1 & a+b+c \\ 2 & 3 & 4a+3b+2c \\ 3 & 6 & 10a+6b+3c \end{vmatrix} + ab.0 \qquad [\because c_1 = c_2 \text{ in second det.}]$$

$$= a^2 \begin{vmatrix} 1 & 1 & a+b+c \\ 2 & 3 & 4a+3b+2c \\ 3 & 6 & 10a+6b+3c \end{vmatrix}$$

$$= a^2 \begin{vmatrix} 1 & 1 & a \\ 2 & 3 & 4a \\ 3 & 6 & 10a \end{vmatrix} + a^2 \begin{vmatrix} 1 & 1 & b \\ 2 & 3 & 3b \\ 3 & 6 & 6b \end{vmatrix} + a^2.c \begin{vmatrix} 1 & 1 & c \\ 2 & 3 & 2c \\ 3 & 6 & 3c \end{vmatrix}$$

$$= a^2.a \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix} + a^2.b \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 6 & 6 \end{vmatrix} + a^2.c \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 6 & 3 \end{vmatrix}$$

$$= a^3 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix} + a^2b.0 + a^2c.0 \qquad \left[\because c_2 = c_3 \text{ in second det.}, \; c_1 = c_3 \text{ in third det.}\right]$$

$$= a^3 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ we get

$$a^3 \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 3 & 7 \end{vmatrix}$$

Expanding along R_1 we get

$$= a^3.1(7-6) - 0 + 0$$
$$= a^3.$$

14. Given $\quad x^m . y^n = (x+y)^{m+n}$

Taking logarithm of both sides we get

$$\log(x^m . y^n) = \log(x+y)^{m+n}$$

$$\Rightarrow \quad \log x^m + \log y^n = (m+n).\log(x+y)$$

$$\Rightarrow \quad m\log x + n\log y = (m+n).\log(x+y)$$

Differentiating both sides w.r.t. x we get

$$\frac{m}{x} + \frac{n}{y}.\frac{dy}{dx} = \frac{m+n}{x+y}.\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{m}{x} - \frac{m+n}{x+y} = \left(\frac{m+n}{x+y} - \frac{n}{y}\right)\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{mx+my-mx-nx}{x(x+y)} = \left(\frac{my+ny-nx-ny}{y(x+y)}\right).\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{my-nx}{x(x+y)} = \frac{my-nx}{y(x+y)}.\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{my-nx}{x(x+y)}.\frac{y(x+y)}{my-nx} = \frac{y}{x}$$

15. We have, $y = e^{a\cos^{-1} x}$

Taking log on both sides

$$\log y = a \log \cos^{-1} x$$

Differentiating w.r.t. x, we have

$$\frac{1}{y}\frac{dy}{dx} = a \times \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{-ay}{\sqrt{1-x^2}} \qquad \ldots(i)$$

Again differentiating w.r.t. x, we have

$$\frac{d^2y}{dx^2} = -a\left[\frac{\sqrt{1-x^2}\,\frac{dy}{dx} - y \times \frac{1}{2\sqrt{1-x^2}} \times -2x}{(1-x^2)}\right]$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} = -a\left[\sqrt{1-x^2} \times \frac{-ay}{\sqrt{1-x^2}} + \frac{xy}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} = a^2 y - \frac{axy}{\sqrt{1-x^2}}$$

$$\therefore \quad (1-x^2)\frac{d^2y}{dx^2} = a^2 y + x\frac{dy}{dx} \text{ [From } (i)]$$

We have,

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0$$

OR

Given, $\quad x\sqrt{1+y}+y\sqrt{1+x}=0$

$\Rightarrow \quad x\sqrt{1+y}=-y\sqrt{1+x}$

Squaring both sides, we have

$$x^2(1+y)=y^2(1+x)$$

$\Rightarrow \quad x^2+x^2y=y^2+xy^2 \Rightarrow x^2-y^2=xy(y-x)$

$\Rightarrow \quad x+y+xy=0 \qquad [\because x\neq y]$

$\Rightarrow \quad y=-\dfrac{x}{1+x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{(1+x)(-1)+x}{(1+x)^2}=\frac{-1}{(1+x)^2}$$

16. Here $\quad f(x)=\log(1+x)-\dfrac{2x}{2+x} \qquad$ [Where $y=f(x)$]

$\Rightarrow \quad f'(x)=\dfrac{1}{1+x}-2\left[\dfrac{(2+x).1-x}{(2+x)^2}\right]$

$$=\frac{1}{1+x}-\frac{2(2+x-x)}{(2+x)^2}=\frac{1}{1+x}-\frac{4}{(2+x)^2}$$

$$=\frac{4+x^2+4x-4-4x}{(x+1)(x+2)^2}=\frac{x^2}{(x+1)(x+2)^2}$$

For $f(x)$ being increasing function

$\Rightarrow \quad f'(x)>0$

$\Rightarrow \quad \dfrac{x^2}{(x+1)(x+2)^2}>0 \qquad \Rightarrow \quad \dfrac{1}{x+1}\cdot\dfrac{x^2}{(x+2)^2}>0$

$\Rightarrow \quad \dfrac{1}{x+1}>0 \qquad \left[\because \dfrac{x^2}{(x+2)^2}>0\right]$

$\Rightarrow \quad x+1>0 \qquad [\because 1>0]$

$\Rightarrow \quad x+1>0 \quad$ or $\quad x>-1$

i.e., $f(x)=y=\log(1+x)-\dfrac{2x}{2+x}$ is increasing function in its domain $x>-1$ *i.e.* $(-1,\infty)$.

OR

Given, curve $\quad ay^2 = x^3$

We have, $\quad 2ay\dfrac{dy}{dx} = 3x^2$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2}{2ay}$$

$$\Rightarrow \quad \frac{dy}{dx} \text{ at } (am^2, am^3) = \frac{3 \times a^2m^4}{2a \times am^3} = \frac{3m}{2}$$

$$\therefore \quad \text{Slope of normal} = -\frac{1}{\text{Slope of tangent}} = -\frac{1}{\frac{3m}{2}} = -\frac{2}{3m}$$

Equation of normal at the point (am^2, am^3) is given by

$$\frac{y - am^3}{x - am^2} = -\frac{2}{3m}$$

$$\Rightarrow \quad 3my - 3am^4 = -2x + 2am^2$$

$$\Rightarrow \quad 2x + 3my - am^2(2 + 3m^2) = 0$$

Hence, equation of normal is

$$2x + 3my - am^2(2 + 3m^2) = 0$$

17. $\displaystyle\int x^2 \tan^{-1} x\, dx = \tan^{-1} x \cdot \frac{x^3}{3} - \int \frac{1}{1+x^2} \cdot \frac{x^3}{3}\, dx$

$$= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3}\int\left(x - \frac{x}{x^2+1}\right)dx$$

$$\left[\begin{array}{l} x^2+1 \,\big)\, x^3 \,\big(\, x \\ \quad\;\; \pm x^3 \pm x \\ \quad\;\; \overline{\quad\;\; -x} \end{array}\right]$$

$$= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3}\left[\int x\,dx - \int \frac{x}{x^2+1}\,dx\right]$$

$$= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3}\cdot\frac{x^2}{2} + \frac{1}{3}\int \frac{dz}{2z}$$

$$\left[\begin{array}{l} \text{Let } x^2 + 1 = z \\ \Rightarrow 2x\,dx = dz \\ \Rightarrow x\,dx = \dfrac{dz}{2} \end{array}\right]$$

$$= \frac{x^3 \tan^{-1} x}{3} - \frac{x^2}{6} + \frac{1}{6}\log|z| + c$$

$$= \frac{x^3 \tan^{-1} x}{3} - \frac{x^2}{6} + \frac{1}{6}\log\left|x^2 + 1\right| + c$$

OR

Let $\dfrac{3x-1}{(x+2)^2} = \dfrac{A}{x+2} + \dfrac{B}{(x+2)^2}$

$\Rightarrow \quad \dfrac{3x-1}{(x+2)^2} = \dfrac{A(x+2)+B}{(x+2)^2}$

$\Rightarrow \quad 3x-1 = A(x+2)+B$

$\Rightarrow \quad 3x-1 = Ax + (2A+B)$

Equating coefficients, we get

$A = 3, \quad 2A + B = -1$

$\Rightarrow \quad 2\times 3 + B = -1$

$B = -7$

$\Rightarrow \quad \dfrac{3x-1}{(x+2)^2} = \dfrac{3}{x+2} - \dfrac{7}{(x+2)^2}$

$\therefore$

$\Rightarrow \quad \displaystyle\int \frac{3x-1}{(x+2)^2}\,dx = \int \frac{3}{x+2}\,dx - \int \frac{7}{(x+2)^2}\,dx$

$$= 3\log|x+2| - 7\,\frac{(x+2)^{-1}}{-1} + c$$

$$= 3\log|x+2| + \frac{7}{(x+2)} + c$$

18. Given $\left(\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}} - \dfrac{y}{\sqrt{x}}\right)\dfrac{dx}{dy} = 1, \quad x \neq 0$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{e^{-2\sqrt{x}}}{\sqrt{x}} - \dfrac{y}{\sqrt{x}}$

$\Rightarrow \quad \dfrac{dy}{dx} + \dfrac{1}{\sqrt{x}}.y = \dfrac{e^{-2\sqrt{x}}}{\sqrt{x}}$

It is linear equation of form $\dfrac{dy}{dx} + py = Q.$

Where $P = \dfrac{1}{\sqrt{x}}, \quad Q = \dfrac{e^{-2\sqrt{x}}}{\sqrt{x}}$

$\therefore \quad I.F. = e^{\int Pdx}$

$$= e^{\int \frac{1}{\sqrt{x}}dx}$$

$$= e^{\int x^{-\frac{1}{2}}\,dx}$$

$$= e^{\frac{x^{\frac{1}{2}+1}}{-\frac{1}{2}+1}} = e^{2\sqrt{x}}$$

Therefore General solution is

$$y.e^{2\sqrt{x}} = \int Q \times I.F\, dx + c$$

$$\Rightarrow \quad y.e^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}}.e^{2\sqrt{x}}\, dx + c$$

$$\Rightarrow \quad y.e^{2\sqrt{x}} = \int \frac{dx}{\sqrt{x}} + c \qquad \Rightarrow \qquad y.e^{2\sqrt{x}} = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$\Rightarrow \quad y.e^{2\sqrt{x}} = 2\sqrt{x} + c$$

19. Given

$$3e^x \tan y\, dx + (2 - e^x) \sec^2 y\, dy = 0$$

$$\Rightarrow \quad (2 - e^x) \sec^2 y\, dy = -3e^x \tan y\, dx$$

$$\Rightarrow \quad \frac{\sec^2 y}{\tan y} dy = \frac{-3e^x}{2 - e^x} dx \qquad \Rightarrow \qquad \int \frac{\sec^2 y\, dy}{\tan y} = 3\int \frac{-e^x\, dx}{2 - e^x}$$

$$\Rightarrow \quad \log|\tan y| = 3 \log|2 - e^x| + \log c$$

$$\Rightarrow \quad \log|\tan y| = \log c|(2 - e^x)|^3$$

$$\Rightarrow \quad \tan y = c(2 - e^x)^3$$

Putting $x = 0,\ y = \frac{\pi}{4}$ we get

$$\Rightarrow \quad \tan\frac{\pi}{4} = c(2 - e^{\circ})^3$$

$$1 = 8c \qquad \Rightarrow \qquad c = \frac{1}{8}$$

Therefore particular solution is

$$\tan y = \frac{(2 - e^x)^3}{8}.$$

20. Q $\quad \vec{\beta}_1$ is parallel to $\vec{\alpha}$

$\Rightarrow \quad \vec{\beta}_1 = \lambda\vec{\alpha}$ where λ is any scalar quantity.

$$\Rightarrow \quad \vec{\beta}_1 = 3\lambda\hat{i} + 4\lambda\hat{j} + 5\lambda\hat{k}$$

Also If, $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$

$$\Rightarrow \quad 2\hat{i} + \hat{j} - 4\hat{k} = (3\lambda\hat{i} + 4\lambda\hat{j} + 5\lambda\hat{k}) + \vec{\beta}_2$$

$$\Rightarrow \quad \vec{\beta}_2 = (2 - 3\lambda)\hat{i} + (1 - 4\lambda)\hat{j} - (4 + 5\lambda)\hat{k}$$

It is given $\vec{\beta}_2 \perp \vec{\alpha}$

$$\Rightarrow \quad (2 - 3\lambda).3 + (1 - 4\lambda).4 - (4 + 5\lambda).5 = 0$$

$$\Rightarrow \quad 6 - 9\lambda + 4 - 16\lambda - 20 - 25\lambda = 0$$

$\Rightarrow \quad -10-50\lambda = 0 \qquad \Rightarrow \qquad \lambda = \frac{-1}{5}$

Therefore $\vec{\beta}_1 = -\frac{3}{5}\hat{i} + \frac{4}{5}\hat{j} - \hat{k}$

$$\vec{\beta}_2 = \left(2+\frac{3}{5}\right)\hat{i} + \left(1+\frac{4}{5}\right)\hat{j} - (4-1)\hat{k}$$

$$= \frac{13}{5}\hat{i} + \frac{9}{5}\hat{j} - 3\hat{k}$$

Therefore required expression is

$$(2\hat{i}+\hat{j}-4\hat{k}) = \left(-\frac{3}{5}\hat{i} - \frac{4}{5}\hat{j} - \hat{k}\right) + \left(\frac{13}{5}\hat{i} + \frac{9}{5}\hat{j} - 3\hat{k}\right)$$

21. Let required cartesian equation of line be

$$\frac{x-1}{a} = \frac{y-2}{b} = \frac{z-3}{c} \qquad \text{...}(i)$$

Given planes are

$$\vec{r}.(\hat{i}-\hat{j}+2\hat{k}) = 5 \qquad \text{...}(ii)$$

$$\vec{r}.(3\hat{i}+\hat{j}+\hat{k}) = 6 \qquad \text{...}(iii)$$

Since line (*i*) is parallel to plane (*ii*) and normal vector of plane (*ii*) is $\hat{i}-\hat{j}+2\hat{k}$

$\Rightarrow \qquad a - b + 2c = 0 \qquad \text{...}(iv)$

Similarly line (*i*) is parallel to plane (*iii*) and normal vector of plane (*iii*) is $3\hat{i}+\hat{j}+\hat{k}$

$\Rightarrow \qquad 3a + b + c = 0 \qquad \text{...}(v)$

From (*iv*) and (*v*)

$$\frac{a}{-1-2} = \frac{b}{6-1} = \frac{c}{1+3}$$

$$\frac{a}{-3} = \frac{b}{5} = \frac{c}{4} = \lambda$$

$\Rightarrow \qquad a = -3\lambda, b = 5\lambda, c = 4\lambda$

Putting value of *a*, *b* and *c* in (*i*) we get required cartesian equation of line

$$\frac{x-1}{-3\lambda} = \frac{y-2}{5\lambda} = \frac{z-3}{4\lambda} \qquad \Rightarrow \qquad \frac{x-1}{-3} = \frac{y-2}{5} = \frac{z-3}{4}$$

Its vector equation is

$$\vec{r} = (\hat{i}+2\hat{j}+3\hat{k}) + \lambda(-3\hat{i}+5\hat{j}+4\hat{k})$$

22. Here, number of throws = 4

$$P(\text{doublet}) = p = \frac{6}{36} = \frac{1}{6}$$

$$P(\text{not doublet}) = q = \frac{30}{36} = \frac{5}{6}$$

Let X denotes number of successes, then

$$P(X=0) = {}^4C_0 p^0 q^4 = 1 \times 1 \times \left(\frac{5}{6}\right)^4 = \frac{625}{1296}$$

$$P(X=1) = {}^4C_1 \frac{1}{6} \times \left(\frac{5}{6}\right)^3 = 4 \times \frac{125}{1296} = \frac{500}{1296}$$

$$P(X=2) = {}^4C_2 \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^2 = 6 \times \frac{25}{1296} = \frac{150}{1296}$$

$$P(X=3) = {}^4C_3 \left(\frac{1}{6}\right)^3 \times \frac{5}{6} = \frac{20}{1296}$$

$$P(X=4) = {}^4C_4 \left(\frac{1}{6}\right)^4 = \frac{1}{1296}$$

Therefore the probability distribution of X is

X or x_i	0	1	2	3	4
$P(X)$ or p_i	$\frac{625}{1296}$	$\frac{500}{1296}$	$\frac{150}{1296}$	$\frac{20}{1296}$	$\frac{1}{1296}$

$\therefore$ Mean (M) $= \sum x_i p_i$

$$= 0 \times \frac{625}{1296} + 1 \times \frac{500}{1296} + 2 \times \frac{150}{1296} + 3 \times \frac{20}{1296} + 4 \times \frac{1}{1296}$$

$$= \frac{500}{1296} + \frac{300}{1296} + \frac{60}{1296} + \frac{4}{1296} = \frac{864}{1296} = \frac{2}{3}$$

SECTION–C

23. Given equations

$$x - y + z = 4$$
$$2x + y - 3z = 0$$
$$x + y + z = 2$$

We can write this system of equations as

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

Let $AX = B$

where $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 1(1+3) - (-1)(2+3) + 1(2-1) = 4 + 5 + 1 = 10$$

Now $\quad X = A^{-1}B$

For A^{-1}, we have

$$\text{Cofactors matrix of } A = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

Thus, $\quad X = A^{-1}.B = \dfrac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \dfrac{1}{10}\begin{bmatrix} 16+0+4 \\ -20+0+10 \\ 4-0+6 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

The required solution is

$\therefore \quad x = 2, \; y = -1, z = 1$

OR

For B^{-1}

$$|B| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix}$$

$$= 1(3-0) - 2(-1-0) - 2(2-0)$$

$$= 3 + 2 - 4 = 1 \neq 0$$

i.e., B is invertible matrix

$\Rightarrow \quad B^{-1}$ exist.

Now $\; C_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} = 3 - 0 = 3$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -(-1-0) = 1$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} = 2 - 0 = 2$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} 2 & -2 \\ -2 & 1 \end{vmatrix} = -(2-4) = 2$$

$$C_{22}=(-1)^{2+2}\begin{vmatrix}1 & -2\\ 0 & 1\end{vmatrix}=1-0=1$$

$$C_{23}=(-1)^{2+3}\begin{vmatrix}1 & 2\\ 0 & -2\end{vmatrix}=-(-2-0)=2$$

$$C_{31}=(-1)^{3+1}\begin{vmatrix}2 & -2\\ 3 & 0\end{vmatrix}=0+6=6$$

$$C_{32}=(-1)^{3+2}\begin{vmatrix}1 & -2\\ -1 & 0\end{vmatrix}=-(0-2)=2$$

$$C_{33}=(-1)^{3+3}\begin{vmatrix}1 & 2\\ -1 & 3\end{vmatrix}=(3+2)=5$$

$$\therefore \quad Adj\,B=\begin{bmatrix}3 & 1 & 2\\ 2 & 1 & 2\\ 6 & 2 & 5\end{bmatrix}'=\begin{bmatrix}3 & 2 & 6\\ 1 & 1 & 2\\ 2 & 2 & 5\end{bmatrix}$$

$$\Rightarrow\; B^{-1}=\frac{1}{|B|}(adj\,B)$$

$$=\frac{1}{1}\begin{bmatrix}3 & 2 & 6\\ 1 & 1 & 2\\ 2 & 2 & 5\end{bmatrix}=\begin{bmatrix}3 & 2 & 6\\ 1 & 1 & 2\\ 2 & 2 & 5\end{bmatrix}$$

Now $(AB)^{-1}=B^{-1}.A^{-1}$

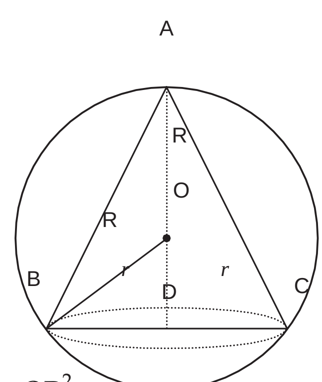
$$=\begin{bmatrix}3 & 2 & 6\\ 1 & 1 & 2\\ 2 & 2 & 5\end{bmatrix}.\begin{bmatrix}3 & -1 & 1\\ -15 & 6 & -5\\ 5 & -2 & 2\end{bmatrix}=$$

$$\begin{bmatrix}9-30+30 & -3+12-12 & 3-10+12\\ 3-15+10 & -1+6-4 & 1-5+4\\ 6-30+25 & -2+12-10 & 2-10+10\end{bmatrix}$$

$$=\begin{bmatrix}9 & -3 & 5\\ -2 & 1 & 0\\ 1 & 0 & 2\end{bmatrix}$$

24. Let h be the altitude of cone inscribed in a sphere of radius R. Also let r be radius of base of cone.

If V be volume of cone then



$$V=\frac{1}{3}\pi r^2 h$$

$$V=\frac{1}{3}\pi(2hR-h^2).h \qquad [\text{In } \Delta OBD \;\; BD^2=OB^2-OD^2$$

$$V = \frac{\pi}{3}(2h^2R - h^3)$$

$$\Rightarrow \quad \frac{dV}{dh} = \frac{\pi}{3}(4hR - 3h^2)$$

$$\Rightarrow r^2 = R^2 - (h-R)^2$$

$$\Rightarrow r^2 = R^2 - h^2 - R^2 + 2hR$$

$$\Rightarrow r^2 = 2hR - h^2$$

For maximum or minimum value

$$\frac{dV}{dh} = 0$$

$$\Rightarrow \quad \frac{\pi}{3}(4hR - 3h^2) = 0$$

$$\Rightarrow \quad 4hR - 3h^2 = 0$$

$$\Rightarrow \quad h(4R - 3h) = 0$$

$$\Rightarrow \quad h = 0, \quad h = \frac{4R}{3}.$$

Now $\dfrac{d^2V}{dh^2} = \dfrac{\pi}{3}(4R - 6h)$

$$\left.\frac{d^2V}{dh^2}\right]_{h=0} = +ve \text{ and } \left.\frac{d^2V}{dh^2}\right]_{h=\frac{4R}{3}} = -ve$$

Hence for $h = \dfrac{4R}{3}$, volume of cone is maximum.

25. Obviously $x^2 + y^2 = 4$ is a circle having centre at (0, 0) and radius 2 units.

For graph of line $x = \sqrt{3}y$

x	0	1
y	0	0.58

For intersecting point of given circle and line

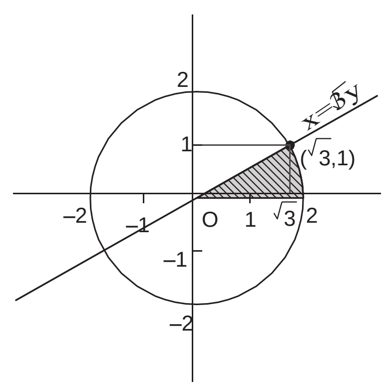
Putting $x = \sqrt{3}\,y$ in $x^2 + y^2 = 4$ we get

$$(\sqrt{3}y)^2 + y^2 = 4$$

$$\Rightarrow \quad 3y^2 + y^2 = 4$$

$$\Rightarrow \quad 4y^2 = 4 \quad \Rightarrow \quad y = \pm 1$$

$$\therefore \quad x = \pm\sqrt{3}$$



Intersecting points are $(\sqrt{3}, 1), (-\sqrt{3}, -1)$.

Shaded region is required region.

Now required area $= \displaystyle\int_0^{\sqrt{3}} \frac{x}{\sqrt{3}}\,dx + \int_{\sqrt{3}}^{2} \sqrt{4 - x^2}$

$$= \frac{1}{\sqrt{3}}\left[\frac{x^2}{2}\right]_0^{\sqrt{3}} + \left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{\sqrt{3}}^2$$

$$= \frac{1}{2\sqrt{3}}(3-0) + \left[2\sin^{-1}1 - \left(\frac{\sqrt{3}}{2} + 2\sin^{-1}\frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{\sqrt{3}}{2} + \left[2\frac{\pi}{2} - \frac{\sqrt{3}}{2} - \frac{2\pi}{3}\right]$$

$$= \frac{\sqrt{3}}{2} + \pi - \frac{\sqrt{3}}{2} - \frac{2\pi}{3} = \pi - \frac{2\pi}{3} = \frac{\pi}{3} \text{ sq. unit.}$$

26. Here $a = 1$, $b = 3$, $h = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$

$\Rightarrow \qquad nh = 2$

Also $f(x) = x^2 + x$

By definition $\int_a^b f(x)dx = \lim_{h\to 0} h\{f(a) + f(a+h) + \ldots\ldots\ldots\ldots + f(a+(n-1)h\}$

$$\int_1^3 f(x)dx = \lim_{h\to 0} h\{f(1) + f(1+h) + \ldots\ldots\ldots\ldots + f(1+(n-1)h\}$$

Now $f(1) = 1^2 + 1 = 2$

$$f(1+h) = (1+h)^2 + (1+h) = 1^2 + h^2 + 2h + 1 + h = 2 + 3h + h^2$$

$$f(1+2h) = (1+2h)^2 + (1+2h) = 1^2 + 2^2h^2 + 4h + 1 + 2h = 2 + 6h + 2^2h^2$$

$$f(1+(n-1)h = \{1+(n-1)h\}^2 + \{1+(n-1)h\}$$

$$= 2 + 3(n-1)h + (n-1)^2h^2$$

Hence

$$\int_1^3 (x^2+x)dx = \lim_{h\to 0} h\{2 + (2+3h+h^2) + (2+6h+2^2h^2) + \ldots\ldots + (2+3(n-1)h + (n-1)^2.h^2)\}$$

$$= \lim_{h\to 0} h.[\{2n + 3h\{1+2+\ldots(n-1)\} + h^2\{1^2 + 2^2 + \ldots\ldots + (n-1)^2\}]$$

$$= \lim_{h\to 0} h\left\{2n + 3h.\frac{(n-1).n}{2} + h^2\frac{(n-1).n(2n-1)}{6}\right\}$$

$$= \lim_{h\to 0}\left\{2n.h + 3\frac{n^2.h^2\left(1-\frac{1}{n}\right)}{2} + \frac{n^3h^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{6}\right\}$$

$$= \lim_{n\to\infty}\left\{4 + \frac{12}{2}\left(1 - \frac{1}{n}\right) + \frac{8}{6}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)\right\} \qquad \left[\begin{array}{l} \because\ nh = 2 \\ h \to 0 \Rightarrow n \to \infty \end{array}\right]$$

$$= 4 + 6\,(1-0) + \frac{4}{3}\,(1-0)\,(2-0)$$

$$= 4 + 6 + \frac{4}{3} \times 2 = 10 + \frac{8}{3} = \frac{38}{3}$$

OR

Let $\quad I = \int_0^{\pi/2} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x} dx$

$$= \int_0^{\pi/2} \frac{\cos^2 x}{\cos^2 x + 4(1-\cos^2 x)}\, dx$$

$$= \int_0^{\pi/2} \frac{\cos^2 x}{4 - 3\cos^2 x}\, dx = -\frac{1}{3}\int_0^{\pi/2} \frac{4 - 3\cos^2 x - 4}{4 - 3\cos^2 x}\, dx$$

$$= -\frac{1}{3}\int_0^{\pi/2}\left(1 - \frac{4}{4 - 3\cos^2 x}\right)dx = -\frac{1}{3}\int_0^{\pi/2} dx + \frac{4}{3}\int_0^{\pi/2} \frac{dx}{4 - 3\cos^2 x}$$

$$= \frac{-1}{3}[x]_0^{\pi/2} + \frac{4}{3}\int_0^{\pi/2} \frac{\sec^2 x\, dx}{4\sec^2 x - 3}$$

$$= -\frac{1}{3}\cdot\frac{\pi}{2} + \frac{4}{3}\int_0^{\pi/2} \frac{\sec^2 x\, dx}{4(1+\tan^2 x) - 3}$$

$$\left[\begin{array}{l} \text{Let } \tan x = z \Rightarrow \sec^2 x\, dx = dz \\ \text{Also, } x = \frac{\pi}{2} \Rightarrow z = \infty;\ x = 0 \Rightarrow z = 0 \end{array}\right]$$

$$= -\frac{\pi}{6} + \frac{4}{3}\int_0^{\infty} \frac{dz}{4 + 4z^2 - 3}$$

$$= -\frac{\pi}{6} + \frac{4}{3 \times 4}\int_0^{\infty} \frac{dz}{z^2 + \left(\frac{1}{2}\right)^2}$$

$$= -\frac{\pi}{6} + \frac{1}{3}\left[2 \cdot \tan^{-1}\frac{z}{1/2}\right]_0^{\infty}$$

$$= -\frac{\pi}{6} + \frac{2}{3}[\tan^{-1} 2z]_0^{\infty}$$

$$= -\frac{\pi}{6} + \frac{2}{3}[\tan^{-1}\infty - \tan^{-1} 0]$$

$$= -\frac{\pi}{6} + \frac{2}{3}\left[\frac{\pi}{2} - 0\right]$$

$$= -\frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{6}.$$

27. Let the equation of plane through (2, 1, –1) be

$$a(x-2)+b(y-1)+c(z+1)=0 \qquad \text{...}(i)$$

$\because$ (i) passes through (–1, 3, 4)

$$\therefore \quad a(-1-2)+b(3-1)+c(4+1)=0$$

$$\Rightarrow \quad -3a+2b+5c=0 \qquad \text{...}(ii)$$

Also since plane (i) is perpendicular to plane $x-2y+4z=10$

$$\therefore \quad a-2b+4c=0 \qquad \text{...}(iii)$$

From (ii) and (iii) we get

$$\frac{a}{8+10}+\frac{b}{5+12}=\frac{c}{6-2}$$

$$\Rightarrow \quad \frac{a}{18}=\frac{b}{17}=\frac{c}{4}=\lambda \ (\text{say})$$

$$\Rightarrow \quad a=18\lambda, b=17\lambda, c=4\lambda,$$

Putting the value of a, b, c in (i) we get

$$18\lambda(x-2)+17\lambda(y-1)+4\lambda(z+1)=0$$

$$\Rightarrow \quad 18x-36+17y-17+4z+4=0$$

$$\Rightarrow \quad 18x+17y+4z=49$$

$\therefore$ Required vector equation of plane is

$$\vec{r}.(18\hat{i}+17\hat{j}+4\hat{k})=49 \qquad \text{...}(iv)$$

Obviously plane (iv) contains the line

$$\vec{r}=(-\hat{i}+3\hat{j}+4\hat{k})+\lambda(3\hat{i}-2\hat{j}-5\hat{k}) \qquad \text{...}(v)$$

if point $(-\hat{i}+3\hat{j}+4\hat{k})$ satisfy equation (iv) and vector $(18\hat{i}+17\hat{j}+4\hat{k})$ is perpendicular to $(3\hat{i}-2\hat{j}+5\hat{k})$.

Here, $(-\hat{i}+3\hat{j}+4\hat{k}).(18\hat{i}+17\hat{j}+4\hat{k})=-18+51+16=49$

Also, $(18\hat{i}+17\hat{j}+4\hat{k}).(3\hat{i}-2\hat{j}-5\hat{k})=54-34-20=0$

Therefore (iv) contains line (v).

28. Let x and y units of packet of mixes are purchased from S and T respectively. If Z is total cost then

$$Z=10x+4y \qquad \text{...}(i)$$

is objective function which we have to minimize

Here constraints are.

$$4x+y\geq 80 \qquad \text{...}(ii)$$

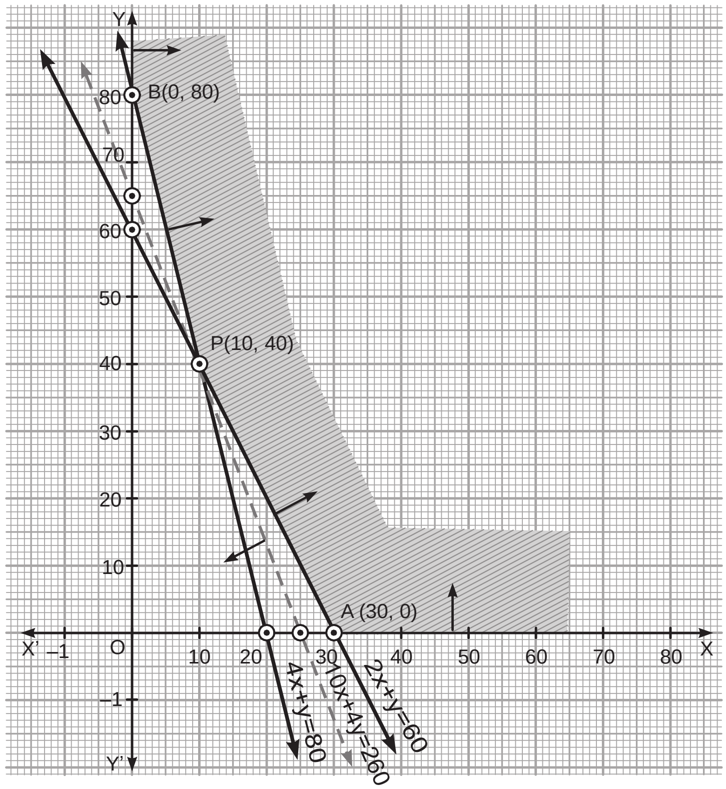
$$2x+y\geq 60 \qquad \text{...}(iii)$$

Also, $$x\geq 0 \qquad \text{...}(iv)$$

$$y\geq 0 \qquad \text{...}(v)$$

On plotting graph of above constraints or inequalities (ii), (iii) , (iv) and (v) we get shaded region having corner point A, P, B as feasible region.

For coordinate of P



Point of intersection of

$$2x + y = 60 \qquad \text{...}(vi)$$

and $\quad 4x + y = 80 \qquad \text{...}(vii)$

$(vi) - (vii) \quad \Rightarrow 2x + y - 4x - y = 60 - 80$

$\Rightarrow -2x = -20 \quad \Rightarrow \quad x = 10$

$\Rightarrow y = 40$

Q $\quad$ co-ordinate of $P \equiv (10, 40)$

Now the value of Z is evaluated at corner point in the following table

Corner point	$Z = 10x + 4y$	
A (30, 0)	300	
P (10, 40)	260	← Minimum
B (0, 80)	320	

Since feasible region is unbounded. Therefore we have to draw the graph of the inequality.

$$10x + 4y < 260 \qquad \text{...}(viii)$$

Since the graph of inequality (*viii*) does not have any point common.

So the minimum value of Z is 260 at (10, 40).

i.e., minimum cost of each bottle is ` 260 if the company purchases 10 packets of mixes from S and 40 packets of mixes from supplier T.

29. Let E_1, E_2, A be events such that

E_1 = student selected is girl

E_2 = student selected is Boy

A = student selected is taller than 1.75 metres.

Here $P(E_1/A)$ is required.

Now $P(E_1)=\dfrac{60}{100}=\dfrac{3}{5}$, $\quad P(E_2)=\dfrac{40}{100}=\dfrac{2}{5}$

$P\left(\dfrac{A}{E_1}\right)=\dfrac{1}{100}$, $\quad P\left(\dfrac{A}{E_2}\right)=\dfrac{4}{100}$

$$\therefore \quad P\left(\frac{E_1}{A}\right)=\frac{P(E_1).P\left(\frac{A}{E_1}\right)}{P(E_1).P\left(\frac{A}{E_1}\right)+P(E_2).P\left(\frac{A}{E_2}\right)}$$

$$=\frac{\frac{3}{5}\times\frac{1}{100}}{\frac{3}{5}\times\frac{1}{100}+\frac{2}{5}\times\frac{4}{100}}=\frac{\frac{3}{500}}{\frac{3}{500}+\frac{8}{500}}=\frac{3}{500}\times\frac{500}{11}=\frac{3}{11}$$

Set–II

9. Let $e\in R$ be identity element.

$\therefore \quad a*e=a \quad \forall\, a\in R$

$\Rightarrow \quad \dfrac{3ae}{7}=a \quad \Rightarrow \quad e=\dfrac{7a}{3a}$

$\Rightarrow \quad e=\dfrac{7}{3}$

10. $\displaystyle\int\frac{2}{1+\cos 2x}\,dx=\int\frac{2}{2\cos^2 x}\,dx$

$\displaystyle =\int\sec^2 x\,dx=\tan x+c$

19. Given $x^{13}y^7=(x+y)^{20}$

Taking logarithm of both sides, we get

$\log(x^{13}y^7)=\log(x+y)^{20}$

$\Rightarrow \quad \log x^{13}+\log y^7=20\log(x+y)$

$\Rightarrow \quad 13\log x+7\log y=20\log(x+y)$

Differentiating both sides w.r.t. x we get

$$\frac{13}{x}+\frac{7}{y}.\frac{dy}{dx}=\frac{20}{x+y}.\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{13}{x} - \frac{20}{x+y} = \left(\frac{20}{x+y} - \frac{7}{y}\right)\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{13x + 13y - 20x}{x(x+y)} = \left(\frac{20y - 7x - 7y}{(x+y).y}\right)\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{13y - 7x}{x(x+y)} = \left(\frac{13y - 7x}{x(x+y)}\right).\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{13y - 7x}{x(x+y)} \times \frac{y(x+y)}{13y - 7x} \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{y}{x}$$

20. Given $e^x\sqrt{1-y^2}\,dx + \frac{y}{x}dy = 0$

$$\frac{y}{x}dy = -e^x\sqrt{1-y^2}\,dx \qquad \Rightarrow \qquad \frac{y}{\sqrt{1-y^2}}dy = -x\,e^x\,dx$$

Integrating both sides we get

$$\int \frac{y}{\sqrt{1-y^2}}dy = -\int x\,e^x\,dx$$

$$\Rightarrow \quad \int \frac{-z\,dz}{z} = -[x.e^x - \int e^x\,dx] + c \qquad [\text{Let } 1 - y^2 = z^2 \Rightarrow -2ydy = 2zdz \Rightarrow ydy = -zdz]$$

$$\Rightarrow \quad -z = -x\,e^x + e^x + c$$

$$\Rightarrow \quad -\sqrt{1-y^2} = -x\,e^x + e^x + c \qquad \Rightarrow x\,e^x - e^x - \sqrt{1-y^2} = c$$

Putting $x = 0, y = 1$ we get

$$\Rightarrow \quad -1 - \sqrt{1-1} = c \quad \Rightarrow \quad c = -1$$

Hence particular solution is

$$\Rightarrow \quad x\,e^x - e^x - \sqrt{1-y^2} = -1$$

$$\Rightarrow \quad e^x(x-1) - \sqrt{1-y^2} + 1 = 0$$

21. Q $\quad \vec{\beta}_1$ is parallel to $\vec{\alpha}$

$$\Rightarrow \quad \vec{\beta}_1 = \lambda\vec{\alpha} \qquad \Rightarrow \qquad \vec{\beta}_1 = 3\lambda\hat{i} - \lambda\hat{j}$$

Also $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$

$$\Rightarrow \quad 2\hat{i} + \hat{j} - 3\hat{k} = (3\lambda\hat{i} - \lambda\hat{j}) + \vec{\beta}_2$$

$$\Rightarrow \quad \vec{\beta}_2 = (2\hat{i} + \hat{j} - 3\hat{k}) - (3\lambda\hat{i} - \lambda\hat{j}) = (2 - 3\lambda)\hat{i} + (1 + \lambda)\hat{j} - 3\hat{k}$$

It is given $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$

$$\therefore \quad (2 - 3\lambda)3 + (1 + \lambda).(-1) + (-3).0 = 0$$

$$\Rightarrow \quad 6 - 9\lambda - 1 - \lambda = 0$$

$$\Rightarrow \quad 5 - 10\lambda = 0 \quad \Rightarrow \qquad \lambda = \frac{5}{10} = \frac{1}{2}$$

$\therefore \qquad \vec{\beta}_1 = 3 \times \frac{1}{2}\hat{i} - \frac{1}{2}\hat{j} = \frac{3}{2}\hat{i} - \frac{1}{2}\hat{j}$

$\qquad \vec{\beta}_2 = \left(2 - 3 \times \frac{1}{2}\right)\hat{i} + \left(1 + \frac{1}{2}\right)\hat{j} - 3\hat{k} = \frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}$

Therefore required expression is

$$2\hat{i} + \hat{j} - 3\hat{k} = \left(\frac{3}{2}\hat{i} - \frac{1}{2}\hat{j}\right) + \left(\frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}\right)$$

22. Let the cartesian equation of the line passing through the point P (3, 0, 1) be

$$\frac{x-3}{a} = \frac{y-0}{b} = \frac{z-1}{c} \qquad \text{...}(i)$$

Given planes are

$$\vec{r}.(\hat{i} + 2\hat{j}) = 0 \qquad \text{...}(ii)$$

$$\vec{r}.(3\hat{i} - \hat{k}) = 0 \qquad \text{...}(iii)$$

Since line (i) is parallel to plane (ii) and (iii)

$\Rightarrow \qquad (a\hat{i} + b\hat{j} + c\hat{k}).(\hat{i} + 2\hat{j}) = 0 \;\Rightarrow\; a + 2b + 0.c = 0 \qquad \text{...}(iv)$

and $\qquad (a\hat{i} + b\hat{j} + c\hat{k}).(3\hat{i} - \hat{k}) = 0 \;\Rightarrow\; 3a + 0.b - c = 0 \qquad \text{...}(v)$

From (iv) and (v)

$$\frac{a}{-2-0} = \frac{b}{0+1} = \frac{c}{0-6}$$

$\Rightarrow \qquad \frac{a}{-2} = \frac{b}{1} = \frac{c}{-6} = \lambda \text{ (say)}$

$\Rightarrow \qquad a = -2\lambda,\; b = \lambda,\; c = -6\lambda$

Putting the value of $a = -2\lambda$, $b = \lambda$, $c = -6\lambda$ in (i) we get required cartesian equation of line

$$\frac{x-3}{-2\lambda} = \frac{y}{\lambda} = \frac{z-1}{-6\lambda} \qquad \Rightarrow \qquad \frac{x-3}{-2} = \frac{y}{1} = \frac{z-1}{-6}$$

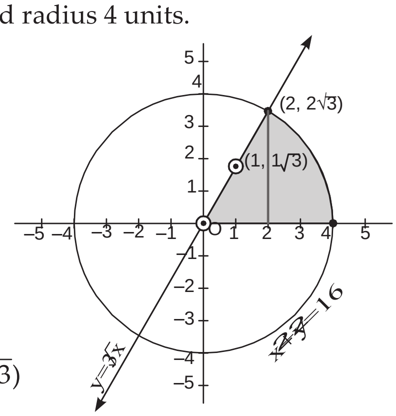
Therefore required vector equation is

$$\vec{r} = (-3\hat{i} + \hat{k}) + \lambda(-2\hat{i} + \hat{j} - 6\hat{k})$$

28. Obviously $x^2 + y^2 = 16$ is a circle having centre at (0, 0) and radius 4 units.

For graph of line $y = \sqrt{3}x$

x	0	1
y	0	$\sqrt{3} = 1.732$



For intersecting point of given circle and line

Putting $y = \sqrt{3}x$ in $x^2 + y^2 = 16$ we get

$\qquad x^2 + (\sqrt{3}x)^2 = 16$

$\Rightarrow \qquad 4x^2 = 16 \qquad \Rightarrow \qquad x = \pm 2$

$\therefore \qquad y = \pm 2\sqrt{3}.$

Therefore, intersecting point of circle and line is $(\pm 2, \pm 2\sqrt{3})$

Now shaded region is required region

$\therefore$ Required Area $= \int_0^2 \sqrt{3}\,x\,dx + \int_2^4 \sqrt{16 - x^2}\,dx.$

$$= \sqrt{3}\left[\frac{x^2}{2}\right]_0^2 + \left[\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_2^4$$

$$= \frac{\sqrt{3}}{2} \times 4 + \left[\frac{x}{2}\sqrt{16-x^2} + 8\sin^{-1}\frac{x}{4}\right]_2^4$$

$$= 2\sqrt{3} + \left[0 + \frac{8\pi}{2} - \left(\sqrt{12} + \frac{8\pi}{6}\right)\right] = 2\sqrt{3} + \left[4\pi - \sqrt{12} - \frac{4\pi}{3}\right]$$

$$= 2\sqrt{3} + 4\pi - 2\sqrt{3} - \frac{4\pi}{3} = 4\pi - \frac{4\pi}{3} = \frac{8\pi}{3} \text{ sq. unit.}$$

29. Let the equation of plane through (3, 4, 2) be

$a(x-3) + b(y-4) + c(z-2) = 0$ ...(*i*)

$\because$ (*i*) passes through (7, 0, 6)

$\therefore$ $a(7-3) + b(0-4) + c(6-2) = 0$

$\Rightarrow$ $4a - 4b + 4c = 0$

$\Rightarrow$ $a - b + c = 0$ ...(*ii*)

Also, since plane (i) is perpendicular to plane $2x - 5y - 15 = 0$

$2a - 5b + 0c = 0$ ...(*iii*)

From (*ii*) and (*iii*) we get

$\frac{a}{5} = \frac{b}{2} = \frac{c}{-3} = \lambda$ (say) $\Rightarrow$ $a = 5\lambda$, $b = 2\lambda$, $c = -3\lambda$.

Putting the value of a, b, c in (*i*) we get

$5\lambda(x-3) + 2\lambda(y-4) - 3\lambda(z-2) = 0$

$\Rightarrow$ $5x - 15 + 2y - 8 - 3z + 6 = 0$

$\Rightarrow$ $5x + 2y - 3z = 17$

$\therefore$ Required vector equation of plane is $\vec{r}\cdot(5\hat{i} + 2\hat{j} - 3\hat{k}) = 17$ ...(*iv*)

Obviously plane (*iv*) contains the line

$\vec{r} = (\hat{i} + 3\hat{j} - 2\hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ ...(*v*)

if point $(\hat{i} + 3\hat{j} - 2\hat{k})$ satisfy the equation (*iv*) and vector $(5\hat{i} + 2\hat{j} - 3\hat{k})$ is perpendicular to $(\hat{i} - \hat{j} + \hat{k})$.

Here $(\hat{i} + 3\hat{j} - 2\hat{k})\cdot(5\hat{i} + 2\hat{j} - 3\hat{k}) = 5 + 6 + 6 = 17$

Also $(5\hat{i} + 2\hat{j} - 3\hat{k})\cdot(\hat{i} - \hat{j} + \hat{k}) = 5 - 2 - 3 = 0$

Therefore (*iv*) contains line (*v*).

Set–III

9. Let e be the identity for $*$ in Z.

$\therefore \quad a * e = a \qquad \forall\, a \in Z$

$\Rightarrow \quad a + e + 2 = a$

$\Rightarrow \quad e = a - a - 2$

$\Rightarrow \quad e = -2$

19. Given

$$x^{16}y^9 = (x^2 + y)^{17}$$

Taking logarithm of both sides, we get

$$\log(x^{16}y^9) = \log(x^2 + y)^{17}$$

$$\Rightarrow \quad \log x^{16} + \log y^9 = 17\log(x^2 + y)$$

$$\Rightarrow \quad 16\log x + 9\log y = 17\log(x^2 + y)$$

Differentiating both sides w.r.t. x, we get

$$\Rightarrow \quad \frac{16}{x} + \frac{9}{y}\cdot\frac{dy}{dx} = \frac{17}{x^2 + y}\left(2x + \frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{16}{x} + \frac{9}{y}\cdot\frac{dy}{dx} = \frac{34x}{x^2 + y} + \frac{17}{x^2 + y}\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad \left(\frac{9}{y} - \frac{17}{x^2 + y}\right)\frac{dy}{dx} = \frac{34x}{x^2 + y} - \frac{16}{x}$$

$$\Rightarrow \quad \left(\frac{9x^2 + 9y - 17y}{y(x^2 + y)}\right)\cdot\frac{dy}{dx} = \frac{34x^2 - 16x^2 - 16y}{x(x^2 + y)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{18x^2 - 16y}{x(x^2 + y)} \times \frac{y(x^2 + y)}{9x^2 - 8y} = \frac{2(9x^2 - 8y).y}{x(9x^2 - 8y)} = \frac{2y}{x}$$

20. Given $\quad (x^2 - yx^2)dy + (y^2 + x^2y^2)dx = 0$

$$\Rightarrow \quad x^2(1 - y)dy + y^2(1 + x^2)dx = 0$$

$$\Rightarrow \quad \frac{(1 - y).dy}{y^2} = \left(\frac{1 + x^2}{x^2}\right)dx$$

Integrating both sides we get

$$\Rightarrow \quad \int \frac{1 - y}{y^2}\,dy = \int \frac{1 + x^2}{x^2}\,dx$$

$$\Rightarrow \quad \int \frac{1}{y^2}\,dy - \int \frac{y}{y^2}\,dy = \int \frac{1}{x^2}\,dx + \int dx$$

$$\Rightarrow \quad \int y^{-2}dy - \int \frac{1}{y}\,dy = \int x^{-2}dx + \int dx$$

$$\Rightarrow \quad \frac{y^{-2+1}}{-2+1} - \log y = \frac{x^{-2+1}}{-2+1} + x + c$$

$$\Rightarrow \quad -\frac{1}{y} - \log y = -\frac{1}{x} + x + c \qquad \ldots(i)$$

Putting $x = 1, y = 1$ we get

$$\Rightarrow \quad -\frac{1}{1} - \log 1 = -\frac{1}{1} + 1 + c$$

$$\Rightarrow \quad -1 - 0 = -1 + 1 + c \quad \Rightarrow \quad c = -1$$

Putting $c = -1$ in (i) we get particular solution

$$-\frac{1}{y} - \log y = -\frac{1}{x} + x - 1$$

$$\Rightarrow \quad \log y = \frac{1}{x} - x + 1 - \frac{1}{y} \quad \Rightarrow \quad \log y = \frac{y - x^2 y + xy - x}{xy}$$

21. Plane determined by the points $A(3, -1, 2)$, $B(5, 2, 4)$ and $C(-1, -1, 6)$ is

$$\begin{vmatrix} x-3 & y+1 & z-2 \\ 5-3 & 2+1 & 4-2 \\ -1-3 & -1+1 & 6-2 \end{vmatrix} = 0 \quad \Rightarrow \quad \begin{vmatrix} x-3 & y+1 & z-2 \\ 2 & 3 & 2 \\ -4 & 0 & 4 \end{vmatrix} = 0$$

$$\Rightarrow \quad (x-3)\begin{vmatrix} 3 & 2 \\ 0 & 4 \end{vmatrix} - (y+1)\begin{vmatrix} 2 & 2 \\ -4 & 4 \end{vmatrix} + (z-2)\begin{vmatrix} 2 & 3 \\ -4 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \quad 12x - 36 - 16y - 16 + 12z - 24 = 0$$

$$\Rightarrow \quad 3x - 4y + 3z - 19 = 0$$

Distance of this plane from point $P(6, 5, 9)$ is

$$\left| \frac{(3 \times 6) - (4 \times 5) + (3 \times 9) - 19}{\sqrt{(3)^2 + (4)^2 + (3)^2}} \right| = \left| \frac{18 - 20 + 27 - 19}{\sqrt{9 + 16 + 9}} \right| = \frac{6}{\sqrt{34}} \text{ units.}$$

22. Let two adjacent sides of a parallelogram be

$$\vec{a} = 2\hat{i} - 4\hat{j} + 5\hat{k} \text{ and } \vec{b} = \hat{i} - 2\hat{j} - 3\hat{k}$$

$$\text{Now } \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ 1 & -2 & -3 \end{vmatrix} = 22\hat{i} + 11\hat{j}$$

$$\Rightarrow \quad \text{Area of given parallelogram} = |\vec{a} \times \vec{b}|$$

$$= \sqrt{(22)^2 + (11)^2} = \sqrt{484 + 121} = \sqrt{605}$$

$$= 11\sqrt{5} \text{ square unit.}$$

Let $\vec{a}$ and $\vec{b}$ be represented by $\vec{AB}$ and $\vec{AD}$ respectively.

$$\therefore \quad \vec{BC} = \vec{b}$$

$$\Rightarrow \quad \vec{AC} = \vec{AB} + \vec{BC}$$

$$\Rightarrow \quad \vec{AC} = \vec{a} + \vec{b} = (2\hat{i} - 4\hat{j} + 5\hat{k}) + (\hat{i} - 2\hat{j} - 3\hat{k}) = 3\hat{i} - 6\hat{j} + 2\hat{k}$$

Also $|\vec{AC}| = \sqrt{3^2 + (-6)^2 + 2^2}$

$$= \sqrt{9 + 36 + 4} = \sqrt{49} = 7$$

$\therefore$ Required unit vector parallel to one diagonal is

$$= \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k})$$

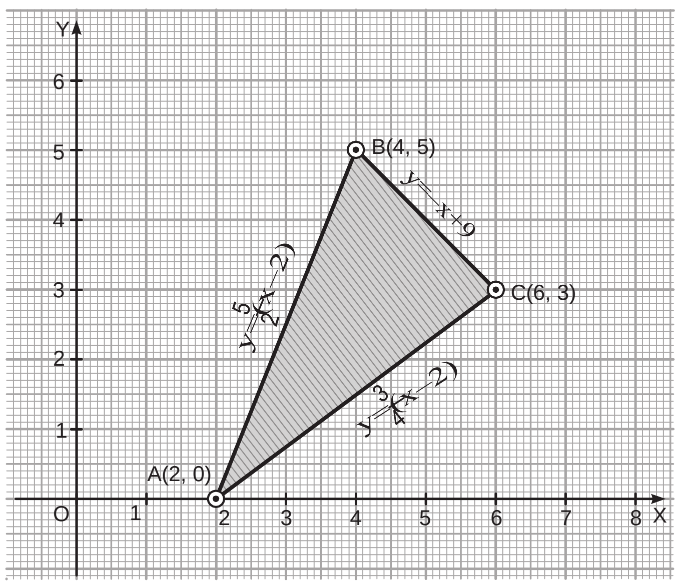
SECTION C

28. Vertices of ΔABC are A (2, 0), B (4, 5), C (6, 3).

Equation of line AB is

$$\frac{y-0}{x-2} = \frac{5-0}{4-2} \quad \Rightarrow \quad \frac{y}{x-2} = \frac{5}{2}$$

$$\Rightarrow \quad y = \frac{5}{2}(x-2) \qquad \ldots(i)$$



Equation of line BC is

$$\frac{y-5}{x-4} = \frac{3-5}{6-4} \quad \Rightarrow \quad y - 5 = \frac{-2}{2}(x-4)$$

$$\Rightarrow \quad y = -x + 4 + 5$$

$$\Rightarrow \quad y = -x + 9 \qquad \ldots(ii)$$

Equation of line AC

$$\frac{y-0}{x-2} = \frac{3-0}{6-2} \quad \Rightarrow \quad \frac{y}{x-2} = \frac{3}{4}$$

$$\Rightarrow \quad y = \frac{3}{4}(x-2) \qquad \ldots(iii)$$

Now Area of ΔABC = Area of region bounded by line (i), (ii) and (iii)

$$= \int_2^4 \frac{5}{2}(x-2)\,dx + \int_4^6 (-x+9)\,dx - \int_2^6 \frac{3}{4}(x-2)\,dx$$

$$= \frac{5}{2}\left[\frac{(x-2)^2}{2}\right]_2^4 - \left[\frac{(x-9)^2}{2}\right]_4^6 - \frac{3}{4}\left[\frac{(x-2)^2}{2}\right]_2^6$$

$$= \frac{5}{4}(4-0) - \frac{1}{2}(9-25) - \frac{3}{8}(16-0)$$

$$= 5 + 8 - 6 \quad = 7 \text{ sq. unit}$$

29. Let the equation of plane through (2, 2, 1) be

$$a(x-2) + b(y-2) + c(z-1) = 0 \qquad ...(i)$$

Q (*i*) passes through (9, 3, 6)

$\therefore$ $a(9-2) + b(3-2) + c(6-1) = 0$

$\Rightarrow$ $7a + b + 5c = 0 \qquad ...(ii)$

Also since plane (*i*) is perpendicular to plane $2x + 6y + 6z = 1$

$$2a + 6b + 6c = 0 \qquad ...(iii)$$

From (*ii*) and (*iii*)

$$\frac{a}{6-30} = \frac{b}{10-42} = \frac{c}{42-2}$$

$\Rightarrow$ $\dfrac{a}{-24} = \dfrac{b}{-32} = \dfrac{c}{40}$

$\Rightarrow$ $\dfrac{a}{-3} = \dfrac{b}{-4} = \dfrac{c}{5} = \mu$ (say)

$\Rightarrow$ $a = -3\mu,\ b = -4\mu,\ c = 5\mu$

Putting the value of *a, b, c* in (i) we get

$$-3\mu(x-2) - 4\mu(y-2) + 5\mu(z-1) = 0$$

$\Rightarrow$ $-3x + 6 - 4y + 8 + 5z - 5 = 0$

$\Rightarrow$ $-3x - 4y + 5z = -9$

It is required equation of plane.

Its vector form is

$$\vec{r}\cdot(-3\hat{i} - 4\hat{j} + 5\hat{k}) = -9 \qquad ...(iv)$$

Obviously, plane (iv) contains the line

$$\vec{r} = (4\hat{i} + 3\hat{j} + 3\hat{k}) + \lambda(7\hat{i} + \hat{j} + 5\hat{k}) \qquad ...(v)$$

if point $(4\hat{i} + 3\hat{j} + 3\hat{k})$ satisfy equation (*iv*) and vector $(7\hat{i} + \hat{j} + 5\hat{k})$ is perpendicular to $-3\hat{i} - 4\hat{j} + 5\hat{k}$.

Here $(4\hat{i} + 3\hat{j} + 3\hat{k})\cdot(-3\hat{i} - 4\hat{j} + 5\hat{k}) = -12 - 12 + 15 = -9$

Also $(7\hat{i} + \hat{j} + 5\hat{k})\cdot(-3\hat{i} - 4\hat{j} + 5\hat{k}) = -21 - 4 + 25 = 0$

Therefore plane (*iv*) contains line (*v*).

CBSE Examination Papers (Delhi–2013)

Time allowed: 3 hours **Maximum marks: 100**

General Instructions: *As given in CBSE Sample Question Paper.*

Set–I

SECTION–A

Question numbers 1 to 10 carry 1 mark each.

1. Write the principal value of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right)$.

2. Write the value of $\tan\left(2\tan^{-1}\frac{1}{5}\right)$.

3. Find the value of a if $\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$

4. If $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$, then write the value of x.

5. If $\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$, then find the matrix A.

6. Write the degree of the differential equation $x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0$.

7. If $\vec{a} = x\hat{i} + 2\hat{j} - z\hat{k}$ and $\vec{b} = 3\hat{i} - y\hat{j} + \hat{k}$ are two equal vectors, then write the value of $x + y + z$.

8. If a unit vector $\vec{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find the value of θ.

9. Find the Cartesian equation of the line which passes through the point $(-2, 4, -5)$ and is parallel to the line $\frac{x+3}{3} = \frac{4-y}{5} = \frac{z+8}{6}$.

10. The amount of pollution content added in air in a city due to x-diesel vehicles is given by $P(x) = 0.005x^3 + 0.02x^2 + 30x$. Find the marginal increase in pollution content when 3 diesel vehicles are added and write which value is indicated in the above question.

SECTION–B

Question numbers 11 to 22 carry 4 marks each.

11. Show that the function f in $A = \mathbb{R} - \left\{\frac{2}{3}\right\}$ defined as $f(x) = \frac{4x+3}{6x-4}$ is one-one and onto. Hence find f^{-1}.

12. Find the value of the following:

$$\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right], \ |x|<1, y>0 \text{ and } xy<1.$$

OR

Prove that: $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$.

13. Using properties of determinants, prove the following:

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2.$$

14. Differentiate the following function with respect to x:

$$(\log x)^x + x^{\log x}.$$

15. If $y = \log\left[x + \sqrt{x^2+a^2}\right]$, show that $(x^2+a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$.

16. Show that the function $f(x) = |x-3|, x \in \mathbb{R}$, is continuous but not differentiable at $x = 3$.

OR

If $x = a\sin t$ and $y = a\left(\cos t + \log\tan\frac{t}{2}\right)$, find $\frac{d^2y}{dx^2}$.

17. Evaluate: $\int \frac{\sin(x-a)}{\sin(x+a)}\,dx$

OR

Evaluate: $\int \frac{5x-2}{1+2x+3x^2}\,dx$

18. Evaluate : $\int \frac{x^2}{(x^2+4)(x^2+9)}\,dx$

19. Evaluate : $\int_0^4 (|x| + |x-2| + |x-4|)\,dx$

20. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}+\vec{b}| = |\vec{a}|$, then prove that vector $2\vec{a}+\vec{b}$ is perpendicular to vector $\vec{b}$.

21. Find the coordinates of the point, where the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{2}$ intersects the plane $x-y+z-5=0$. Also find the angle between the line and the plane.

OR

Find the vector equation of the plane which contains the line of intersection of the planes $\vec{r}.(\hat{i}+2\hat{j}+3\hat{k})-4=0$ and $\vec{r}.(2\hat{i}+\hat{j}-\hat{k})+5=0$ and which is perpendicular to the plane $\vec{r}.(5\hat{i}+3\hat{j}-6\hat{k})+8=0$.

22. ***A speaks truth in 60% of the cases, while B in 90% of the cases. In what percent of cases are they likely to contradict each other in stating the same fact? In the cases of contradiction do you think, the statement of B will carry more weight as he speaks truth in more number of cases than A?***

SECTION–C

Question numbers 23 to 29 carry 6 marks each.

23. ***A school wants to award its students for the values of Honesty, Regularity and Hard work with a total cash award of ` 6,000. Three times the award money for Hardwork added to that given for honesty amounts to ` 11,000. The award money given for Honesty and Hardwork together is double the one given for Regularity. Represent the above situation algebraically and find the award money for each value, using matrix method. Apart from these values, namely, Honesty, Regularity and Hardwork, suggest one more value which the school must include for awards.***

24. Show that the height of the cylinder of maximum volume, that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

OR

Find the equation of the normal at a point on the curve $x^2=4y$ which passes through the point (1, 2). Also find the equation of the corresponding tangent.

25. Using integration, find the area bounded by the curve $x^2=4y$ and the line $x=4y-2$.

OR

Using integration, find the area of the region enclosed between the two circles $x^2+y^2=4$ and $(x-2)^2+y^2=4$.

26. Show that the differential equation $2ye^{x/y}\,dx+(y-2xe^{x/y})\,dy=0$ is homogeneous. Find the particular solution of this differential equation, given that $x=0$ when $y=1$.

27. Find the vector equation of the plane passing through three points with position vectors $\hat{i}+\hat{j}-2\hat{k}$, $2\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$. Also find the coordinates of the point of intersection of this plane and the line $\vec{r}=3\hat{i}-\hat{j}-\hat{k}+\lambda(2\hat{i}-2\hat{j}+\hat{k})$.

28. ***A cooperative society of farmers has 50 hectares of land to grow two crops A and B. The profits from crops A and B per hectare are estimated as `10,500 and `9,000 respectively. To control weeds, a liquid herbicide has to be used for crops A and B at the rate of 20 litres and***

10 litres per hectare, respectively. Further not more than 800 litres of herbicide should be used in order to protect fish and wildlife using a pond which collects drainage from this

land. Keeping in mind that the protection of fish and other wildlife is more important than earning profit, how much land should be allocated to each crop so as to maximize the total profit? Form an LPP from the above and solve it graphically. Do you agree with the message that the protection of wildlife is utmost necessary to preserve the balance in environment?

29. ***Assume that the chances of a patient having a heart attack is 40%. Assuming that a meditation and yoga course reduces the risk of heart attack by 30% and prescription of certain drug reduces its chance by 25%. At a time a patient can choose any one of the two options with equal probabilities. It is given that after going through one of the two options, the patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga. Interpret the result and state which of the above stated methods is more beneficial for the patient.***

Set–II

Only those questions, not included in Set I, are given.

9. Write the degree of the differential equation $\left(\frac{dy}{dx}\right)^4 + 3x\frac{d^2y}{dx^2} = 0.$

16. P, speaks truth in 70% of the cases and Q in 80% of the cases. In what percent of cases are they likely to agree in stating the same fact? Do you think when they agree, means both are speaking truth?

18. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$, find a vector $\vec{c}$, such that $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a} \cdot \vec{c} = 3$.

19. Evaluate: $\int_1^4 [|x-1| + |x-2| + |x-3|]\,dx.$

20. Evaluate: $\int \frac{x^2+1}{(x^2+4)(x^2+25)}\,dx.$

28. Show that the differential equation $x\frac{dy}{dx}\sin\left(\frac{y}{x}\right) + x - y\sin\left(\frac{y}{x}\right) = 0$ is homogeneous. Find the particular solution of this differential equation, given that $x = 1$ when $y = \frac{\pi}{2}$.

29. Find the vector equation of the plane determined by the points A (3, –1, 2), B (5, 2, 4) and C (–1, –1, 6). Also find the distance of point P (6, 5, 9) from this plane.

Set–III

Only those questions, not included in Set I and Set II, are given.

2. Write a unit vector in the direction of the sum of vectors $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} + 3\hat{k}$.

4. Write the degree of the differential equation $x\left(\frac{d^2y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^4 + x^3 = 0.$

11. ***A speaks truth in 75% of the cases, while B in 90% of the cases. In what percent of cases are they likely to contradict each other in stating the same fact?***

Do you think that statement of B is true?

13. Using vectors, find the area of the triangle ABC with vertices A (1, 2, 3), B (2, –1, 4) and C (4, 5, –1).

14. Evaluate: $\int_2^5 [\,|x-2|+|x-3|+|x-5|\,]\,dx$.

15. Evaluate: $\int \frac{2x^2+1}{x^2(x^2+4)}\,dx$.

25. Find the coordinate of the point where the line through (3, –4, –5) and (2, –3, 1) crosses the plane, passing through the points (2, 2, 1), (3, 0, 1) and (4, –1, 0).

26. Show that the differential equation $(x\,e^{y/x} + y)\,dx = x\,dy$ is homogeneous. Find the particular solution of this differential equation, given that $x = 1$ when $y = 1$.

Solutions

Set–I

SECTION–A

Question numbers 1 to 10 carry 1 mark each.

1. $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)=\tan^{-1}\left(\tan\frac{\pi}{4}\right)+\cos^{-1}\left(\cos\left(\pi-\frac{\pi}{3}\right)\right)$

$$=\tan^{-1}\left(\tan\frac{\pi}{4}\right)+\cos^{-1}\left(\cos\frac{2\pi}{3}\right)\qquad \left[\because \frac{\pi}{4}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \text{ and } \frac{2\pi}{3}\in[0,\pi]\right]$$

$$=\frac{\pi}{4}+\frac{2\pi}{3}=\frac{3\pi+8\pi}{12}=\frac{11\pi}{12}$$

2. Let $2\tan^{-1}\frac{1}{5}=\theta$

$\Rightarrow \quad \tan^{-1}\frac{1}{5}=\frac{\theta}{2} \qquad \Rightarrow \quad \tan\frac{\theta}{2}=\frac{1}{5}$

Now, $\tan\left(2\tan^{-1}\frac{1}{5}\right)=\tan\theta=\dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

$$=\frac{2\times\frac{1}{5}}{1-\left(\frac{1}{5}\right)^2}=\frac{2}{5}\times\frac{25}{24}=\frac{5}{12}$$

3. Given: $\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d\end{bmatrix}=\begin{bmatrix}-1 & 5\\ 0 & 13\end{bmatrix}$

$\Rightarrow$
$$a-b=-1 \qquad \ldots(i)$$
$$2a+c=5 \qquad \ldots(ii)$$
$$2a-b=0 \qquad \ldots(iii)$$
$$3c+d=13 \qquad \ldots(iv)$$

From (iii) $2a=b \Rightarrow a=\frac{b}{2}$

Putting in (i) we get $\frac{b}{2}-b=-1$

$\Rightarrow \quad \frac{b}{2}=1 \Rightarrow b=2$

$\therefore \quad a=1$

(*ii*) $\Rightarrow$ $c = 5 - 2 \times 1 = 5 - 2 = 3$

(*iv*) $\Rightarrow$ $d = 13 - 3 \times (3) = 13 - 9 = 4$

i.e. $a = 1,\ b = 2, c = 3,\ d = 4$

4. Given $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$

$$\Rightarrow \quad (x+1)(x+2)-(x-1)(x-3)=12+1$$
$$\Rightarrow \quad x^2+2x+x+2-x^2+3x+x-3=13$$
$$\Rightarrow \quad 7x-1=13$$
$$\Rightarrow \quad 7x=14$$
$$\Rightarrow \quad x=2$$

5. Given $\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$

$$\Rightarrow \quad A = \begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix} = \begin{bmatrix} 8 & -3 & 5 \\ -2 & -3 & -6 \end{bmatrix}$$

6. Degree = 2

7. $\because \quad \vec{a} = \vec{b}$

$$x\hat{i}+2\hat{j}-z\hat{k} = 3\hat{i}-y\hat{j}+\hat{k}$$

Equating, we get, $x=3$,

$$-y=2 \Rightarrow y=-2$$
$$-z=1 \Rightarrow z=-1$$
$$\therefore \quad x+y+z=3-2-1=0$$

8. Let l, m, n be Direction cosines of $\vec{a}$

$$\therefore \quad l=\cos\frac{\pi}{3}=\frac{1}{2}; \quad m=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}; \quad n=\cos\theta$$
$$\because \quad l^2+m^2+n^2=1$$
$$\Rightarrow \quad \left(\frac{1}{2}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2+\cos^2\theta=1$$
$$\Rightarrow \quad \frac{1}{4}+\frac{1}{2}+\cos^2\theta=1$$
$$\Rightarrow \quad \cos^2\theta=1-\left(\frac{1}{4}+\frac{1}{2}\right)=1-\frac{3}{4}=\frac{1}{4}$$
$$\Rightarrow \quad \cos\theta=\frac{1}{2} \quad \Rightarrow \quad \theta=\frac{\pi}{3}$$

9. Equation of required line is

$$\frac{x-(-2)}{3}=\frac{y-4}{-5}=\frac{z-(-5)}{6}$$ [$\because$ Direction ratios of given line are 3, – 5, 6]

$$\Rightarrow \quad \frac{x+2}{3}=\frac{y-4}{-5}=\frac{z+5}{6}$$

10. We have to find $[P'(x)]_{x=3}$

Now, $P(x) = 0.005x^3 + 0.02x^2 + 30x$

$\therefore \quad P'(x) = 0.015x^2 + 0.04x + 30$

$\Rightarrow \quad [P'(x)]_{x=3} = 0.015 \times 9 + 0.04 \times 3 + 30$

$\qquad\qquad = 0.135 + 0.12 + 30 = 30.255$

This question indicates "how increase in number of diesel vehicles increase the air pollution, which is harmful for living body."

SECTION–B

11. Let $x_1, x_2 \in A$

Now $f(x_1) = f(x_2) \Rightarrow \dfrac{4x_1 + 3}{6x_1 - 4} = \dfrac{4x_2 + 3}{6x_2 - 4}$

$$\Rightarrow 24x_1 x_2 + 18x_2 - 16x_1 - 12 = 24x_1 x_2 + 18x_1 - 16x_2 - 12$$

$$\Rightarrow -34x_1 = -34x_2 \quad \Rightarrow \quad x_1 = x_2$$

Hence f is one-one function

For onto

Let $y = \dfrac{4x+3}{6x-4} \qquad \Rightarrow \qquad 6xy - 4y = 4x + 3$

$\Rightarrow \quad 6xy - 4x = 4y + 3 \qquad \Rightarrow \qquad x(6y - 4) = 4y + 3$

$\Rightarrow \quad x = \dfrac{4y+3}{6y-4}$

$\Rightarrow \quad \forall\ y \in \text{codomain}\ \exists x \in \text{Domain} \left[Q \;\; x \neq \dfrac{2}{3}\right] \Rightarrow \quad f$ in onto function.

Thus f is one-one onto function.

Also, $\qquad f^{-1}(x) = \dfrac{4x+3}{6x-4}$

12. $\tan \dfrac{1}{2}\left[\sin^{-1} \dfrac{2x}{1+x^2} + \cos^{-1} \dfrac{1-y^2}{1+y^2}\right]$

$$= \tan \frac{1}{2}[2\tan^{-1} x + 2\tan^{-1} y]$$

$$= \tan(\tan^{-1} x + \tan^{-1} y)$$

$$= \tan\left(\tan^{-1} \frac{x+y}{1-xy}\right) = \frac{x+y}{1-xy}$$

$\left[\textbf{Note: } \sin^{-1} \dfrac{2x}{1+x^2} = 2\tan^{-1} x = \cos^{-1} \dfrac{1-x^2}{1+x^2}\right]$

OR

Refer to Q. No. 17 page -47.

13. Refer to Q. No. 4 page -100.

14. Let $y = (\log x)^x + x^{\log x}$

$\Rightarrow \quad y = u + v$ where $u = (\log x)^x$, $v = x^{\log x}$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \ldots(i)$$

Now $u = (\log x)^x$

Taking logarithm of both sides, we get

$$\log u = x . \log(\log x)$$

Differentiating both sides w.r.t.x, we get

$$\frac{1}{u}.\frac{du}{dx} = x.\frac{1}{\log x}.\frac{1}{x} + \log(\log x) \qquad \Rightarrow \qquad \frac{du}{dx} = u\left\{\frac{1}{\log x} + \log(\log x)\right\}$$

$$\Rightarrow \quad \frac{du}{dx} = (\log x)^x \left\{\frac{1}{\log x} + \log(\log x)\right\} \qquad \ldots(ii)$$

Again $\quad v = x^{\log x}$

Taking logarithm of both sides , we get

$$\log v = \log x^{\log x}$$

$$\Rightarrow \quad \log v = \log x . \log x \qquad \Rightarrow \qquad \log v = (\log x)^2$$

Differentiating both sides w.r.t.x, we get

$$\frac{1}{v}\frac{dv}{dx} = 2\log x . \frac{1}{x}$$

$$\Rightarrow \quad \frac{dv}{dx} = 2x^{\log x}.\frac{\log x}{x} \qquad \ldots(iii)$$

Putting $\frac{du}{dx}$ and $\frac{dv}{dx}$ from (ii) and (iii) in (i) we get

$$\frac{dy}{dx} = (\log x)^x \left\{\frac{1}{\log x} + \log(\log x)\right\} + 2\frac{\log x . x^{\log x}}{x}$$

15. Given $y = \log\left[x + \sqrt{x^2 + a^2}\right]$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + a^2}} \times \left[1 + \frac{2x}{2\sqrt{x^2 + a^2}}\right] \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{x + \sqrt{x^2 + a^2}}{\left(x + \sqrt{x^2 + a^2}\right)\left(\sqrt{x^2 + a^2}\right)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{x^2 + a^2}} \qquad \ldots(i)$$

Differentiating again w.r.t. x we get

$$\frac{d^2y}{dx^2} = -\frac{1}{2}(x^2 + a^2)^{-\frac{3}{2}}.2x = \frac{-x}{(x^2 + a^2)^{\frac{3}{2}}}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} \quad \frac{-x}{(x^2+a^2).\quad x^2+a^2} \quad \Rightarrow \quad (x^2+a^2)\frac{d\quad y}{dx^2} = -\frac{x}{x^2+a^2}$$

$$\Rightarrow \quad (x^2 + a^2) \frac{d^2 y}{dx^2} + x . \frac{dy}{dx} = 0 \qquad \text{[from (i)]}$$

16. Here, $f(x) = |x - 3|$

$$f(x) = \begin{cases} -(x-3) & , x < 3 \\ 0 & , x = 3 \\ (x-3) & , x > 3 \end{cases}$$

Now, $\lim\limits_{x \to 3^+} f(x) = \lim\limits_{h \to 0} f(3 + h)$ [Let $x = 3 + h$ and $x \to 3^+ \Rightarrow h \to 0$]

$$= \lim_{h \to 0} (3 + h - 3) = \lim_{h \to 0} h = 0$$

$$\lim_{x \to 3^+} f(x) = 0 \qquad ...(i)$$

$$\lim_{x \to 3^-} f(x) = \lim_{h \to 0} f(3 - h) \qquad \text{[Let } x = 3 - h \text{ and } x \to 3^- \Rightarrow h \to 0]$$

$$= \lim_{h \to 0} -(3 - h - 3) = \lim_{h \to 0} h = 0$$

$$\lim_{x \to 3^+} f(x) = 0 \qquad ...(ii)$$

Also, $f(3) = 0$ ...(*iii*)

From equation (*i*), (*ii*) and (*iii*)

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(x) = f(3)$$

Hence, $f(x)$ is continuous at $x = 3$

At $x = 3$

$$\text{RHD} = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{(3 + h - 3) - 0}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} \qquad [\because |h| = h, |0| = 0]$$

$$= \lim_{h \to 0} 1$$

$$\text{RHD} = 1 \qquad ...(iv)$$

$$\text{LHD} = \lim_{h \to 0} \frac{f(3-h) - f(3)}{-h} = \lim_{h \to 0} \frac{-(3 - h - 3) - 0}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h} \qquad [\because |h| = h]$$

$$= \lim_{h \to 0} (-1)$$

$$\text{LHD} = -1 \qquad ...(v)$$

Equation (*iv*) and (*v*) $\Rightarrow$ RHD $\neq$ LHD at $x = 3$.

Hence $f(x)$ is not differentiable at $x = 3$

Therefore, $f(x) = |x - 3|, x \in R$ is continuous but not differentiable at $x = 3$.

OR

Here, $x = a\sin t$, $y = a\left[\cos t + \log\left(\tan\frac{t}{2}\right)\right]$

∵ $x = a\sin t$

Differentiating both sides w.r.t. t, we get

$$\frac{dx}{dt} = a\cos t \qquad ...(i)$$

Again, ∵ $y = a\left[\cos t + \log\left(\tan\frac{t}{2}\right)\right]$

Differentiating both sides w.r.t. t, we get

$$\frac{dy}{dt} = a\left[-\sin t + \frac{1}{\tan\frac{t}{2}}.\sec^2\frac{t}{2}.\frac{1}{2}\right]$$

$$\Rightarrow \quad \frac{dy}{dt} = a\left[-\sin t + \frac{1}{\sin t}\right]$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{a(1-\sin^2 t)}{\sin t}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{a\cos^2 t}{\sin t} \qquad ...(ii)$$

$$\because \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{a\cos^2 t}{\sin t} \times \frac{1}{a\cos t} \qquad \text{[From } (i) \text{ and } (ii)\text{]}$$

$$\Rightarrow \quad \frac{dy}{dx} = \cot t$$

Differentiating again w.r.t. x we get

$$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2 t.\frac{dt}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\operatorname{cosec}^2 t.\frac{1}{a\cos t} = \frac{-\operatorname{cosec}^2 t}{a\cos t}$$

17. Let $I = \int \frac{\sin(x-a)}{\sin(x+a)}\,dx$

Let $x + a = t \Rightarrow x = t - a$

$\Rightarrow \quad dx = dt$

$$\therefore \quad I = \int \frac{\sin(t-2a)}{\sin t}\,dt$$

$$= \int \frac{\sin t.\cos 2a - \cos t.\sin 2a}{\sin t}\,dt$$

$\sin t$

$= \cos 2a \int dt - \int \sin 2a \, . \cot t \, dt = \cos 2a$

$.t - \sin 2a . \log|\sin t| + C = \cos 2a . (x + a) - \sin 2a . \log|\sin (x + a)| + C$

$= x \cos 2a + a \cos 2a - (\sin 2a) \log|\sin (x + a)| + C$

OR

Refer to Q. No. 10 page 282.

18. Let $I = \int \frac{x^2}{(x^2+4)(x^2+9)}\,dx$

Let $x^2 = t$

$$\therefore \quad \frac{x^2}{(x^2+4)(x^2+9)} = \frac{t}{(t+4)(t+9)}$$

Now $\frac{t}{(t+4)(t+9)} = \frac{A}{t+4} + \frac{B}{t+9} = \frac{A(t+9)+B(t+4)}{(t+4)(t+9)}$

$\Rightarrow \quad t = (A+B)t + (9A+4B)$

Equating we get

$A + B = 1, \quad 9A + 4B = 0$

Solving above two equations, we get

$$A = -\frac{4}{5}, \; B = \frac{9}{5}$$

$$\therefore \quad \frac{x^2}{(x^2+4)(x^2+9)} = -\frac{4}{5(x^2+4)} + \frac{9}{5(x^2+9)}$$

$$I = -\frac{4}{5}\int \frac{dx}{x^2+2^2} + \frac{9}{5}\int \frac{dx}{x^2+3^2}$$

$$= -\frac{4}{5} \times \frac{1}{2}\tan^{-1}\frac{x}{2} + \frac{9}{5} \times \frac{1}{3}\tan^{-1}\frac{x}{3} + C$$

$$= -\frac{2}{5}\tan^{-1}\frac{x}{2} + \frac{3}{5}\tan^{-1}\frac{x}{3} + C$$

19. Let $I = \int_0^4 (|x| + |x-2| + |x-4|)\,dx$

$$= \int_0^4 |x|\,dx + \int_0^4 |x-2|\,dx + \int_0^4 |x-4|\,dx$$

$$= \int_0^4 |x|\,dx + \left[\int_0^2 |x-2|\,dx + \int_2^4 |x-2|\,dx\right] + \int_0^4 |x-4|\,dx \qquad \text{[By properties]}$$

$$= \int_0^4 x\,dx + \int_0^2 -(x-2)\,dx + \int_2^4 (x-2)\,dx + \int_0^4 -(x-4)\,dx$$

$$\left[\begin{array}{l} \because \; |x| = x, \text{ if } 0 \le x \le 4 \\ \quad |x-2| = -(x-2), \text{ if } 0 \le x \le 2 \\ \quad |x-2| = (x-2), \text{ if } 2 \le x \le 4 \\ \quad |x-4| = -(x-4), \text{ if } 0 \le x \le 4 \end{array}\right]$$

$$= \left[\frac{x^2}{2}\right]_0^4 - \left[\frac{(x-2)^2}{2}\right]_0^2 + \left[\frac{(x-2)^2}{2}\right]_2^4 - \left[\frac{(x-4)^2}{2}\right]_0^4$$

$$= \frac{}{2} \times 16 - \frac{}{2} \times (0-4) + \frac{}{2}(4-0) - \frac{}{2} \times (0-16)$$

$$= 8 + 2 + 2 + 8 = 20$$

20. Q $\quad |\vec{a}+\vec{b}| = |\vec{a}| \quad \Rightarrow \quad |\vec{a}+\vec{b}|^2 = |\vec{a}|^2$

$\Rightarrow \quad (\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = |\vec{a}|^2$

$\Rightarrow \quad \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b} = |\vec{a}|^2$

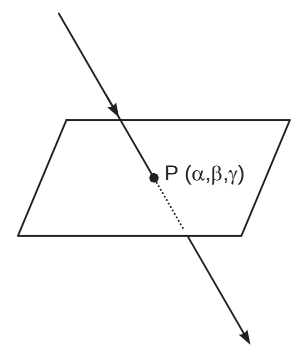
$\Rightarrow \quad |\vec{a}|^2 + 2\,\vec{a}.\vec{b} + \vec{b}.\vec{b} = |\vec{a}|^2 \qquad$ [Q $\vec{a}.\vec{b} = \vec{b}.\vec{a}$]

$\Rightarrow \quad 2\,\vec{a}.\vec{b} + \vec{b}.\vec{b} = 0 \quad \Rightarrow \quad (2\vec{a}+\vec{b}).\vec{b} = 0$

$\Rightarrow \quad (2\vec{a}+\vec{b})$ is perpendicular to $\vec{b}$.

21. Let the given line

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2} \qquad \ldots(i)$$



intersect the plane $x - y + z - 5 = 0$ (ii) at point $P(\alpha, \beta, \gamma)$

Q $\quad P(\alpha, \beta, \gamma)$ lie on line (i)

$\therefore \quad \dfrac{\alpha-2}{3} = \dfrac{\beta+1}{4} = \dfrac{\gamma-2}{2} = \lambda$ (say)

$\alpha = 3\lambda + 2; \quad \beta = 4\lambda - 1; \quad \gamma = 2\lambda + 2$

Also $P(\alpha, \beta, \gamma)$ lies on plane (ii)

$\therefore \quad (3\lambda+2) - (4\lambda-1) + (2\lambda+2) - 5 = 0$

$\Rightarrow \quad 3\lambda + 2 - 4\lambda + 1 + 2\lambda + 2 - 5 = 0$

$\Rightarrow \quad \lambda = 0$

$\therefore \quad \alpha = 2, \; \beta = -1, \gamma = 2$

Hence, co-ordinate of required point = (2, – 1, 2)

Now to find angle between line (i) and plane (ii)

If θ be the required angle, then

$$\sin\theta = \left|\frac{\vec{b}.\vec{n}}{|\vec{b}|.|\vec{n}|}\right|$$

$$\therefore \quad \sin\theta = \left|\frac{1}{\sqrt{9+16+4}\,.\sqrt{1^2+(-1)^2+1^2}}\right| = \left|\frac{1}{\sqrt{29}\,.\sqrt{3}}\right|$$

$\left[\begin{array}{l} \text{Q } \vec{b} = 3\hat{i} + 4\hat{j} + 2\hat{k} \\ \quad \vec{n} = \hat{i} - \hat{j} + \hat{k} \\ \therefore \vec{b}.\vec{n} = 3 - 4 + 2 = 1 \end{array}\right]$

$$\sin\theta = \frac{1}{\sqrt{87}} \quad \Rightarrow \quad \theta = \sin^{-1}\left(\frac{1}{\sqrt{87}}\right)$$

OR

Refer to Q. No. 4 page 451.

22. Refer to Q. No. 6 page 500.

Yes, the statement of B will carry more weight as the probability of B to speak truth is more than that of A.

SECTION–C

23. Let x, y and z be the awarded money for honesty, Regularity and hardwork.

From question

$$x+y+z=6000 \qquad \ldots(i)$$
$$x+3z=11000 \qquad \ldots(ii)$$
$$x+z=2y \Rightarrow x-2y+z=0 \qquad \ldots(iii)$$

The above system of three equations may be written in matrix form as

$$AX=B,$$

where $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 0 & 3\\ 1 & -2 & 1\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$, $B=\begin{bmatrix}6000\\ 11000\\ 0\end{bmatrix}$

Now $|A|=\begin{vmatrix}1 & 1 & 1\\ 1 & 0 & 3\\ 1 & -2 & 1\end{vmatrix}=1(0+6)-1(1-3)+1(-2-0)$

$$=6+2-2=6\neq 0$$

Hence A^{-1} exist

If A_{ij} is co-factor of a_{ij} then

$$A_{11}=(-1)^{1+1}\begin{vmatrix}0 & 3\\ -2 & 1\end{vmatrix}=0+6=6$$

$$A_{12}=(-1)^{1+2}\begin{vmatrix}1 & 3\\ 1 & 1\end{vmatrix}=-(1-3)=2; \qquad A_{13}=(-1)^{1+3}\begin{vmatrix}1 & 0\\ 1 & -2\end{vmatrix}=(-2-0)=-2$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix}1 & 1\\ -2 & 1\end{vmatrix}=-(1+2)=-3; \qquad A_{22}=(-1)^{2+2}\begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix}=0$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix}1 & 1\\ 1 & -2\end{vmatrix}=(-2-1)=3; \qquad A_{31}=(-1)^{3+1}\begin{vmatrix}1 & 1\\ 0 & 3\end{vmatrix}=3-0=3$$

$$A_{32}=(-1)^{3+2}\begin{vmatrix}1 & 1\\ 1 & 3\end{vmatrix}=-(3-1)=-2; \qquad A_{33}=(-1)^{3+3}\begin{vmatrix}1 & 1\\ 1 & 0\end{vmatrix}=0-1=-1$$

$$adj\,A=\begin{bmatrix}6 & 2 & -2\\ -3 & 0 & 3\\ 3 & -2 & -1\end{bmatrix}^T=\begin{bmatrix}6 & -3 & 3\\ 2 & 0 & -2\\ -2 & 3 & -1\end{bmatrix}$$

$\therefore \quad A^{-1}=\frac{1}{|A|}.\text{adj } A=\frac{1}{6}\begin{bmatrix}6 & -3 & 3\\ 2 & 0 & -2\\ -2 & 3 & -1\end{bmatrix}$

Q $\quad AX=B \Rightarrow X=A^{-1}B$

$\Rightarrow \quad \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{6}\begin{bmatrix}6 & -3 & \\ 2 & 0 & \\ -2 & 3 & \end{bmatrix}$

3

6
0
0
0

−

2

1
1
0
0
0

−1 ⌋ ⌊ 0 ⌋

$$= \frac{1}{6}\begin{bmatrix} 36000-33000+0 \\ 12000+0+0 \\ -12000+33000+0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3000 \\ 12000 \\ 21000 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 500 \\ 2000 \\ 3500 \end{bmatrix}$$

$$\Rightarrow \quad x = 500, \; y = 2000, \; z = 3500$$

Except above three values, school must include discipline for award as discipline has great importance in student's life.

24. Refer to page 240 Q. No. 15

OR

Let the point of contact of tangent to the given curve be (x_0, y_0)

Now the given curve is $x^2 = 4y$

$$\Rightarrow \quad 2x = 4\frac{dy}{dx} \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{x}{2}$$

Now slope of tangent to the given curve at $(x_0, y_0) = \left[\frac{dy}{dx}\right]_{(x_0, y_0)} = \frac{x_0}{2}$

$\therefore$ Slope of normal to the given curve at $(x_0, y_0) = -\frac{1}{\text{Slope of tangent at}(x_0 y_0)}$

$$= -\frac{1}{\frac{x_0}{2}} = -\frac{2}{x_0}$$

Hence equation of required normal is

$$(y - y_0) = -\frac{2}{x_0}(x - x_0) \qquad \ldots(i)$$

$\because$ (i) passes through (1, 2)

$$(2 - y_0) = -\frac{2}{x_0}(1 - x_0)$$

$$\Rightarrow \quad 2x_0 - x_0 y_0 = -2 + 2x_0$$

$$\Rightarrow \quad x_0 y_0 = 2 \qquad \ldots(ii)$$

Also $\because$ (x_0, y_0) lie on given curve $x^2 = 4y$

$$\Rightarrow \quad x_0^2 = 4y_0 \quad \Rightarrow \quad y_0 = \frac{x_0^2}{4} \qquad \ldots(iii)$$

Putting the value of y_0 from (iii) in (ii) we get

$$x_0 \cdot \frac{x_0^2}{4} = 2 \qquad \Rightarrow \qquad x_0^3 = 8$$

$$\Rightarrow \quad x_0 = 2$$

$$\therefore \quad y_0 = \frac{x_0^2}{4} = \frac{2^2}{4} = 1$$

Therefore, the equation of required normal is

$$(y-1) = -\frac{2}{2}(x-2)$$

$\Rightarrow \quad y-1 = -x+2 \quad \Rightarrow \quad x+y-3=0$

Also, equation of required tangent is

$$(y-1) = \frac{2}{2}(x-2) \quad \Rightarrow \quad y-1 = x-2 \quad \Rightarrow \quad x-y-1=0$$

25. Refer to Q. No. 7 page 329.

OR

Refer to Q. No. 9 page 330.

26. Given: $2y.e^{x/y}\,dx + (y-2x\,e^{x/y})\,dy = 0$

$$\Rightarrow \quad \frac{dx}{dy} = -\frac{y-2x\,e^{x/y}}{2y.e^{x/y}} \quad \Rightarrow \quad \frac{dx}{dy} = \frac{2x\;e^{x/y}-y}{2y.e^{x/y}}$$

Let $\quad F(x,y) = \dfrac{2x.e^{x/y}-y}{2y.e^{x/y}}$

$$\therefore \quad F(\lambda x, \lambda y) = \frac{2\lambda\,x.e^{\lambda x/\lambda y} - \lambda\,y}{2\lambda\,y\;.e^{\lambda x/\lambda y}} = \lambda^0\,\frac{2\,x\,e^{x/y}-y}{2\,y\;\;e^{x/y}} = \lambda^0\,.F(x,y)$$

Hence, given differential equation is homogeneous.

Now, $\quad \dfrac{dx}{dy} = \dfrac{2x\,e^{x/y}-y}{2y.e^{x/y}} \qquad \ldots(i)$

Let $x = vy \quad \Rightarrow \quad \dfrac{dx}{dy} = v + y.\dfrac{dv}{dy}$

$$\therefore (i) \Rightarrow \quad v + y.\frac{dv}{dy} = \frac{2vy.e^{\frac{vy}{y}} - y}{2y.e^{\frac{vy}{y}}}$$

$$\Rightarrow \quad y.\frac{dv}{dy} = \frac{y(2v\,e^v - 1)}{2y.e^v} - v \quad \Rightarrow \quad y.\frac{dv}{dy} = \frac{2v\,e^v - 1}{2e^v} - v$$

$$\Rightarrow \quad y\frac{dv}{dy} = -\frac{1}{2e^v} \quad \Rightarrow \quad 2y\,e^v\,dv = -\,dy$$

$$\Rightarrow \quad 2\int e^v\,dv = -\int\frac{dy}{y} \quad \Rightarrow \quad 2e^v = -\log y + C$$

$$\Rightarrow \quad 2e^{\frac{x}{y}} = \log y = C$$

When $x=0,\; y=1$

$\therefore \qquad 2e^0 + \log 1 = C$ or $C = 2$

Hence, the required solution is

$$2e^{x/y} + \log y = 2 \qquad \Rightarrow \qquad \log C = 2$$

27. The equation of plane passing through three points $\hat{i}+\hat{j}-2\hat{k}$, $2\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$ *i.e.*, (1, 1, –2), (2, – 1, 1) and (1, 2, 1) is

$$\begin{vmatrix} x-1 & y-1 & z+2 \\ 2-1 & -1-1 & 1+2 \\ 1-1 & 2-1 & 1+2 \end{vmatrix}=0 \quad\Rightarrow\quad \begin{vmatrix} x-1 & y-1 & z+2 \\ 1 & -2 & 3 \\ 0 & 1 & 3 \end{vmatrix}=0$$

$\Rightarrow\quad (x-1)(-6-3)-(y-1)(3-0)+(z+2)(1+0)=0$

$\Rightarrow\quad -9x+9-3y+3+z+2=0$

$\Rightarrow\quad 9x+3y-z=14 \qquad \ldots(i)$

Its vector form is,

$$\vec{r}\cdot(9\hat{i}+3\hat{j}-\hat{k})=14$$

The given line is

$$\vec{r}=(3\hat{i}-\hat{j}-\hat{k})+\lambda(2\hat{i}-2\hat{j}+\hat{k})$$

Its cartesian form is

$$\frac{x-3}{2}=\frac{y+1}{-2}=\frac{z+1}{1} \qquad \ldots(ii)$$

Let the line (*ii*) intersect plane (*i*) at (α,β,γ)

$\because\quad (\alpha,\beta,\gamma)$ lie on (*ii*)

$$\frac{\alpha-3}{2}=\frac{\beta+1}{-2}=\frac{\gamma+1}{1}=\lambda\,(\text{say})$$

$\Rightarrow\quad \alpha=2\lambda+3;\quad \beta=-2\lambda-1;\quad \gamma=\lambda-1$

Also, point (α,β,γ) lie on plane (*i*)

$\Rightarrow\quad 9\alpha+3\beta-\gamma=14$

$\Rightarrow\quad 9(2\lambda+3)+3(-2\lambda-1)-(\lambda-1)=14$

$\Rightarrow\quad 18\lambda+27-6\lambda-3-\lambda+1=14 \quad\Rightarrow\quad 11\lambda+25=14$

$\Rightarrow\quad 11\lambda=14-25 \quad\Rightarrow\quad 11\lambda=-11$

$\Rightarrow\quad \lambda=-1$

Therefore point of intersection ≡ (1, 1, – 2).

28. Let x and y hectare of land be allocated to crop A and B respectively. If Z is the profit then

$$Z=10500x+9000y \qquad \ldots(i)$$

We have to maximize Z subject to the constraints

$x+y\le 50 \qquad \ldots(ii)$

$20x+10y\le 800 \Rightarrow 2x+y\le 80 \qquad \ldots(iii)$

$x\ge 0,\ y\ge 0 \qquad \ldots(iv)$

The graph of system of inequalities (ii) to (iv) are drawn, which gives feasible region OABC with corner points O (0, 0), A (40, 0), B (30, 20) and C (0, 50).

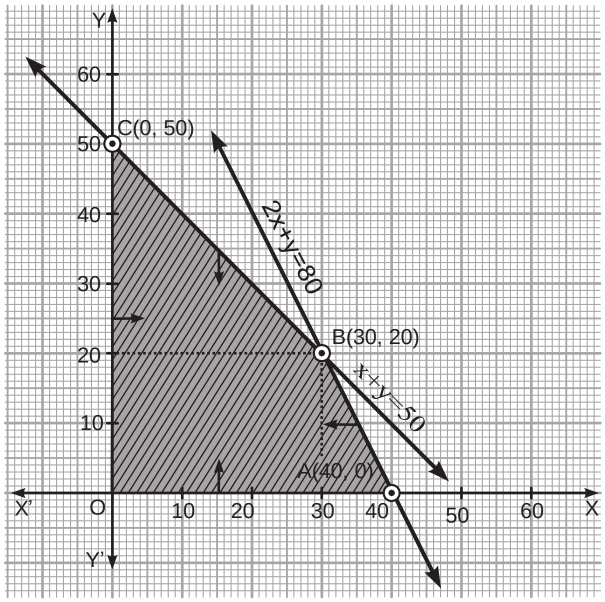
Graph for $x+y=50$

x	0	50

y	50	0

Graph for $2x + y = 80$

x	0	40
y	80	0



Feasible region is bounded.

Now,

Corner point	$Z = 10500x + 9000y$
O (0, 0)	0
A (40, 0)	420000
B (30, 20)	495000 ← Maximum
C (0, 50)	450000

Hence the co-operative society of farmers will get the maximum profit of ` 4,95,000 by allocating 30 hectares for crop A and 20 hectares for crop B.

Yes, because excess use of herbicide can make drainage water poisonous and thus it harm the life of water living creature and wildlife.

29. Let E_1, E_2, A be events defined as

$E_1 =$ treatment of heart attack with Yoga and meditation

$E_2 =$ treatment of heart attack with certain drugs.

$A =$ Person getting heart attack.

$P(E_1) = \frac{1}{2}, P(E_2) = \frac{1}{2}$

Now $P\left(\frac{A}{E_1}\right) = 40\% - \left(40 \times \frac{30}{100}\right)\% = 40\% - 12\% = 28\% = \frac{28}{100}$

$$P\left(\frac{A}{E_2}\right) = 40\% - \left(40 \times \frac{25}{100}\right)\% = 40\% - 10\% = 30\% = \frac{30}{100}$$

We have to find $P\left(\frac{E_1}{A}\right)$

$$\therefore \quad P\left(\frac{E_1}{A}\right) = \frac{P(E_1) \cdot P\left(\frac{A}{E_1}\right)}{P(E_1) \cdot P\left(\frac{A}{E_1}\right) + P(E_2) \cdot P\left(\frac{A}{E_2}\right)}$$

$$= \frac{\frac{1}{2} \times \frac{28}{100}}{\frac{1}{2} \times \frac{28}{100} + \frac{1}{2} \times \frac{30}{100}} = \frac{28}{100} \times \frac{100}{58} = \frac{14}{29}$$

The problem emphasises the importance of Yoga and meditation.

Treatment with Yoga and meditation is more beneficial for the heart patient.

Set–II

9. Degree = 1

16. Let E_1 and E_2 be two events such that

$E_1 = P$ speaks truth

$E_2 = Q$ speaks truth

Now $P(E_1) = \frac{70}{100} = \frac{7}{10} \quad \Rightarrow P(\bar{E}_1) = 1 - \frac{7}{10} = \frac{3}{10}$

$P(E_2) = \frac{80}{100} = \frac{4}{5} \quad \Rightarrow P(\bar{E}_2) = 1 - \frac{4}{5} = \frac{1}{5}$

P (P and Q stating the same fact)

= P (speak truth and Q speak truth or P does not speak truth and Q does not speak truth)

= P (both speak truth) + P (both do not speak truth)

$= \frac{7}{10} \times \frac{4}{5} + \frac{3}{10} \times \frac{1}{5} = \frac{28}{50} + \frac{3}{50} = \frac{31}{50}$

No, both can tell a lie.

18. Refer to page 412, Q. No. 12.

19. Let $I = \int_1^3 [\,|x-1| + |x-2| + |x-3|\,]\,dx = \int_1^3 |x-1|\,dx + \int_1^3 |x-2|\,dx + \int_1^3 |x-3|\,dx$

$$= \int_1^3 |x-1|\,dx + \int_1^2 |x-2|\,dx + \int_2^3 |x-2|\,dx + \int_1^3 |x-3|\,dx$$

[By properties of definite integral]

$$= \int_1^3 (x-1)dx + \int_1^2 -(x-2)dx + \int_2^3 (x-2)dx + \int_1^3 -(x-3)dx$$

$$\left\{\begin{array}{l} x-1 \ge 0, \text{ if } 1 \le x \le 3 \\ x-2 \le 0, \text{ if } 1 \le x \le 2 \\ x-2 \ge 0, \text{ if } 2 \le x \le 3 \\ x-3 \le 0, \text{ if } 1 \le x \le 3 \end{array}\right\}$$

$\leq 3 \rfloor$

$$= \left[\frac{(x-1)^2}{2}\right]_1^3 - \left[\frac{(x-2)^2}{2}\right]_1^2 + \left[\frac{(x-2)^2}{2}\right]_2^3 - \left[\frac{(x-3)^2}{2}\right]_1^3$$

$$= \left(\frac{4}{2} - 0\right) - \left(0 - \frac{1}{2}\right) + \left(\frac{1}{2} - 0\right) - \left(0 - \frac{4}{2}\right) = 2 + \frac{1}{2} + \frac{1}{2} + 2 = 5$$

20. Let $I = \int \frac{x^2 + 1}{(x^2 + 4)\,(x^2 + 25)}\,dx$

Let $x^2 = y$

$\Rightarrow \quad \frac{x^2+1}{(x^2 + 4)(x^2 + 25)} = \frac{y+1}{(y+4)(y+25)}$

Now, $\frac{y+1}{(y+4)(y+25)} = \frac{A}{y+4} + \frac{B}{y+25} \Rightarrow \frac{y+1}{(y+4)(y+5)} = \frac{A(y+25)+B(y+4)}{(y+4)(y+5)}$

$\Rightarrow \quad y+1 = (A+B)\,y + (25A + 4B)$

Equating we get

$A + B = 1$ and $25A + 4B = 1$

$\Rightarrow \quad A = -\frac{1}{7},\ B = \frac{8}{7}$

$\therefore \quad \frac{x^2 + 1}{(x^2 + 4)(x^2 + 25)} = \frac{-1}{7\,(x^2 + 4)} + \frac{8}{7\,(x^2 + 25)}$

$\therefore \quad I = \int \left[-\frac{1}{7\,(x^2 + 4)} + \frac{8}{7\,(x^2 + 25)}\right] dx = -\frac{1}{7}\int \frac{dx}{x^2 + 2^2} + \frac{8}{7}\int \frac{dx}{x^2 + 5^2}$

$$= -\frac{1}{7} \times \frac{1}{2}\tan^{-1}\frac{x}{2} + \frac{8}{7} \times \frac{1}{5}\tan^{-1}\frac{x}{5} + C = -\frac{1}{14}\tan^{-1}\frac{x}{2} + \frac{8}{35}\tan^{-1}\frac{x}{5} + C$$

28. Given differential equation is

$$x\frac{dy}{dx}\sin\frac{y}{x} + x - y\sin\frac{y}{x} = 0$$

Dividing both sides by $x\sin\frac{y}{x}$, we get

$$\frac{dy}{dx} + \operatorname{cosec}\frac{y}{x} - \frac{y}{x} = 0$$

$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} - \operatorname{cosec}\frac{y}{x} \qquad \ldots(i)$

Let $\quad F(x, y) = \frac{y}{x} - \operatorname{cosec}\frac{y}{x}$

$\therefore \quad F(\lambda x, \lambda y) = \frac{\lambda y}{\lambda x} - \operatorname{cosec}\frac{\lambda y}{\lambda x} = \lambda^0\left[\frac{y}{x} - \operatorname{cosec}\frac{y}{x}\right] = \lambda^0\,F(x, y)$

Hence, differential equation (i) *is* homogeneous

$y \qquad dy \qquad dv$

Let $y = vx \quad \Rightarrow \quad \frac{}{x} = v \quad \Rightarrow \quad \frac{}{dx} = v + x.\frac{}{dx}$

Now equation (*i*) becomes

$$v + x.\frac{dv}{dx} = \frac{vx}{x} - \operatorname{cosec}\frac{vx}{x}$$

$$v + x.\frac{dv}{dx} = v - \operatorname{cosec} v \Rightarrow \quad x.\frac{dv}{dx} = -\operatorname{cosec} v$$

$$\Rightarrow \quad -\sin v\, dv = \frac{dx}{x} \quad \Rightarrow \quad -\int \sin v\, dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad \cos v = \log|x| + C \quad \Rightarrow \quad \cos\frac{y}{x} = \log|x| + C$$

Given $y = \frac{\pi}{2}$, $x = 1$

$$\therefore \quad \cos\frac{\pi}{2} = \log 1 + C \quad \Rightarrow \quad 0 = 0 + C \quad \Rightarrow \quad C = 0$$

Hence, particular solution is

$$\cos\frac{y}{x} = \log|x| + 0 \quad \textit{i.e.} \quad \cos\frac{y}{x} = \log|x|$$

29. Refer to page 444 Q. No. 27.

Set–III

2. $\vec{a} + \vec{b} = (2\hat{i} - \hat{j} + 2\hat{k}) + (-\hat{i} + \hat{j} + 3\hat{k}) = \hat{i} + 5\hat{k}$

$\therefore$ Unit vector in the direction of $\vec{a} + \vec{b} = \dfrac{\hat{i} + 5\hat{k}}{\sqrt{1^2 + 5^2}}$

$$= \frac{1}{\sqrt{26}}(\hat{i} + 5\hat{k}) = \frac{1}{\sqrt{26}}\hat{i} + \frac{5}{\sqrt{26}}\hat{k}$$

4. Degree = 3

11. Let E_1 and E_2 be two events such that

$E_1 = A$ speaks truth

$E_2 = B$ speaks truth

Now $P(E_1) = \frac{75}{100} = \frac{3}{4} \Rightarrow P(\bar{E}_1) = 1 - \frac{3}{4} = \frac{1}{4}$

$$P(E_2) = \frac{90}{100} = \frac{9}{10} \Rightarrow P(\bar{E}_2) = 1 - \frac{9}{10} = \frac{1}{10}$$

$\therefore$ $P(A \text{ and } B \text{ contradict to each other}) = P(E_1) \times P(\bar{E}_2) + P(\bar{E}_1) \times P(E_2)$

$$= \frac{3}{4} \times \frac{1}{10} + \frac{1}{4} \times \frac{9}{10} = \frac{12}{40} = \frac{3}{10}$$

It is not necessary that the statement of B is always true, it may be false also.

13. Given $A(1, 2, 3)$, $B(2, -1, 4)$ and $C(4, 5, -1)$

Now $\vec{AB} = (2-1)\hat{i} + (-1-2)\hat{j} + (4-3)\hat{k} = \hat{i} - 3\hat{j} + \hat{k}$

$$\vec{AC} = (4-1)\hat{i} + (5-2)\hat{j} + (-1-3)\hat{k} = 3\hat{i} + 3\hat{j} - 4\hat{k}$$

Area of given triangle $= \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{1}{2} \left| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 3 & 3 & -4 \end{vmatrix} \right|$

$$= \frac{1}{2} |(12-3)\hat{i} - (-4-3)\hat{j} + (3+9)\hat{k}|$$

$$= \frac{1}{2} |9\hat{i} + 7\hat{j} + 12\hat{k}| = \frac{1}{2} \sqrt{9^2 + 7^2 + 12^2}$$

$$= \frac{1}{2} \sqrt{274} \text{ sq. unit}$$

14. Let $I = \int_2^5 [|x-2| + |x-3| + |x-5|]\, dx$

$$= \int_2^5 |x-2|\, dx + \int_2^5 |x-3|\, dx + \int_2^5 |x-5|\, dx$$

$$= \int_2^5 |x-2|\, dx + \int_2^3 |x-3|\, dx + \int_3^5 |x-3|\, dx + \int_2^5 |x-5|\, dx$$

[By properties of Definite Integral]

$$= \int_2^5 (x-2)\, dx + \int_2^3 -(x-3)\, dx + \int_3^5 (x-3)\, dx + \int_2^5 -(x-5)\, dx$$

$$\begin{Bmatrix} x-2 \ge 0, \text{ if } 2 \le x \le 5 \\ x-3 \le 0, \text{ if } 2 \le x \le 3 \\ x-3 \ge 0, \text{ if } 3 \le x \le 5 \\ x-5 \le 0, \text{ if } 2 \le x \le 5 \end{Bmatrix}$$

$$= \left[\frac{(x-2)^2}{2}\right]_2^5 - \left[\frac{(x-3)^2}{2}\right]_2^3 + \left[\frac{(x-3)^2}{2}\right]_3^5 - \left[\frac{(x-5)^2}{2}\right]_2^5$$

$$= \left(\frac{9}{2} - 0\right) - \left(0 - \frac{1}{2}\right) + (2 - 0) - \left(0 - \frac{9}{2}\right) = \frac{9}{2} + \frac{1}{2} + 2 + \frac{9}{2}$$

$$= \frac{9 + 1 + 4 + 9}{2} = \frac{23}{2}$$

$$I = \frac{23}{2}$$

15. Let $I = \int \frac{2x^2 + 1}{x^2 (x^2 + 4)}\, dx$

Let $x^2 = y$ $\quad = \quad$

$\therefore \qquad \Rightarrow$

Now,

$$\frac{2x^2+1}{x^2(x^2+4)} = \frac{2y+1}{y(y+4)}$$

$$\frac{2y+1}{y(y+4)} = \frac{A}{y} + \frac{B}{y+4}$$

$$2y+1 = A(y+4) + By$$

$\Rightarrow \quad 2y+1=(A+B)\,y+4A$

$\Rightarrow \quad 4A=1 \Rightarrow A=\frac{1}{4}$

and $\quad A+B=2 \Rightarrow B=2-\frac{1}{4}=\frac{7}{4}$

$\therefore \quad I=\int \frac{1}{4x^2}\,dx+\int \frac{7\,dx}{4(x^2+4)}=\frac{1}{4}\,\frac{x^{-2+1}}{-2+1}+\frac{7}{4}\times\frac{1}{2}\tan^{-1}\frac{x}{2}+C$

$=-\frac{1}{4x}+\frac{7}{8}\tan^{-1}\frac{x}{2}+C$

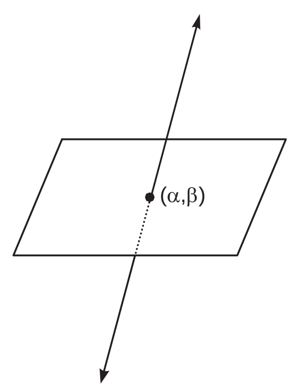
25. The equation of line through the points (3, – 4, – 5) and (2, – 3, 1) is given by

$$\frac{x-3}{2-3}=\frac{y+4}{-3+4}=\frac{x+5}{1+5}$$

$$\Rightarrow \quad \frac{x-3}{-1}=\frac{y+4}{1}=\frac{z+5}{6} \qquad \ldots(i)$$

The equation of plane determined by points (2, 2, 1), (3, 0, 1) and (4, – 1, 0) is

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ 3-2 & 0-2 & 1-1 \\ 4-2 & -1-2 & 0-1 \end{vmatrix}=0 \quad \Rightarrow \quad \begin{vmatrix} x-2 & y-2 & z-1 \\ 1 & -2 & 0 \\ 2 & -3 & -1 \end{vmatrix}=0$$



$\Rightarrow \quad (x-2)(2-0)-(y-2)(-1-0)+(z-1)(-3+4)$

$\Rightarrow \quad 2x-4+y-2+z-1=0$

$\Rightarrow \quad 2x+y+z-7=0 \qquad \ldots(ii)$

Let line (*i*) cross the plane (*ii*) at (α,β,γ)

Q Point (α,β,γ) lies on line (*i*)

$\therefore \quad \frac{\alpha-3}{-1}=\frac{\beta+4}{1}=\frac{\gamma+5}{6}=\lambda$ (say)

$\Rightarrow \quad \alpha=-\lambda+3;\ \beta=\lambda-4;\ \gamma=6\lambda-5$

Also, point (α,β,γ) lies on plane (ii)

$\therefore \quad 2\alpha+\beta+\gamma-7=0$

$\Rightarrow \quad 2(-\lambda+3)+(\lambda-4)+6\lambda-5-7=0$

$\Rightarrow \quad -2\lambda+6+\lambda-4+6\lambda-12=0$

$\Rightarrow \quad 5\lambda-10=0 \quad \Rightarrow \quad \lambda=2$

Hence, the coordinate of the point, where line (*i*) cross the plane (*ii*) is (1, – 2, 7)

26. Given differential equation is

$$\left(x.e^{\frac{y}{x}}+y\right)dx=x\,dy \quad \Rightarrow \quad \frac{dy}{dx}=\frac{x.e^{\frac{y}{x}}+y}{x} \qquad \ldots(i)$$

Let $\quad F(x,y)=\frac{x.e^{\frac{y}{x}}+y}{x}$

$\therefore \quad F(\lambda x,\lambda y)=\frac{\lambda x.e^{\frac{\lambda y}{\lambda x}}+\lambda y}{\lambda x}=\lambda^0\,\frac{x.e^{\frac{y}{x}}+y}{x}=\lambda^0\,F(x,y)$

Hence, given differential equation (*i*) is homogenous.

Let $y = vx$

$\Rightarrow \quad \dfrac{dy}{dx} = v + x.\dfrac{dv}{dx}$

Now given differential equation (*i*) is becomes

$$v + x\frac{dv}{dx} = \frac{x.e^{\frac{vx}{x}} + vx}{x} \quad \Rightarrow \quad v + x.\frac{dv}{dx} = e^{v} + v$$

$$\Rightarrow \quad x.\frac{dv}{dx} = e^{v} \quad \Rightarrow \quad \frac{dv}{e^{v}} = \frac{dx}{x}$$

$$\Rightarrow \quad \int e^{-v} dv = \int \frac{dx}{x} \quad \Rightarrow \quad \frac{e^{-v}}{-1} = \log x + C$$

$$\Rightarrow \quad -e^{-\frac{y}{x}} = \log x + C \quad \Rightarrow \quad -\frac{1}{e^{\frac{y}{x}}} = \log x + C$$

$$\Rightarrow \quad e^{\frac{y}{x}}.\log x + C.e^{\frac{y}{x}} + 1 = 0$$

Given that $x = 1$ when $y = 1$

$\therefore \quad e \log 1 + Ce + 1 = 0 \quad \Rightarrow \quad C = -\dfrac{1}{e}$

$\therefore$ The required particular solution is

$$e^{\frac{y}{x}}.\log x - \frac{1}{e}e^{\frac{y}{x}} + 1 = 0$$

or $\quad e^{\frac{y}{x}} \log x - e^{\frac{y}{x}-1} + 1 = 0$

zzz

CBSE Examination Papers
(All India–2013)

Time allowed: 3 hours **Maximum marks: 100**

General Instructions: *As given in CBSE Sample Question Paper.*

Set–I

SECTION–A

Question numbers 1 to 10 carry 1 mark each.

1. Write the principal value of $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$.

2. Write the value of $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$.

3. For what value of x, is the matrix $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix}$ a skew-symmetric matrix?

4. If matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $A^2 = kA$, then write the value of k.

5. Write the differential equation representing the family of curves $y = mx$, where m is an arbitrary constant.

6. If A_{ij} is the cofactor of the element a_{ij} of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$, then write the value of $a_{32} \cdot A_{32}$.

7. P and Q are two points with position vectors $3\vec{a} - 2\vec{b}$ and $\vec{a} + \vec{b}$ respectively. Write the position vector of a point R which divides the line segment PQ in the ratio 2:1 externally.

8. Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 15$.

9. Find the length of the perpendicular drawn from the origin to the plane $2x - 3y + 6z + 21 = 0$.

10. ***The money to be spent for the welfare of the employees of a firm is proportional to the rate of change of its total revenue (marginal revenue). If the total revenue (in rupees) received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$, find the marginal revenue when $x = 5$, and write which value does the question indicate.***

SECTION–B

Question numbers 11 to 22 carry 4 marks each.

11. Consider $f : R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of f given by $f^{-1}(y) = \sqrt{y-4}$, where R_+ is the set of all non-negative real numbers.

12. Show that: $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$

OR

Solve the following equation: $\cos(\tan^{-1} x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$

13. Using properties of determinants, prove the following:

$$\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y)$$

14. If $y^x = e^{y-x}$, prove that $\frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$.

15. Differentiate the following with respect to x:

$$\sin^{-1}\left(\frac{2^{x+1}.3^x}{1+(36)^x}\right)$$

16. Find the value of k, for which

$$f(x) = \begin{cases} \dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if} -1 \le x < 0 \\ \dfrac{2x+1}{x-1}, & \text{if } 0 \le x < 1 \end{cases}$$

is continuous at $x = 0$.

OR

If $x = a\cos^3\theta$ and $y = a\sin^3\theta$, then find the value of $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{6}$.

17. Evaluate: $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha} dx$.

Evaluate: $\int$

R **O** $\frac{x+2}{\sqrt{x \quad +2x+3}}\,dx$

18. Evaluate : $\int \frac{dx}{x(x^5+3)}$

19. Evaluate : $\int_0^{2\pi} \frac{1}{1+e^{\sin x}}\,dx$

20. If $\vec{a} = \hat{i} - \hat{j} + 7\hat{k}$ and $\vec{b} = 5\hat{i} - \hat{j} + \lambda\hat{k}$, then find the value of λ, so that $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular vectors.

21. Show that the lines

$$\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k});$$

$$\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k});$$

are intersecting. Hence find their point of intersection.

OR

Find the vector equation of the plane through the points (2, 1, – 1) and (– 1, 3, 4) and perpendicular to the plane $x - 2y + 4z = 10$.

22. ***The probabilities of two students A and B coming to the school in time are $\frac{3}{7}$ and $\frac{5}{7}$ respectively. Assuming that the events, 'A coming in time' and 'B coming in time' are independent, find the probability of only one of them coming to the school in time.***

Write at least one advantage of coming to school in time.

SECTION–C

Question numbers 23 to 29 carry 6 marks each.

23. Find the area of the greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

OR

Find the equations of tangents to the curve $3x^2 - y^2 = 8$, which pass through the point $\left(\frac{4}{3}, 0\right)$.

24. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.

25. Find the particular solution of the differential equation $(\tan^{-1} y - x)\, dy = (1 + y^2)\, dx$, given that when $x = 0$, $y = 0$.

26. Find the equation of the plane passing through the line of intersection of the planes $\vec{r}\cdot(\hat{i} + 3\hat{j}) - 6 = 0$ and $\vec{r}\cdot(3\hat{i} - \hat{j} - 4\hat{k}) = 0$, whose perpendicular distance from origin is unity.

OR

Find the vector equation of the line passing through the point (1, 2, 3) and parallel to the planes $\vec{r}\cdot(\hat{i} - \hat{j} + 2\hat{k}) = 5$ and $\vec{r}\cdot(3\hat{i} + \hat{j} + \hat{k}) = 6$.

27. In a hockey match, both teams A and B scored same number of goals up to the end of the game, so to decide the winner, the referee asked both the captains to throw a die alternately and decided that the team, whose captain gets a six first, will be declared the winner. If the captain of team A was asked to start, find their respective probabilities of winning the match and state whether the decision of the referee was fair or not.

28. ***A manufacturer considers that men and women workers are equally efficient and so he pays them at the same rate. He has 30 and 17 units of workers (male and female) and capital respectively, which he uses to produce two types of goods A and B. To produce one unit of A, 2 workers and 3 units of capital are required while 3 workers and 1 unit of capital is required to produce one unit of B. If A and B are priced at ₹ 100 and ₹ 120 per***

unit respectively, how should he use his resources to maximise the total revenue? Form the above as an LPP and solve graphically.

Do you agree with this view of the manufacturer that men and women workers are equally efficient and so should be paid at the same rate?

29. *The management committee of a residential colony decided to award some of its members (say x) for honesty, some (say y) for helping others and some others (say z) for supervising the workers to keep the colony neat and clean. The sum of all the awardees is 12. Three times the sum of awardees for cooperation and supervision added to two times the number of awardees for honesty is 33. If the sum of the number of awardees for honesty and supervision is twice the number of awardees for helping others, using matrix method, find the number of awardees of each category. Apart from these values, namely, honesty, cooperation and supervision, suggest one more value which the management of the colony must include for awards.*

Set–II

Only those questions, not included in Set I, are given.

9. If matrix $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ and $A^2 = pA$, then write the value of p.

10. A and B are two points with position vectors $2\vec{a} - 3\vec{b}$ and $6\vec{b} - \vec{a}$ respectively. Write the position vector of a point P which divides the line segment AB internally in the ratio 1 : 2.

19. If $x^y = e^{x-y}$, prove that $\dfrac{dy}{dx} = \dfrac{\log x}{(1 + \log x)^2}$.

20. Evaluate: $\displaystyle\int \frac{dx}{x(x^3 + 8)}$

21. Evaluate: $\displaystyle\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx$

22. If $\vec{p} = 5\hat{i} + \lambda\hat{j} - 3\hat{k}$ and $\vec{q} = \hat{i} + 3\hat{j} - 5\hat{k}$, then find the value of λ, so that $\vec{p} + \vec{q}$ and $\vec{p} - \vec{q}$ are perpendicular vectors.

28. Find the area of the region $\{(x, y) : y^2 \le 6ax$ and $x^2 + y^2 \le 16a^2\}$ using method of integration.

29. Show that the differential equation $\left[x \sin^2\left(\dfrac{y}{x}\right) - y\right] dx + x\, dy = 0$ is homogeneous. Find the particular solution of this differential equation, given that $y = \dfrac{\pi}{4}$ when $x = 1$.

Set–III

Only those questions, not included in Set I and Set II, are given.

9. If matrix $A = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$ and $A^2 = \lambda A$, then write the value of λ.

10. L and M are two points with position vectors $2\vec{a} - \vec{b}$ and $\vec{a} + 2\vec{b}$ respectively. Write the position vector of a point N which divides the line segment LM in the ratio 2:1 externally.

19. Using vectors, find the area of the triangle ABC, whose vertices are A (1, 2, 3), B (2, –1, 4) and C (4, 5, –1).

20. Evaluate: $\int \frac{dx}{x(x^3+1)}$.

21. If $x \sin(a+y) + \sin a \cos(a+y) = 0$, prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.

22. Using properties of determinants, prove the following:

$$\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix} = 3(x+y+z)(xy+yz+zx).$$

28. Find the area of the region $\{(x, y) : y^2 \le 4x, 4x^2 + 4y^2 \le 9\}$ using method of integration.

29. Find the particular solution of the differential equation.

$$\frac{dx}{dy} + x\cot y = 2y + y^2 \cot y, (y \ne 0), \text{ given that } x = 0 \text{ when } y = \frac{\pi}{2}.$$

Solutions

Set–I

SECTION–A

Question numbers 1 to 10 carry 1 mark each.

1. $\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3}) = \tan^{-1}\left(\tan\frac{\pi}{3}\right) - \cot^{-1}\left(-\cot\frac{\pi}{6}\right)$

$$= \tan^{-1}\left(\tan\frac{\pi}{3}\right) - \cot^{-1}\left(\cot\left(\pi - \frac{\pi}{6}\right)\right) = \tan^{-1}\left(\tan\frac{\pi}{3}\right) - \cot^{-1}\left(\cot\frac{5\pi}{6}\right)$$

$$= \frac{\pi}{3} - \frac{5\pi}{6} \qquad \left[\because \frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ and } \frac{5\pi}{6} \in (0, \pi)\right]$$

$$= \frac{2\pi - 5\pi}{6} = -\frac{\pi}{2}$$

2. $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right] = \tan^{-1}\left(2\sin\left(2\times\frac{\pi}{6}\right)\right) \qquad \left[\because \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}\right]$

$$= \tan^{-1}\left(2\sin\frac{\pi}{3}\right) = \tan^{-1}\left(2\times\frac{\sqrt{3}}{2}\right) = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$$

3. A will be skew symmetric matrix if

$$A = -A'$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -1 & x \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -x \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$$

Equating, we get $x = 2$

4. Given $A^2 = kA$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow 2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow \qquad k = 2$$

5. $y = mx$

Differentiating both sides w.r.t. x, we get

dy

$$\frac{}{dx} = m$$

Hence, required differential equation is

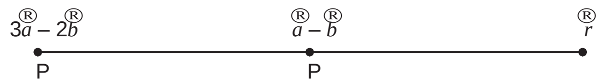
$$y = \frac{dy}{dx} . x \qquad \Rightarrow \qquad ydx - xdy = 0$$

6. $a_{32} . A_{32} = 5 \times (-1)^{3+2} \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix}$

$= -5\,(8 - 30) = -5 \times -22 = 110$

7. If $\vec{r}$ is the position vector of R then by section formula

$$\vec{r} = \frac{2(\vec{a} + \vec{b}) - 1.(3\vec{a} - 2\vec{b})}{2 - 1}$$



$$= \frac{2\vec{a} + 2\vec{b} - 3\vec{a} + 2\vec{b}}{1} = 4\vec{b} - \vec{a}$$

8. Given $(\vec{x} - \vec{a}).(\vec{x} + \vec{a}) = 15$

$\Rightarrow \quad (\vec{x})^2 - (\vec{a})^2 = 15$

$\Rightarrow \quad \vec{x}.\vec{x} - \vec{a}.\vec{a} = 15 \qquad \Rightarrow \quad |\vec{x}|^2 - |\vec{a}|^2 = 15$

$\Rightarrow \quad |\vec{x}|^2 - 1 = 15 \qquad \Rightarrow \quad |\vec{x}|^2 = 16$

$\Rightarrow \quad |\vec{x}| = 4$ [∵ –ve value is not acceptable]

9. Given plane is

$2x - 3y + 6z + 21 = 0$

$\therefore$ Length of $\perp$ar from origin (0, 0, 0) $= \left| \dfrac{0 \times 2 + 0 \times (-3) + 0 \times 6 + 21}{\sqrt{2^2 + (-3)^2 + 6^2}} \right|$

$$= \left| \frac{21}{\sqrt{4 + 9 + 36}} \right| = \frac{21}{\sqrt{49}} = \frac{21}{7} = 3$$

Note: If p is perpendicular distance from (α, β, γ) to plane $ax + by + cz + d = 0$ then

$$p = \left| \frac{a\alpha + b\beta + c\gamma + d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

10. Given: $R(x) = 3x^2 + 36x + 5$

$\Rightarrow \quad R'(x) = 6x + 36$

$\therefore$ Marginal revenue (when $x = 5$) $= R'(x)\big]_{x=5} = 6 \times 5 + 36 =$ ₹66.

The question indicates the value of welfare, which is necessary for each society.

SECTION–B

11. For one-one

Let $\quad x_1, x_2 \in R_+$ (Domain)

$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1^2 + 4 = x_2^2 + 4$

$\Rightarrow \quad x_1^2 = x_2^2$

$\Rightarrow$ $\qquad x_1 = x_2 \qquad$ [Qx_1, x_2 are +ve real number]

$\therefore$ f is one-one function.

For onto

Let $y \in [4, \infty)$ s.t.

$$y = f(x) \quad \forall\, x \in R_+ \qquad \text{(set of non-negative reals)}$$

$$\Rightarrow \qquad y = x^2 + 4$$

$$\Rightarrow \qquad x = \sqrt{y-4} \qquad [\because x \text{ is + ve real number}]$$

Obviously, $\forall\, y \in [4, \infty]$, x is real number $\in R$ (domain)

i.e., all elements of codomain have pre image in domain.

$\Rightarrow$ f is onto.

Hence f is invertible being one-one onto.

For inverse function: If f^{-1} is inverse of f, then

$$fof^{-1} = I \qquad \text{(Identity function)}$$

$$\Rightarrow \qquad fof^{-1}(y) = y \quad \forall\, y \in [4, \infty)$$

$$\Rightarrow \qquad f(f^{-1}(y)) = y$$

$$\Rightarrow \qquad (f^{-1}(y))^2 + 4 = y \qquad [\because f(x) = x^2 + 4]$$

$$\Rightarrow \qquad f^{-1}(y) = \sqrt{y-4}$$

Therefore, required inverse function is $f^{-1} : [4, \infty] \to R$ defined by

$$f^{-1}(y) = \sqrt{y-4} \quad \forall\, y \in [4, \infty).$$

12. Let $\sin^{-1}\frac{3}{4} = \theta \quad \Rightarrow \quad \sin\theta = \frac{3}{4}$

$$\Rightarrow \qquad \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{3}{4}$$

$$\Rightarrow \qquad 3 + 3\tan^2\frac{\theta}{2} = 8\tan\frac{\theta}{2}$$

$$\Rightarrow \qquad \tan\frac{\theta}{2} = \frac{8 \pm \sqrt{64-36}}{6}$$

$$\Rightarrow \qquad \tan\frac{\theta}{2} = \frac{8 \pm 2\sqrt{7}}{6}$$

$$\Rightarrow \qquad \tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$$

Given $\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$

$$\Rightarrow \qquad \cos(\tan^{-1}x) = \cos\left(\frac{\pi}{2} - \cot^{-1}\frac{3}{4}\right)$$

$$\Rightarrow \qquad \tan^{-1}x = \frac{\pi}{2} - \cot^{-1}\frac{3}{4}$$

$\left[\theta \in \left(-\frac{}{2}, \frac{}{2}\right)\right]$

$\left|\quad \left| \quad - \quad - \right| \right|$

$\because \sin 2x = \frac{x}{1 + \tan \quad x} \Big|$

$\left[\qquad \frac{2 \tan}{2} \right]$

$3 \tan^2 \theta - 8 \tan \theta + 3 = 0$

$- \qquad -$

$\tan \frac{}{2} =$

$\frac{\theta}{} \quad \frac{\sqrt{\ }}{}$

$8 \pm \quad 28$

6

$\frac{\theta}{} \quad \frac{\sqrt{\ }}{}$

$\tan \frac{}{2} =$

$4 \pm \quad 7$

3

$\because \theta = \sin^{-1} 3 \Big]$

$\left[\qquad - \right]$

OR

$\Rightarrow \quad \dfrac{\pi}{2} - \cot^{-1} x = \dfrac{\pi}{2} - \cot^{-1} \dfrac{3}{4}$ $\qquad \Rightarrow \quad \cot^{-1} x = \cot^{-1} \dfrac{3}{4}$

$\Rightarrow \quad x = \dfrac{3}{4}$

Note: $\sin\theta = \cos\left(\dfrac{\pi}{2} - \theta\right)$

$\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$

13. L.H.S. $= \begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$

$= \begin{vmatrix} 3(x+y) & 3(x+y) & 3(x+y) \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$ [Applying $R_1 = R_1 + R_2 + R_3$]

$= 3(x+y) \begin{vmatrix} 1 & 1 & 1 \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$ [Taking $3(x+y)$ common from R_1]

$= 3(x+y) \begin{vmatrix} 0 & 0 & 1 \\ y & -y & x+y \\ y & 2y & x \end{vmatrix}$ [Applying $C_1 \to C_1 - C_3, C_2 \to C_2 - C_3$]

Expanding along R_1 we get

$= 3(x+y)\{1(2y^2 + y^2)\}$

$= 9y^2(x+y) = \text{RHS}$

14. Given $y^x = e^{y-x}$

Taking logarithm both sides we get

$\log y^x = \log e^{y-x}$

$\Rightarrow \quad x.\log y = (y-x).\log e$ $\qquad \Rightarrow \quad x.\log y = (y-x)$

$\Rightarrow \quad x(1+\log y) = y$ $\qquad \Rightarrow \quad x = \dfrac{y}{1+\log y}$

Differentiating both sides w.r.t. y. We get

$\dfrac{dx}{dy} = \dfrac{(1+\log y).1 - y.\left(0 + \dfrac{1}{y}\right)}{(1+\log y)^2}$

$= \dfrac{1+\log y - 1}{(1+\log y)^2} = \dfrac{\log y}{(1+\log y)^2}$ $\qquad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{(1+\log y)^2}{\log y}$

Note:

$(i)\ \log_e mn = \log_e m + \log_e n$ |

$(ii)\ \log_e n = \log_e m - \log_e n$ |

$(iii)\ \log_e m^n = n \log_e m$

15. Let $y = \sin^{-1}\left(\dfrac{2^{x+1}.3^x}{1+(36)^x}\right) = \sin^{-1}\left(\dfrac{2.2^x.3^x}{1+(6^2)^x}\right) = \sin^{-1}\left(\dfrac{2.6^x}{1+(6^x)^2}\right)$

Let $6^x = \tan\theta \Rightarrow \theta = \tan^{-1}(6^x)$

$\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right)$

$\therefore \quad y = \sin^{-1}$ $\qquad y = \sin^{-1}(\sin 2\theta)$

$\Rightarrow \Rightarrow \quad y = 2\theta \qquad y = 2.\tan^{-1}(6^x)$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{2.6^x.\log_e 6}{1+36^x}$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{2}{1+(6^x)^2}.\log_e 6.6^x \quad \Rightarrow$

16. $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{h\to 0} f(0-h)$ $\quad$ [Let $x = 0-h, x \to 0^- \Rightarrow h \to 0$]

$= \lim\limits_{h\to 0} f(-h) = \lim\limits_{h\to 0} \dfrac{\sqrt{1+k(-h)} - \sqrt{1-k(-h)}}{-h}$

$= \lim\limits_{h\to 0} \dfrac{\sqrt{1-kh} - \sqrt{1+kh}}{-h} \times \dfrac{\sqrt{1-kh} + \sqrt{1+kh}}{\sqrt{1-kh} + \sqrt{1+kh}}$

$= \lim\limits_{h\to 0} \dfrac{(1-kh)-(1+kh)}{-h\left\{\sqrt{1-kh} + \sqrt{1+kh}\right\}} = \lim\limits_{h\to 0} \dfrac{2k}{\left\{\sqrt{1-kh} + \sqrt{1+kh}\right\}}$

$= \dfrac{2k}{2}$

$\Rightarrow \quad \lim\limits_{x\to 0^-} f(x) = k \qquad$ ....(*i*)

Again $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{h\to 0} f(0+h) \qquad$ [Let $x = 0+h, x \to 0^+ \Rightarrow h \to 0$]

$= \lim\limits_{h\to 0} f(-h) = \lim\limits_{h\to 0} \dfrac{2h+1}{h-1} = \dfrac{1}{-1}$

$\Rightarrow \quad \lim\limits_{x\to 0^+} f(x) = -1 \qquad$ ....(*ii*)

Also $f(0) = \dfrac{2\times 0+1}{0-1} = -1$

Q $\quad f$ is continuous at $x = 0$

$\therefore \quad \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = f(0) \Rightarrow \quad k = -1.$

OR

Given: $x = a\cos^3\theta$

Differentiating both sides w.r.t. θ we get

$\dfrac{dx}{d\theta} = -3a\cos^2\theta.\sin\theta$ Also $y = a\sin^3\theta$

Differentiating both sides w.r.t. θ we get

$\dfrac{dy}{d\theta} = 3a\sin^2\theta.\cos\theta$

...(*i*)

...(*ii*)

Now $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{3a\sin^2\theta\,.\cos\theta}{-3a\cos^2\theta\,.\sin\theta}$

$\Rightarrow \quad \dfrac{dy}{dx} = -\tan\theta$

$\Rightarrow \quad \dfrac{d^2y}{dx^2} = -\sec^2\theta.\dfrac{d\theta}{dx}$

$$= \frac{-\sec^2\theta}{-3a\cos^2\theta\,.\sin\theta} = \frac{1}{3a}\sec^4\theta\,.\,\text{cosec}\,\theta$$

$$\therefore \left.\frac{d^2y}{dx^2}\right]_{x=\pi/6} = \frac{1}{3a}\sec^4\frac{\pi}{6}.\,\text{cosec}\,\frac{\pi}{6} = \frac{1}{3a}.\left(\frac{2}{\sqrt{3}}\right)^4 \times 2 = \frac{32}{27a}$$

17. Let $I = \int \dfrac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}\,dx = \int \dfrac{(2\cos^2 x - 1) - (2\cos^2\alpha - 1)}{\cos x - \cos\alpha}\,dx$

$$= \int \frac{2(\cos^2 x - \cos^2\alpha)}{\cos x - \cos\alpha}\,dx \qquad = \int \frac{2(\cos x + \cos\alpha).(\cos x - \cos\alpha)}{(\cos x - \cos\alpha)}\,dx$$

$$= 2\int(\cos x + \cos\alpha)dx = 2\int \cos x\,dx + 2\int \cos\alpha\,dx$$

$$= 2\sin x + 2x\cos\alpha + C$$

OR

Let $I = \displaystyle\int \frac{x+2}{\sqrt{x^2+2x+3}}\,dx$

$$= \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+2x+3}}\,dx = \frac{1}{2}\int \frac{(2x+2)+2}{\sqrt{x^2+2x+3}}\,dx$$

$$= \frac{1}{2}\int \frac{(2x+2)dx}{\sqrt{x^2+2x+3}} + \frac{1}{2}\int \frac{2dx}{\sqrt{x^2+2x+3}}$$

$$I = \frac{1}{2}\mathrm{I}_1 + \mathrm{I}_2 \qquad \text{....}(i)$$

Where $I_1 = \displaystyle\int \frac{(2x+2)dx}{\sqrt{x^2+2x+3}}$ and $I_2 = \displaystyle\int \frac{dx}{\sqrt{x^2+2x+3}}$

Now $I_1 = \displaystyle\int \frac{2x+2}{\sqrt{x^2+2x+3}}\,dx$

Let $x^2 + 2x + 3 = z^2$

$$(2x+2)dx = 2z\,dz \qquad \Rightarrow \qquad I_1 = \int \frac{2z\,dz}{z}$$

$$= 2\int dz = 2z = 2\sqrt{x^2 + 2x + 3} + C_1$$

$$\Rightarrow \qquad I_1 = 2\sqrt{x^2 + 2x + 3} + C_1$$

Again $\quad I_2 = \int \dfrac{dx}{\sqrt{x^2 + 2x + 3}} = \int \dfrac{dx}{\sqrt{(x+1)^2 + (\sqrt{2})^2}}$

$$= \log\left|(x+1) + \sqrt{(x+1)^2 + (\sqrt{2})^2}\right|$$

$$= \log\left|(x+1) + \sqrt{x^2 + 2x + 3}\right| + C_2$$

Putting the value of I_1, and I_2 in (i) we get

$$I = 2\sqrt{x^2 + 2x + 3} + \log\left|(x+1) + \sqrt{x^2 + 2x + 3}\right| + (C_1 + C_2)$$

$$= 2\sqrt{x^2 + 2x + 3} + \log\left|(x+1) + \sqrt{x^2 + 2x + 3}\right| + C.$$

18. Let $\quad I = \int \dfrac{dx}{x(x^5 + 3)} = \int \dfrac{x^4\, dx}{x^5(x^5 + 3)} = \dfrac{1}{5}\int \dfrac{5x^4\, dx}{x^5(x^5 + 3)}$

Let $\quad x^5 = z \qquad \Rightarrow \quad 5x^4 dx = dz$

$\therefore \qquad I = \dfrac{1}{5}\int \dfrac{dz}{z(z+3)}$

$$= \frac{1}{5 \times 3}\int \frac{z + 3 - z}{z(z+3)}\, dz = \frac{1}{15}\int \frac{z+3}{z(z+3)}\, dz - \frac{1}{15}\int \frac{z}{z(z+3)}\, dz$$

$$= \frac{1}{15}\int \frac{dz}{z} - \frac{1}{15}\int \frac{dz}{z+3} = \frac{1}{15}\left\{\log z - \log|z+3|\right\} + C$$

$$= \frac{1}{15}\log\left|\frac{z}{z+3}\right| + C = \frac{1}{15}\log\left|\frac{x^5}{x^5 + 3}\right| + C$$

19. Let $I = \displaystyle\int_0^{2\pi} \frac{1}{1 + e^{\sin x}}\, dx$ ...(i)

Applying properties $\displaystyle\int_0^a f(x)dx = \int_0^a f(a - x)dx$ we get

$$I = \int_0^{2\pi} \frac{dx}{1 + e^{\sin(2\pi - x)}} = \int_0^{2\pi} \frac{dx}{1 + e^{-\sin x}} = \int_0^{2\pi} \frac{dx}{1 + \dfrac{1}{e^{\sin x}}}$$

$I = \displaystyle\int_0^{2\pi} \frac{e^{\sin x}\, dx}{e^{\sin x} + 1}$ ...(ii)

Adding (i) and (ii) we get

$$2I = \int_0^{2\pi} \frac{dx}{1 + e^{\sin x}} + \int_0^{2\pi} \frac{e^{\sin x}\, dx}{1 + e^{\sin x}} = \int_0^{2\pi} \frac{1 + e^{\sin x}}{1 + e^{\sin x}}\, dx$$

$$= \int_0^{2\pi} dx = [x]_0^{2\pi}$$

$$\Rightarrow \quad 2I = 2\pi \qquad \Rightarrow \qquad I = \pi.$$

20. Here $\vec{a} = \hat{i} - \hat{j} + 7\hat{k}\,;\ \vec{b} = 5\hat{i} - \hat{j} + \lambda\hat{k}$

$\therefore\quad \vec{a} + \vec{b} = 6\hat{i} - 2\hat{j} + (7+\lambda)\hat{k};\ \vec{a} - \vec{b} = -4\hat{i} + (7-\lambda)\hat{k}$

$\because\quad (\vec{a} + \vec{b})$ is perpendicular to $(\vec{a} - \vec{b})$

$\Rightarrow\quad (\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = 0 \quad\Rightarrow\quad -24 + (7+\lambda).(7-\lambda) = 0$

$\Rightarrow\quad -24 + 49 - \lambda^2 = 0 \qquad \Rightarrow\quad \lambda^2 = 25$

$\Rightarrow\quad \lambda = \pm 5.$

21. Given lines are

$$\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$$

$$\vec{r} = 5\hat{i} + 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$$

Its corresponding Cartesian forms are

$$\frac{x-3}{1} = \frac{y-2}{2} = \frac{z+4}{2} \qquad ....(i)$$

$$\frac{x-5}{3} = \frac{y+2}{2} = \frac{z-0}{6} \qquad ....(ii)$$

If two lines (*i*) and (*ii*) intersect, let interesting point be (α, β, γ).

$\Rightarrow\quad (\alpha, \beta, \gamma)$ satisfy line (*i*)

$$\therefore\quad \frac{\alpha-3}{1} = \frac{\beta-2}{2} = \frac{\gamma+4}{2} = \lambda \text{ (say)}$$

$\Rightarrow\quad \alpha = \lambda + 3, \beta = 2\lambda + 2, \gamma = 2\lambda - 4$

Also (α, β, γ) will satisfy line (*ii*)

$$\therefore\quad \frac{\alpha-5}{3} = \frac{\beta+2}{2} = \frac{\gamma}{6}$$

$$\Rightarrow\quad \frac{\lambda+3-5}{3} = \frac{2\lambda+2+2}{2} = \frac{2\lambda-4}{6}$$

$$\therefore\quad \underset{\text{I}}{\frac{\lambda-2}{3}} = \underset{\text{II}}{\frac{\lambda+2}{1}} = \underset{\text{III}}{\frac{\lambda-2}{3}}$$

I and II $\Rightarrow \dfrac{\lambda-2}{3} = \dfrac{\lambda+2}{1} \Rightarrow \lambda - 2 = 3\lambda + 6 \Rightarrow \lambda = -4$

II and III $\Rightarrow \dfrac{\lambda+2}{1} = \dfrac{\lambda-2}{3} \Rightarrow \lambda = -4$

$\therefore$ The value of λ is same in both cases.

Hence, both lines intersect each other at point

$(\alpha, \beta, \gamma) \equiv (-4+3, 2\times(-4)+2, 2(-4)-4) \equiv (-1, -6, -12)$

OR

Let the equation of plane through the point (2, 1 –1) be

$$a(x-2) + b(y-1) + c(z+1) = 0 \qquad ....(i)$$

Since it passes through (–1, 3, 4)

$\Rightarrow \quad a(-1-2)+b(3-1)+c(4+1)=0$

$\Rightarrow \quad -3a+2b+5c=0$ ....(*ii*)

Also, line (*i*) is $\perp$ar to $x-2y+4z=10$

$\Rightarrow \quad a-2b+4c=0$ ....(*iii*)

From (*ii*) and (*iii*) we get

$$\frac{a}{8+10}=\frac{b}{5+12}=\frac{c}{6-2}$$

$$\Rightarrow \quad \frac{a}{18}=\frac{b}{17}=\frac{c}{4}=\lambda \text{ (say)}$$

$\Rightarrow \quad a=18\lambda, b=17\lambda, c=4\lambda$

Putting the value of *a*, *b* and *c* in (*i*) we get

$$18\lambda(x-2)+17\lambda(y-1)+4\lambda(z+1)=0$$

$\Rightarrow \quad 18x+17y+4z=49$

$\Rightarrow \quad \vec{r}.(18\hat{i}+17\hat{j}+4\hat{k})=49.$

22. Let E_1 and E_2 be two events such that

E_1 = A coming to the school in time.

E_2 = B coming to the school in time.

Here $P(E_1)=\frac{3}{7}$ and $P(E_2)=\frac{5}{7}$

$$\Rightarrow \quad P(\bar{E}_1)=\frac{4}{7}, P(\bar{E}_2)=\frac{2}{7}$$

P (only one of them coming to the school in time) $= P(E_1)\times P(\bar{E}_2)+P(\bar{E}_1)\times P(E_2)$

$$=\frac{3}{7}\times\frac{2}{7}+\frac{5}{7}\times\frac{4}{7}$$

$$=\frac{6}{49}+\frac{20}{49}=\frac{26}{49}$$

Coming to school in time i.e., punctuality is a part of discipline which is very essential for development of an individual.

SECTION–C

23. Let *ABCD* be rectangle having area *A* inscribed in an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{...(i)}$$

Let the coordinate of *A* be (α, β)

$\therefore$ coordinate of $B \equiv (\alpha, -\beta)$

$C \equiv (-\alpha, -\beta)$

$D \equiv (-\alpha, \beta)$

Now A = Length × Breadth =

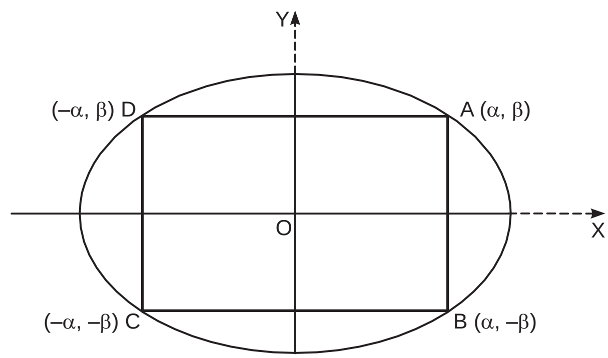
$2\alpha \times 2\beta$ $A = 4\alpha\beta$

$$\Rightarrow \qquad A = 4\alpha \cdot \sqrt{b^2\left(1 - \frac{\alpha^2}{a^2}\right)}$$

$$\left[\begin{array}{l} Q(\alpha, \beta) \text{ lies on ellipse } (i) \\ \therefore \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1 \;\; i.e.\ \beta = \sqrt{b^2\left(1 - \frac{\alpha^2}{a^2}\right)} \end{array}\right]$$

$$\Rightarrow \qquad A^2 = 16\alpha^2\left\{b^2\left(1 - \frac{\alpha^2}{a^2}\right)\right\} \quad \Rightarrow \quad A^2 = \frac{16b^2}{a^2}(a^2\alpha^2 - \alpha^4)$$

$$\Rightarrow \qquad \frac{d(A^2)}{d\alpha} = \frac{16b^2}{a^2}(2a^2\alpha - 4\alpha^3)$$



For maximum or minimum value

$$\frac{d(A^2)}{d\alpha} = 0$$

$$\Rightarrow \quad 2a^2\alpha - 4\alpha^3 = 0$$

$$\Rightarrow \quad 2\alpha(a^2 - 2\alpha^2) = 0$$

$$\Rightarrow \qquad \alpha = 0, \alpha = \frac{a}{\sqrt{2}}$$

Again $\quad \dfrac{d^2(A^2)}{d\alpha^2} = \dfrac{16b^2}{a^2}(2a^2 - 12\alpha^2)$

$$\Rightarrow \qquad \left.\frac{d^2(A^2)}{d\alpha^2}\right]_{\alpha = \frac{a}{\sqrt{2}}} = \frac{16b^2}{a^2}\left(2a^2 - 12 \times \frac{a^2}{2}\right) < 0$$

$\Rightarrow \qquad$ For $\alpha = \dfrac{a}{\sqrt{2}}$, A^2 *i.e.*, A is maximum.

i.e., for greatest area A

$$\alpha = \frac{a}{\sqrt{2}} \text{ and } \beta = \frac{b}{\sqrt{2}} \qquad\qquad (\text{using } (i))$$

$\therefore\;$ Greatest area $= 4\alpha.\beta = 4\dfrac{a}{\sqrt{2}} \times \dfrac{b}{\sqrt{2}} = 2ab$

OR

Let the point of contact be (x_0, y_0)

Now given curve is $3x^2 - y^2 = 8$

Differentiating w.r.t. x we get, $6x - 2y.\dfrac{dy}{dx} = 0$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{6x}{2y} = \frac{3x}{y} \quad \Rightarrow \quad \left.\frac{dy}{dx}\right]_{(x_0, y_0)} = \frac{3x_0}{y_0}$$

Now, equation of required tangent is

$$(y - y_0) = \frac{3x_0}{y_0}(x - x_0) \qquad\qquad ....(i)$$

Q (*i*) passes through $\left(\frac{4}{3}, 0\right)$

$\therefore \quad (0 - y_0) = \dfrac{3x_0}{y_0}\left(\dfrac{4}{3} - x_0\right)$

$\Rightarrow \quad -y_0{}^2 = 4x_0 - 3x_0{}^2$ ....(*ii*)

Also, Q (x_0, y_0) lie on given curve $3x^2 - y^2 = 8$

$\Rightarrow \quad 3x_0^2 - y_0^2 = 8 \quad \Rightarrow \quad y_0^2 = 3x_0^2 - 8$

Putting y_0^2 in (*ii*) we get

$-(3x_0^2 - 8) = 4x_0 - 3x_0^2$

$\Rightarrow \quad 4x_0 = 8 \quad \Rightarrow \quad x_0 = 2$

$\therefore \quad y_0 = \sqrt{3 \times 2^2 - 8} = \sqrt{4} = \pm 2$

Therefore equations of required tangents are

$(y - 2) = \dfrac{3 \times 2}{2}(x - 2)$ and $(y + 2) = \dfrac{3 \times 2}{-2}(x - 2)$

$\Rightarrow \quad y - 2 = 3x - 6$ and $y + 2 = -3x + 6$

$\Rightarrow \quad 3x - y - 4 = 0$ and $3x + y - 4 = 0$

24. Refer to Q. No. 4 Page No. 348.

25. Refer to Q. No. 14 Page 365.

26. The equation of the plane passing through the line of intersection of the planes

$\vec{r}.(\hat{i} + 3\hat{j}) - 6 = 0$ and $\vec{r}.(3\hat{i} - \hat{j} - 4\hat{k}) = 0$ is

$$\left[\vec{r}.(\hat{i} + 3\hat{j}) - 6\right] + \lambda\left[\vec{r}.(3\hat{i} - \hat{j} - 4\hat{k})\right] = 0$$

$\Rightarrow \quad \vec{r}.[(1 + 3\lambda)i + (3 - \lambda)\hat{j} - 4\lambda\hat{k}] - 6 = 0$ ...(*i*)

$\because \quad$ Plane (*i*) is at unit distance from origin (0, 0, 0)

$$\therefore \quad \left|\frac{0 + 0 - 0 - 6}{\sqrt{(1 + 3\lambda)^2 + (3 - \lambda)^2 + (-4\lambda)^2}}\right| = 1$$

$$\Rightarrow \quad \frac{6}{\sqrt{1 + 9\lambda^2 + 6\lambda + 9 + \lambda^2 - 6\lambda + 16\lambda^2}} = 1$$

$$\Rightarrow \quad \frac{6}{\sqrt{26\lambda^2 + 10}} = 1 \quad \Rightarrow \quad \frac{36}{26\lambda^2 + 10} = 1 \qquad \text{[Squaring both sides]}$$

$\Rightarrow \quad 26\lambda^2 + 10 = 36$

$\Rightarrow \quad 26\lambda^2 = 26 \quad \Rightarrow \quad \lambda^2 = 1 \quad \Rightarrow \quad \lambda = \pm 1$

Hence, the equations of required planes are

$\vec{r}.(4\hat{i} + 2\hat{j} - 4\hat{k}) = 6 \quad$ and $\quad \vec{r}.(-2\hat{i} + 4\hat{j} + 4\hat{k}) = 6$

OR

The required line is parallel to the planes

$\rightarrow$

$$r\,.(\hat{i} - \hat{j} + 2\hat{k}) = 5$$

$\vec{r}.(3\hat{i}+\hat{j}+\hat{k})=6$

$\therefore$ Parallel vector of required line $=(\hat{i}-\hat{j}+2\hat{k})\times(3\hat{i}+\hat{j}+\hat{k})$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 2\\ 3 & 1 & 1\end{vmatrix}=(-1-2)\hat{i}-(1-6)\hat{j}+(1+3)\hat{k}$$

$$=-3\hat{i}+5\hat{j}+4\hat{k}$$

Therefore, the vector equation of required line is

$$(\hat{i}+2\hat{j}+3\hat{k})+\lambda(-3\hat{i}+5\hat{j}+4\hat{k})=0$$

27. Let E_1, E_2 be two events such that

E_1 = the captain of team 'A' gets a six.

E_2 = the captain of team 'B' gets a six.

Here $P\,(E_1)=\dfrac{1}{6}$, $P\,(E_2)=\dfrac{1}{6}$

$P(E_1')=1-\dfrac{1}{6}=\dfrac{6}{5}$, $\mathrm{P}(E_1')=1-\dfrac{1}{6}=\dfrac{5}{6}$

Now P (winning the match by team A) $=\dfrac{1}{6}+\dfrac{5}{6}\times\dfrac{5}{6}\times\dfrac{1}{6}+\dfrac{5}{6}\times\dfrac{5}{6}\times\dfrac{5}{6}\times\dfrac{5}{6}\times\dfrac{1}{6}+\ldots\ldots\ldots\ldots\ldots\ldots$

$$=\frac{1}{6}+\left(\frac{5}{6}\right)^2.\frac{1}{6}+\left(\frac{5}{6}\right)^4.\frac{1}{6}+\ldots\ldots$$

$$=\frac{\frac{1}{6}}{1-\left(\frac{5}{6}\right)^2}=\frac{1}{6}\times\frac{36}{11}=\frac{6}{11}$$

$\therefore$ P (winning the match by team B) $=1-\dfrac{6}{11}=\dfrac{5}{11}$

[**Note:** If a be the first term and r the common ratio then sum of infinite terms]

$$S_\infty=\frac{a}{1-r}$$

The decision of refree was not fair because the probability of winning match is more for that team who start to throw dice.

28. Let x, y unit of goods A and B are produced respectively.

Let Z be total revenue

Here $Z=100x+120y$ ....(i)

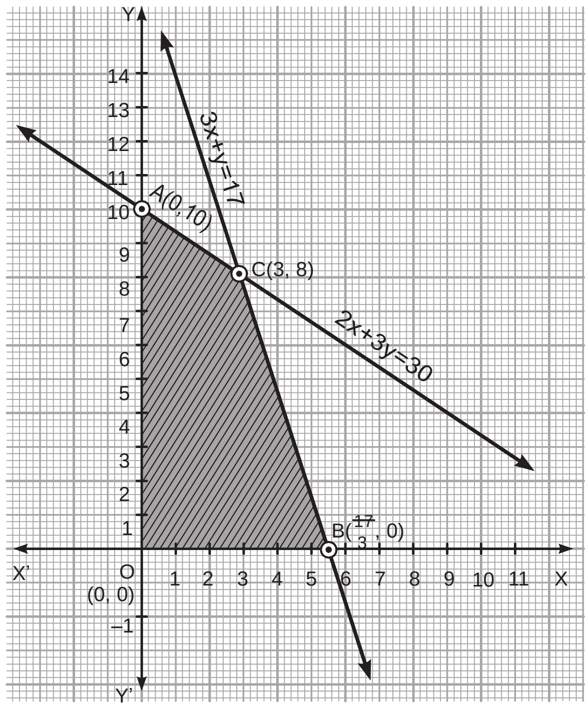
Also $2x+3y\le 30$ ....(ii)

$3x+y\le 17$ ....(iii)

$x\ge 0$ ....(iv)

$y\ge 0$ ....(v)

On plotting graph of above constants or inequalities (ii), (iii), (iv) and (v). We get shaded region as feasible region having corner points A, O, B and C.



For co-ordinate of 'C'

Two equations (*ii*) and (*iii*) are solved and we get coordinate of $C = (3, 8)$

Now the value of Z is evaluated at corner point as:

Corner point	$Z = 100x + 120y$
(0, 10)	1200
(0, 0)	0
$\left(\frac{17}{3}, 0\right)$	$\frac{1700}{3}$
(3, 8)	1260 ←——Maximum

Therefore maximum revenue is `1,260 when 2 workers and 8 units capital are used for production.

Yes, although women workers have less physical efficiency but it can be managed by her other efficiency.

29. According to question

$$x + y + z = 12$$
$$2x + 3y + 3z = 33$$
$$x - 2y + z = 0$$

The above system of linear equation can be written in matrix form as

$AX = B$ | | | | $\lfloor 1 |$ -2

Where $A = \begin{bmatrix} 1 & 1 & \\ 2 & 3 & 1 \end{bmatrix}$ | z | | 0 |

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Now $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & -2 & 1 \end{vmatrix} = 1\,(3+6) - 1\,(2-3) + 1\,(-4-3) = 9 + 1 - 7 = 3$

$A_{11} = 9, \quad A_{12} = 1, \quad A_{13} = -7$

$A_{21} = -3, \quad A_{22} = 0, \quad A_{23} = 3$

$A_{31} = 0, \quad A_{32} = -1, \quad A_{33} = 1$

$$\text{Adj } A = \begin{bmatrix} 9 & 1 & -7 \\ -3 & 0 & 3 \\ 0 & -1 & 1 \end{bmatrix}' = \begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{3}\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}$$

Q $\quad AX = B \Rightarrow X = A^{-1}B$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 108 - 99 \\ 12 + 0 + 0 \\ -84 + 99 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 9 \\ 12 \\ 15 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \Rightarrow x = 3, y = 4, z = 5$$

No. of awards for honesty = 3

No. of awards for helping others = 4

No. of awards for supervising = 5.

The persons, who work in the field of health and hygiene should also be awarded.

SET–II

9. Here $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

Given $A^2 = pA$

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = p\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix} = p\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$\Rightarrow \quad 4\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = p\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$-$

2

2

$$\Rightarrow \qquad p = 4$$

10. Let $\vec{r}$ be the position vector of point P.

By section formula

$$\vec{r} = \frac{1.(6\vec{b} - \vec{a}) + 2.(2\vec{a} - 3\vec{b})}{1+2}$$

$$= \frac{6\vec{b} - \vec{a} + 4\vec{a} - 6\vec{b}}{3} = \frac{3\vec{a}}{3} = \vec{a}$$

19. Given, $x^y = e^{x-y}$

Taking logarithm to the base e both sides, we get $\log x^y = \log e^{x-y}$

Applying law of logarithm, we get $y \log x = (x-y).\log e$

$\Rightarrow \quad y \log x = x - y \qquad [\because \log e = 1]$

$\Rightarrow \quad y(1 + \log x) = x \qquad \Rightarrow \qquad y = \dfrac{x}{1 + \log x}$

Differentiating both sides w.r.t. x we get

$$\frac{dy}{dx} = \frac{(1+\log x).1 - x.(0 + \frac{1}{x})}{(1+\log x)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$$

20. $I = \displaystyle\int \frac{dx}{x(x^3+8)} \qquad \Rightarrow \qquad I = \int \frac{x^2\,dx}{x^3(x^3+8)}$

Let $x^3 = z$

$\Rightarrow \quad 3x^2\,dx = dz \quad \Rightarrow x^2\,dx = \dfrac{dz}{3}$

$$\therefore \quad I = \frac{1}{3}\int \frac{dz}{z(z+8)} = \frac{1}{3\times 8}\int \frac{(z+8) - z}{z(z+8)}\,dz$$

$$= \frac{1}{3\times 8}\int \left[\frac{1}{z} - \frac{1}{z+8}\right] dz = \frac{1}{24}\int \frac{dz}{z} - \frac{1}{24}\int \frac{dz}{z+8}$$

$$= \frac{1}{24}\log|z| - \frac{1}{24}\log|z+8| + C$$

$$I = \frac{1}{24}\log\left|\frac{z}{z+8}\right| + C = \frac{1}{24}\log\left|\frac{x^3}{x^3+8}\right| + C.$$

21. Let $I = \displaystyle\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\,dx \qquad I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)}\,dx$

...(i)

[By
propertie
s]

$$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x}\, dx \qquad ...(ii)$$

Adding (*i*) and (*ii*) we get

$$2I = \pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x}\, dx$$

Let $\cos x = z$

$\Rightarrow \quad -\sin x\, dx = dz$

$x = 0 \;\Rightarrow z = 1$

and $x = \pi \Rightarrow z = -1$

$$\therefore \quad 2I = -\pi \int_1^{-1} \frac{dz}{1 + z^2} = \pi \int_{-1}^{1} \frac{dz}{1 + z^2} = \pi [\tan^{-1} z]^{1}$$

$$= \pi[\tan^{-1} 1 - \tan^{-1}(-1)] = \pi\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{2}.$$

22. Here $\vec{p} = 5\hat{i} + \lambda\hat{j} - 3\hat{k}$

$\vec{q} = \vec{i} + 3\vec{j} - 5\vec{k}$

$\vec{p} + \vec{q} = 6\hat{i} + (3 + \lambda)\hat{j} - 8\hat{k}$

$\vec{p} - \vec{q} = 4\vec{i} + (\lambda - 3)\vec{j} + 2\vec{k}$

Since $(\vec{p} + \vec{q})$ is perpendicular to $(\vec{p} - \vec{q})$.

$\Rightarrow \quad (\vec{p} + \vec{q}) \cdot (\vec{p} - \vec{q}) = 0$

$\Rightarrow \quad (6\hat{i} + (3 + \lambda)\hat{j} - 8\hat{k}) \cdot (4\hat{i} + (\lambda - 3)\hat{j} + 2\hat{k}) = 0$

$\Rightarrow \quad 24 + (\lambda^2 - 9) - 16 = 0$

$\Rightarrow \quad \lambda^2 = 1 \qquad \Rightarrow \qquad \lambda = \pm 1$

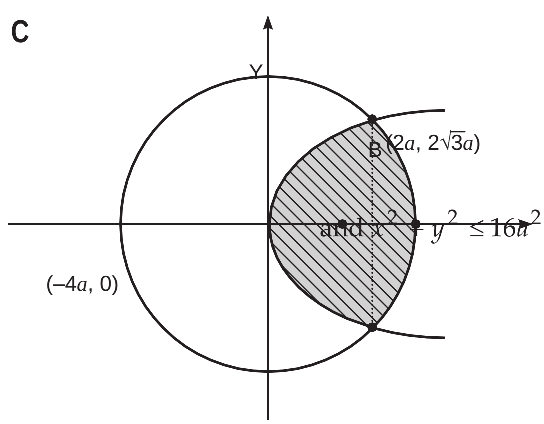
SECTION C

28. Corresponding curves of given region

$\{(x, y): y^2 \le 6ax \text{ and } x^2 + y^2 \le 16a^2\}$ are

$x^2 + y^2 = 16a^2 \qquad ...(i)$

$y = 6ax \qquad ...(ii)$



Obviously, curve (*i*) is a circle having centre (0, 0) and radius 4*a*. While curve (*ii*) is right handed parabola having vertex at (0, 0) and axis along +ve direction of *x*-axis.

Obviously, shaded region *OCAB* is area represented by

$y^2 \le 6ax$

O A($4a$, 0) X

C

Now, intersection point of curve

(i) and (ii)

$x^2 + 6ax = 16a^2$ [Putting the value of y^2 in (i)]

$\Rightarrow \quad x^2 + 6ax - 16a^2 = 0 \quad \Rightarrow \quad x^2 + 8ax - 2ax - 16a^2 = 0$

$\Rightarrow \quad x(x + 8a) - 2a(x + 8a) = 0 \quad \Rightarrow \quad (x + 8a)(x - 2a) = 0$

$\Rightarrow \quad x = 2a, -8a$

$\Rightarrow \quad x = 2a$ [$x = -8a$ is not possible as y^2 is +ve]

$\therefore \quad y = 2\sqrt{3}a$

Since, shaded region is symmetrical about x-axis

$\therefore$ Required area = 2 [Area of $OABO$]

$$= 2\left[\int_0^{2a}\sqrt{6ax}\,dx + \int_{2a}^{4a}\sqrt{16a^2 - x^2}\,dx\right]$$

$$= 2\left[\sqrt{6a}\int_0^{2a}\sqrt{x}\,dx + \int_{2a}^{4a}\sqrt{(4a)^2 - x^2}\,dx\right]$$

$$= 2\sqrt{6a}\times\frac{2}{3}\left[x^{3/2}\right]_0^{2a} + 2\left[\frac{x}{2}\sqrt{16a^2 - x^2} + \frac{1}{2}16a^2\sin^{-1}\frac{x}{4a}\right]_{2a}^{4a}$$

$$= \frac{4\sqrt{6a}}{3}\left[(2a)^{3/2} - 0\right] + 2\left[(0 + 4a^2\pi) - \left(2\sqrt{3}a^2 + \frac{4a^2\pi}{3}\right)\right]$$

$$= \frac{8a^2\sqrt{12}}{3} + 8a^2\pi - 4\sqrt{3}a^2 - \frac{8a^2\pi}{3}$$

$$= \frac{16}{3}\sqrt{3}\,a^2 + \frac{16a^2\pi}{3} - 4\sqrt{3}\,a^2 \Rightarrow \frac{4\sqrt{3}}{3}a^2 + \frac{16a^2\pi}{3}$$

29. Given differential equation is

$$\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + x\,dy = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y - x\sin^2\left(\frac{y}{x}\right)}{x}$$

Let $F(x, y) = \dfrac{y - x\sin^2\frac{y}{x}}{x}$

Then $F(\lambda x, \lambda y) = \dfrac{\lambda y - \lambda x\sin^2\frac{\lambda y}{\lambda x}}{\lambda x}$

$$\therefore \quad = \lambda^0\,\frac{y - x\sin^2\frac{y}{x}}{x}$$

$= \lambda^0 F(x, y)$

Hence, differential equation (i) is homogeneous.

...(i)

Now let $y = vx$

$\therefore \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Putting these value is (*i*) we get

$$v + x\frac{dv}{dx} = \frac{vx - x\sin^2\frac{vx}{x}}{x} \qquad \Rightarrow \qquad v + x\frac{dv}{dx} = \frac{x\{v - \sin^2 v\}}{x}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = v - \sin^2 v \qquad \Rightarrow \qquad x\frac{dv}{dx} = -\sin^2 v$$

$$\Rightarrow \quad \frac{dv}{\sin^2 v} = -\frac{dx}{x}$$

Integrating both sides, we get

$$\Rightarrow \quad \int \operatorname{cosec}^2 v\,dv = -\int \frac{1}{x}\,dx$$

$$\Rightarrow \quad -\cot v = -\log x + C$$

$$\Rightarrow \quad \log x - \cot\left(\frac{y}{x}\right) = C \qquad \ldots(ii)$$

Putting $y = \dfrac{\pi}{4}$ and $x = 1$ we get

$$\log 1 - \cot\frac{\pi}{4} = C \quad \Rightarrow \quad 0 - 1 = C$$

$$\Rightarrow \quad C = -1$$

Hence particular solution is

$$\log x - \cot\left(\frac{y}{x}\right) = -1$$

$$\Rightarrow \quad \log x - \cot\left(\frac{y}{x}\right) + 1 = 0$$

SET–III

9. Here $A = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$

Given $A^2 = \lambda A$

$$\Rightarrow \quad \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} = \lambda\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 18 & -18 \\ -18 & 18 \end{bmatrix} = \lambda\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

3

$$\Rightarrow \quad 6\begin{bmatrix} & -3 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} 3 \end{bmatrix} \quad \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} \quad \Rightarrow \quad \lambda = 6.$$

10. If $\vec{r}$ is the position vector of N then by section formula

$$\vec{r} = \frac{2(\vec{a}+2\vec{b})-1(2\vec{a}-\vec{b})}{2-1} = \frac{2\vec{a}+4\vec{b}-2\vec{a}+\vec{b}}{1} = 5\vec{b}$$

19. Given: $A(1, 2, 3)$, $B(2, -1, 4)$ and $C(4, 5, -1)$

We have $\vec{AB} = \hat{i} - 3\hat{j} + \hat{k}$

$\vec{AC} = 3\hat{i} + 3\hat{j} - 4\hat{k}$

Now area of given triangle $= \dfrac{1}{2}\left|\vec{AB}\times\vec{AC}\right|$

$$= \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 3 & 3 & -4 \end{vmatrix} = \frac{1}{2}\left|(12-3)\hat{i} - (-4-3)\hat{j} + (3+9)\hat{k}\right|$$

$$= \frac{1}{2}\left|9\hat{i}+7\hat{j}+12\hat{k}\right| = \frac{1}{2}\sqrt{9^2+7^2+12^2} = \frac{\sqrt{274}}{2} \text{ sq. unit.}$$

20. Let $I = \displaystyle\int \frac{dx}{x(x^3+1)} = \int \frac{x^2\,dx}{x^3(x^3+1)}$

Let $x^3 = z$

$\Rightarrow \quad 3x^2\,dx = dz \qquad \Rightarrow x^2\,dx = \dfrac{dz}{3}$

$$\therefore \quad I = \frac{1}{3}\int \frac{dz}{z(z+1)} = \frac{1}{3}\int \frac{(z+1)-z}{z(z+1)}\,dz$$

$$= \frac{1}{3}\int\left[\frac{1}{z} - \frac{1}{z+1}\right]dz = \frac{1}{3}\int\frac{dz}{z} - \int\frac{dz}{z+1}$$

$$= \frac{1}{3}\log|z| - \frac{1}{3}\log|z+1| + C$$

$$= \frac{1}{3}\log\left|\frac{z}{z+1}\right| + C = \frac{1}{3}\log\left|\frac{x^3}{x^3+1}\right| + C$$

21. Given: $x\sin(a+y) + \sin a\cos(a+y) = 0$

$$\Rightarrow \quad x = -\frac{\sin a.\cos(a+y)}{\sin(a+y)} \quad \Rightarrow \quad x = -\sin a.\cot(a+y)$$

Differentiating w.r.t. y we get

$$\frac{dx}{dy} = +\sin a.\operatorname{cosec}^2(a+y) = \frac{\sin a}{\sin^2(a+y)}$$

$$\Rightarrow \qquad \frac{dy}{dx} \quad \frac{\sin^2(a+y)}{\sin a}$$

22. LHS $\Delta = \begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$

$$\Delta = \begin{vmatrix} x+y+z & -x+y & -x+z \\ x+y+z & 3y & z-y \\ x+y+z & y-z & 3z \end{vmatrix}$$

Taking out $(x+y+z)$ along C_1, we get

$$\Delta = (x+y+c)\begin{vmatrix} 1 & -x+y & -x+z \\ 1 & 3y & z-y \\ 1 & y-z & 3z \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1; R_3 \to R_3 - R_1$

$$\Delta = (x+y+z)\begin{vmatrix} 1 & -x+y & -x+z \\ 0 & 2y+x & x-y \\ 0 & x-z & x+2z \end{vmatrix}$$

Applying $C_2 \to C_2 - C_3$

$$\Delta = (x+\ y+z)\begin{vmatrix} 1 & y-z & -x+z \\ 0 & 3y & x-y \\ 0 & -3z & x+2z \end{vmatrix}$$

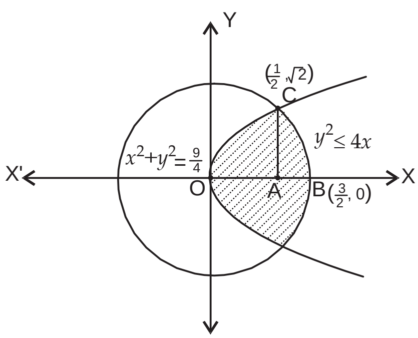
Expanding along I column, we get

$$\Delta = (x+y+z)[(3y\ (x+2z) + 3z\ (x-y)]$$
$$= 3(x+y+z)[xy+2z+2yz+xz-yz]$$
$$= 3(x+y+z)(xy+yz+zx) = \text{R.H.S.}$$

28. We have the region $\{(x, y); y^2 \le 4x, 4x^2 + 4y^2 \le 9\}$

i.e., $\quad y^2 \le 4x \quad$ ...(*i*)

$x^2 + y^2 \le \dfrac{9}{4} \quad$ ...(*ii*)



Clearly, (*i*) is a parabola and (*ii*) is a circle with centre at (0, 0) and radius $\frac{3}{2}$ units.

To find the intersection points of the circle and parabola, we put value of y^2 in (*ii*).

$$x^2 + 4x = \frac{9}{4}$$

$\Rightarrow \qquad 4x^2 + 16x - 9 = 0$

Y'

$$\Rightarrow \quad 4x^2 + 18x - 2x - 9 = 0$$

$$\Rightarrow \quad (2x - 1)(2x + 9) = 0$$

$\Rightarrow \qquad x = \frac{1}{2}, \ \frac{-9}{2}$

when $x = \frac{1}{2}$, $y = \pm\sqrt{2}$

$-\frac{9}{2}$ is not possible as y^2 cannot be –ve.

Required Area $= 2 \times$ Area of $OBCO$

$= 2 \times$ (Area of $OACO$ + Area of $ABCA$)

$$= 2\left[\int_0^{1/2} \sqrt{4x}\,dx + \int_{1/2}^{3/2} \sqrt{\frac{9}{4} - x^2}\,dx\right]$$

$$= 2\left[2\left[\frac{2}{3}x^{3/2}\right]_0^{1/2} + \left[\frac{x}{2}\sqrt{\frac{9}{4} - x^2} + \frac{9}{8}\sin^{-1}\frac{2x}{3}\right]_{1/2}^{3/2}\right]$$

$$= 2\left[\frac{4}{3}\left(\frac{1}{2}\right)^{3/2} + \frac{9\pi}{16} - \frac{1}{4}\sqrt{2} - \frac{9}{8}\sin^{-1}\frac{1}{3}\right]$$

$$= 2\left[\frac{2}{3\sqrt{2}} + \frac{9\pi}{16} - \frac{\sqrt{2}}{4} - \frac{9}{8}\sin^{-1}\frac{1}{3}\right] = \left[\frac{4}{3\sqrt{2}} + \frac{9\pi}{8} - \frac{\sqrt{2}}{2} - \frac{9}{4}\sin^{-1}\frac{1}{3}\right]$$

$$= \frac{1}{3\sqrt{2}} + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3} \text{ sq units.}$$

29. Given differential equation is

$$\frac{dx}{dy} + x\cot y = 2y + y^2\cot y$$

It is in the form of $\frac{dx}{dy} + Px = Q$

Here $P = \cot y$ $Q = 2y + y^2 \cot y$

$\therefore \qquad$ I.F. $= e^{\int \cot y\,dy} = e^{\log(\sin y)} = \sin y \qquad\qquad [\because e^{\log z} = z]$

Hence, general solution is

$$x.\sin y = \int (2y + y^2\cot y)\,.\,\sin y\,dy + C$$

$$= \int 2y\sin y\,dy + \int y^2\cot y\cdot\sin y\,dy + C$$

$$= 2\int y\sin y\,dy + \int y^2\cos y\,dy + C$$

$$= 2\left[\sin y.\frac{y^2}{2} - \int \cos y.\frac{y^2}{2}\,dy\right] + \int y^2\cos y\,dy + C$$

$$= y^2 \sin y - \int y^2 \cos y dy + \int y^2 \cos y \, dy + C$$

$$\Rightarrow \quad x.\sin y = y^2 \sin y + C$$

Putting $x = 0$ and $y = \frac{\pi}{2}$

$$0 = \frac{\pi^2}{4} \times 1 + C \quad \Rightarrow \quad C = -\frac{\pi^2}{4}$$

Hence, particular solution is

$$x.\sin y = y^2 \sin y - \frac{\pi^2}{4}.$$

zzz

CBSE Examination Papers
(Foreign–2013)

Time allowed: 3 hours **Maximum marks: 100**

General Instructions: *As given in CBSE Sample Question Paper.*

Set–I

SECTION–A

Question numbers 1 to 10 carry 1 mark each.

1. Write the principal value of $\tan^{-1}\left(\tan\frac{9\pi}{8}\right)$.

2. Write the value of $\sin\left(2\sin^{-1}\frac{3}{5}\right)$.

3. If A is a 3×3 matrix, whose elements are given by $a_{ij} = \frac{1}{3}\left|-3i + j\right|$, then write the value of a_{23}.

4. If A is a square matrix and $|A| = 2$, then write the value of $|AA'|$, where A' is the transpose of matrix A.

5. If $A = \begin{vmatrix} 3 & 10 \\ 2 & 7 \end{vmatrix}$, then write A^{-1}.

6. Write the differential equation formed from the equation $y = mx + c$, where m and c are arbitrary constants.

7. If $\vec{a}$ is a unit vector and $(\vec{x} - \vec{a})\cdot(\vec{x} + \vec{a}) = 24$, then write the value of $|\vec{x}|$.

8. For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, write the value of the following:
$\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b})$

9. Write the cartesian equation of a plane, bisecting the line segment joining the points A (2, 3, 5) and B (4, 5, 7) at right angles.

10. ***If $C = 0.003x^3 + 0.02x^2 + 6x + 250$ gives the amount of carbon pollution in air in an area on the entry of x number of vehicles, then find the marginal carbon pollution in the air, when 3 vehicles have entered in the area and write which value does the question indicate.***

SECTION–B

Question numbers 11 to 22 carry 4 marks each.

11. Prove that the relation R in the set $A = \{5, 6, 7, 8, 9\}$ given by $R = \{(a, b) : |a - b|$, is divisible by 2$\}$, is an equivalence relation. Find all elements related to the element 6.

12. If $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$, then find the value of x.

OR

If $y = \cot^{-1}\left(\sqrt{\cos x}\right) - \tan^{-1}\left(\sqrt{\cos x}\right)$, then prove that $\sin y = \tan^2\left(\frac{x}{2}\right)$.

13. Using properties of determinants, prove the following:

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2$$

14. Differentiate the following with respect to x:

$x^{\sin x} + (\sin x)^{\cos x}$

15. If $y = \sin(\log x)$, then prove that

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

16. Show that the function $f(x) = 2x - |x|$ is continuous but not differentiable at $x = 0$.

OR

Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\tan^{-1} x$, when $x \neq 0$.

17. Evaluate: $\int \frac{\sin x + \cos x}{9 + 16 \sin 2x}\, dx$

OR

Evaluate: $\int x^2 \log(1 + x)\, dx$

18. Evaluate: $\int \frac{x \tan x}{\sec x + \tan x}\, dx$

19. The magnitude of the vector product of the vector $\hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to $\sqrt{2}$. Find the value of λ.

20. Evaluate: $\int \frac{x^2+1}{x^2 - 5x + 6}\, dx$

21. Find the shortest distance between the following lines:

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1};\ \frac{3-x}{-1} = \frac{y-5}{-2} = \frac{z-7}{1}$$

OR

Find the equation of the plane through the points $(2, 1, -1)$ and $(-1, 3, 4)$ and perpendicular to the plane $x - 2y + 4z = 10$.

22. ***In a group of 50 scouts in a camp, 30 are well trained in first aid techniques while the remaining are well trained in hospitality but not in first aid. Two scouts are selected at random from the group. Find the probability distribution of number of selected scouts who are well trained in first aid. Find the mean of the distribution also.***

Write one more value which is expected from a well trained scout.

SECTION–C

Question numbers 23 to 29 carry 6 marks each.

23. ***10 students were selected from a school on the basis of values for giving awards and were divided into three groups. The first group comprises hard workers, the second group has honest and law abiding students and the third group contains vigilant and obedient students. Double the number of students of the first group added to the number in the second group gives 13, while the combined strength of first and second group is four times that of the third group. Using matrix method, find the number of students in each group. Apart from the values, hard work, honesty and respect for law, vigilance and obedience, suggest one more value, which in your opinion, the school should consider for awards.***

24. Prove that the volume of the largest cone, that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

OR

Show that the normal at any point θ to the curve $x = a\cos\theta + a\theta\sin\theta$, $y = a\sin\theta - a\theta\cos\theta$ is at a constant distance from the origin.

25. Find the area enclosed by the parabola $4y = 3x^2$ and the line $2y = 3x + 12$.

26. Find the particular solution of this differential equation $x^2\frac{dy}{dx} - xy = 1 + \cos\left(\frac{y}{x}\right), x \neq 0$. Find the particular solution of this differential equation, given that when $x = 1$, $y = \frac{\pi}{2}$.

27. Find the image of the point having position vector $\hat{i} + 3\hat{j} + 4\hat{k}$ in the plane $\vec{r}\cdot(2\hat{i} - \hat{j} + \hat{k}) + 3 = 0.$

OR

Find the equation of a plane which is at a distance of $3\sqrt{3}$ units from origin and the normal to which is equally inclined to the coordinate axes.

28. ***An aeroplane can carry a maximum of 200 passengers. A profit of ₹500 is made on each executive class ticket out of which 20% will go to the welfare fund of the employees. Similarly a profit of ₹400 is made on each economy ticket out of which 25% will go for the improvement of facilities provided to economy class passengers. In both cases, the remaining profit goes to the airline's fund. The airline reserves at least 20 seats for executive class. However at least four times as many passengers prefer to travel by economy class than by the executive class. Determine how many tickets of each type must be sold in order to maximise the net profit of the airline. Make the above as an LPP and solve graphically.***

Do you think, more passengers would prefer to travel by such an airline than by others?

29. ***Often it is taken that a truthful person commands, more respect in the society. A man is known to speak the truth 4 out of 5 times. He throws a die and reports that it is actually a***

six. Find the probability that it is actually a six. Do you also agree that the value of truthfulness leads to more respect in the society?

Set–II

Only those questions, not included in Set I, are given

9. If $\vec{p}$ is a unit vector and $(\vec{x}-\vec{p})\cdot(\vec{x}+\vec{p})=48$, then write the value of $|\vec{x}|$.

10. Write the principal value of $\tan^{-1}\left(\tan\dfrac{7\pi}{6}\right)$.

19. Differentiate the following with respect to x:

$$(\sin x)^x + (\cos x)^{\sin x}$$

20. Find a vector of magnitude 6, perpendicular to each of the vectors

$\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$.

21. Prove that the relation R in the set $A=\{1, 2, 3, \ldots, 12\}$ given by $R=\{(a, b):|a-b|$ is divisible by $3\}$, is an equivalence relation. Find all elements related to the element 1.

22. Evaluate:

$$\int \frac{1-x^2}{x-2x^2}\,dx.$$

28. Find the area of the region bounded by the parabola $y^2=2x$ and the line $x-y=4$.

29. Show that the differential equation $(x-y)\dfrac{dy}{dx}=(x+2y)$ is homogeneous and solve it.

Set–III

Only those questions, not included in Set I and Set II are given.

9. Write $\cot^{-1}\left(\dfrac{1}{\sqrt{x^2-1}}\right)$, $|x|>1$ in simplest form.

10. If $\vec{a}$ is a unit vector and $(2\vec{x}-3\vec{a})\cdot(2\vec{x}+3\vec{a})=91$, then write the value of $|\vec{x}|$.

19. Evaluate: $\displaystyle\int \frac{2x^2+3}{x^2+5x+6}\,dx.$

20. Using properties of determinants, prove the following:

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$$

21. Find a unit vector perpendicular to each of the vectors $\vec{a}+2\vec{b}$ and

$2\vec{a}+\vec{b}$, where $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$.

22. Differentiate the following with respect to x:

$$\tan^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right),\ 0<x<\frac{\pi}{2}$$

28. Find the area of the region bounded by the two parabolas $y^2=4ax$ and $x^2=4ay$, when $a>0$.

29. Show that the differential equation $2y\, e^{x/y} dx + (y - 2x\, e^{x/y})\, dy = 0$ is homogeneous. Find the particular solution of this differential equation, given that when $y = 1$, $x = 0$.

Solutions

Set–I

SECTION–A

1. $\tan^{-1}\left(\tan\frac{9\pi}{8}\right) = \tan^{-1}\left(\tan\left(\pi+\frac{\pi}{8}\right)\right)$

$= \tan^{-1}\left(\tan\frac{\pi}{8}\right) = \frac{\pi}{8}$ $\qquad \left[\because \frac{\pi}{8}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right]$

2. Let $\sin\left(2\sin^{-1}\frac{3}{5}\right) = \theta$

$\Rightarrow \quad 2\sin^{-1}\frac{3}{5} = \sin^{-1}\theta$

$\Rightarrow \quad \sin^{-1}\left\{2\times\frac{3}{5}\sqrt{1-\frac{9}{25}}\right\} = \sin^{-1}\theta$ $\qquad \left[\because 2\sin^{-1}x = \sin^{-1}\{2x\sqrt{1-x^2}\}\right]$

$\Rightarrow \quad \sin^{-1}\left\{\frac{6}{5}\times\frac{4}{5}\right\} = \sin^{-1}\theta \quad \Rightarrow \quad \sin^{-1}\left(\frac{24}{25}\right) = \sin^{-1}\theta$

$\Rightarrow \quad \theta = \frac{24}{25} \quad \Rightarrow \quad \sin\left(2\sin^{-1}\frac{3}{5}\right) = \frac{24}{25}$

3. $a_{23} = \frac{1}{3}|-3\times 2+3| = \frac{1}{3}|-6+3|$

$= \frac{1}{3}\times 3 = 1$

4. $|AA'| = |A|.|A'| = |A|.|A| = |A|^2 = 2^2 = 4.$

[Note: $|AB| = |A|.|B|$ and $|A| = |A^T|$, where A and B are square matrices.]

5. Here $A = \begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix}$

$\therefore \quad \text{Adj } A = \begin{bmatrix} 7 & -2 \\ -10 & 3 \end{bmatrix}^T = \begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix}$

Also $|A| = 21 - 20 = 1 \neq 0$

$\therefore \quad A^{-1} = \frac{1}{|A|}\begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix} = \frac{1}{1}\begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix}$

6. Here $y = mx + c$

Differentiating, we get $\quad \frac{dy}{dx} = m$

d^2

Again, differentiating we get

$$\frac{y}{dx^2} = 0 \text{, which is the required differential equation}$$

7. Given: $\left(\vec{x}-\vec{a}\right).\left(\vec{x}+\vec{a}\right)=24$

$\Rightarrow \quad \vec{x}.\vec{x}+\vec{x}.\vec{a}-\vec{a}.\vec{x}-\vec{a}.\vec{a}=24$

$\Rightarrow \quad |\vec{x}|^2-|\vec{a}|^2=24 \qquad \Rightarrow \qquad (\vec{x})^2=25 \qquad \left[\because |\vec{a}|=1\right]$

$\Rightarrow \quad |\vec{x}|=5$

8. $\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})$

$=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}+\vec{c}\times\vec{b}$

$=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}-\vec{a}\times\vec{b}-\vec{a}\times\vec{c}-\vec{b}\times\vec{c}=0$

9. One point of required plane = mid point of given line segment.

$$=\left(\frac{2+4}{2},\frac{3+5}{2},\frac{5+7}{2}\right)=(3,4,6)$$

Also D.r's of Normal to the plane = 4 –2, 5–3, 7–5

= 2, 2, 2

Therefore, required equation of plane is

$2(x-3)+2(y-4)+2(z-6)=0$

$2x+2y+2z=26$ or $x+y+z=13$

10. We have to find *i.e.* $[C'(x)]_{x=3}$

Now $C(x)=0.003x^3+0.02x^2+6x+250$

$\therefore \quad C'(x)=0.009x^2+0.04x+6$

$\left[C'(x)\right]_{x=3}=0.009\times 9+0.04\times 3+6$

$=0.081+0.12+6=6.201$

This question indicates "how increment of vehicles increase the carbon pollution in air, which is harmful for creature.

SECTION–B

11. Here R is a relation defined as

$R=\{(a,b):|a-b| \text{ is divisible by } 2\}$

Reflexivity

Here $(a,a)\in R$ as $|a-a|=|0|=0$ divisible by 2 *i.e.*, R is reflexive.

Symmetry

Let $(a,b)\in R$

$(a,b)\in R \quad \Rightarrow |a-b|$ is divisible by 2

$\Rightarrow a-b=\pm 2m \qquad \Rightarrow b-a=\mp 2m$

$\Rightarrow |b-a|$ is divisible by 2 $\qquad \Rightarrow (b,a)\in R$

Hence R is symmetric

Transitivity Let $(a,b),(b,c)\in R$

Now, $(a, b), (b, c) \in R \quad \Rightarrow |a-b|, |b-c|$ are divisible by 2

$\Rightarrow a - b = \pm 2m$ and $b - c = \pm 2n$

$\Rightarrow a - b + b - c = \pm 2(m + n)$ [
$\Rightarrow (a - c) = \pm 2k$ Qk
$= m + n]$ $\Rightarrow (a - c) = 2k$
$\Rightarrow (a - c)$ is divisible by 2 $\Rightarrow (a, c) \in R$.

Hence R is transitive.

Therefore, R is an equivalence relation.

The elements related to 6 are 6, 8.

12. Refer to Q 21, Page 49.

OR

Given $y = \cot^{-1}\left(\sqrt{\cos x}\right) - \tan^{-1}\left(\sqrt{\cos x}\right)$

$\Rightarrow y = \frac{\pi}{2}$ $- h$

$- \tan^{-1}\left(\cos x\right) - \tan^{-1}\left(\cos x\right) \Rightarrow y = 2$ $\}$

$- 2\tan^{-1}\left(\cos x\right)$ $1 + \cos x$ $\Rightarrow$ $=$

lim

$\Rightarrow y = \sin^{-1}\left(\frac{1 - \cos x}{}\right)$ $(2h$

$\Rightarrow \sin y = \frac{1 - \cos x}{1 + \cos x}$ $\Rightarrow$ $- h)$

$\Rightarrow \sin y = \tan^2 x/2$ 2 $= \lim h$

| **Note:** $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, \quad x \in R$ |

$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, $1 - x^2$

$x \in [-1, 1]$ and $2\tan^{-1} x = \cos^{-1} \frac{1 - x^2}{1 + x^2}, x \geq 0$

13. Refer to Q 6, Page 101.

14. Refer to Q 38, Page 188.

15. Refer to Q 56, Page 198.

16. Here $f(x) = 2x - |x|$ $h \to 0$

For continuity at $x = 0$

$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h)$

$= \lim_{h \to 0} f(h)$ $h \to 0$

$= \lim_{h \to 0} \{2h \quad | \; |$

π $\quad {}_{-1}\left(1-\cos x\right) 2 \quad \left(1 + \cos x\right)$

$y = - - \cos \quad \left| \overline{\qquad\qquad} \right|$

$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$

$2 \sin y = \dfrac{2\sin x/2}{2\cos^2}$

$x\ 2$

$= \lim f(-h)$

$= \lim_{h\to 0}\left\{2(-h) \quad | \quad |\right.$

$\left.- -h\right\} \quad = \lim\{\}2h$

$- h$

$= \lim_{h\to 0}(-3h)$

$h\to 0$

$= 0 \qquad ....(i)$

$= 0 \qquad ....(ii)$

Also $f(0) = 2 \times 0 - |0| = 0$ ....(*iii*)

(*i*), (*ii*) and (*iii*) $\Rightarrow \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x) = f(0)$

Hence $f(x)$ is continuous at $x = 0$

For differentiability at $x = 0$

$$\text{L.H.D.} = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{\{2(-h) - |h|\} - \{2 \times 0 - |0|\}}{-h} = \lim_{h \to 0} \frac{-2h - h - 0}{-h}$$

$$= \lim_{h \to 0} \frac{-3h}{-h} = \lim_{h \to 0} 3$$

$$\text{L.H.D.} = 3 \qquad \text{....}(iv)$$

Again

$$\text{R.H.D.} = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{2h - |h| - 2 \times 0 - |0|}{h}$$

$$= \lim_{h \to 0} \frac{2h - h}{h} = \lim_{h \to 0} \frac{h}{h}$$

$$= \lim_{h \to 0} 1.$$

$$\text{R.H.D.} = 1 \qquad \text{...}(v)$$

From (*iv*) and (*v*)

L.H.D. $\neq$ R.H.D.

Hence, function $f(x) = 2x - |x|$ is not differentiable at $x = 0$

i.e., $f(x)$ is continuous but not differentiable at $x = 0$.

OR

Let $u = \tan^{-1}\left(\dfrac{\sqrt{1 + x^2} - 1}{x}\right)$

$v = \tan^{-1} x$

We have to find $\dfrac{du}{dv}$

Now, $u = \tan^{-1}\left(\dfrac{\sqrt{1 + x^2} - 1}{x}\right)$

Let $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$

$$\therefore \quad u = \tan^{-1}\left[\frac{\sqrt{1 + \tan^2\theta} - 1}{\tan\theta}\right] = \tan^{-1}\left[\frac{\sec\theta - 1}{\tan\theta}\right]$$

t an θ

$$= \tan^{-1}\left[\frac{\frac{1}{\cos\theta} - 1}{\frac{\sin\theta}{\cos\theta}}\right] = \tan^{-1}\left[\frac{1 - \cos\theta}{\cos\theta} \cdot \frac{\cos\theta}{\sin\theta}\right]$$

$$= \tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right] = \tan^{-1}\left[\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right]$$

$$= \tan^{-1}\left[\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right] = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$$

$\therefore \qquad u = \frac{1}{2}\tan^{-1} x$

Differentiating, both sides w.r.t. x we get

$$\frac{du}{dx} = \frac{1}{2(1+x^2)} \qquad \ldots(i)$$

Also, $v = \tan^{-1} x$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{1+x^2} \qquad \ldots(ii)$$

$$\therefore \qquad \frac{du}{dv} = \frac{1}{2(1+x^2)} \times \frac{1+x^2}{1} = \frac{1}{2}$$

17. Let $\quad I = \int \frac{\sin x + \cos x}{9 + 16\sin 2x}\, dx$

Let $\quad \sin x - \cos x = z \quad \Rightarrow \quad (\cos x + \sin x)dx = dz$

Also $\because (\sin x - \cos x) = z \quad \Rightarrow \quad (\sin x - \cos x)^2 = z^2$

$\Rightarrow \qquad (\sin^2 x + \cos^2 x - 2\sin x . \cos x) = z^2$

$\Rightarrow \qquad 1 - \sin 2x = z^2$

$\Rightarrow \qquad \sin 2x = 1 - z^2$

$$\therefore \qquad I = \int \frac{dz}{9 + 16(1-z^2)} = \int \frac{dz}{25 - 16z^2}$$

$$= \frac{1}{16}\int \frac{dz}{\left(\frac{5}{4}\right)^2 - z^2} = \frac{1}{16} . \frac{1}{2 \times \frac{5}{4}} . \log\left|\frac{\frac{5}{4}+z}{\frac{5}{4}-z}\right| + C$$

$$= \frac{1}{40}\log\left|\frac{5+4z}{5-4z}\right| + C = \frac{1}{40}\log\left|\frac{5+4(\sin x - \cos x)}{5-4(\sin x - \cos x)}\right| + C$$

OR

Let $I = \int x^2 \log(1+x)dx$

$$= \log(1+x). \frac{x^3}{3} - \int \frac{1}{1+x} . \frac{x^3}{3}\, dx$$

$$= \log(1+x). \frac{x^3}{3} - \frac{1}{3}\int \frac{x^3}{x+1}\, dx$$

$$= \frac{x^3 \log(1+x)}{3} - \frac{1}{3}\int \left\{(x^2 - x + 1) - \frac{1}{x+1}\right\} dx$$

$$= \frac{x^3 . \log(1+x)}{3} - \frac{1}{3}\int \left(x^2 - x + 1\right)dx + \frac{1}{3}\int \frac{dx}{x+1}$$

$$= \frac{x^3 \log(1+x)}{3} - \frac{x^3}{9} + \frac{x^2}{6} - \frac{x}{3} + \frac{1}{3}\log(x+1) + C$$

18. Refer to Q. 18, Page 299.

19. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$; $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$; $\vec{c} = \lambda\hat{i} + 2\hat{j} + 3\hat{k}$

From question

$$\left|\vec{a} \times \frac{\vec{b} + \vec{c}}{|\vec{b} + \vec{c}|}\right| = \sqrt{2} \Rightarrow \left|\frac{\vec{a} \times (\vec{b} + \vec{c})}{|\vec{b} + \vec{c}|}\right| = \sqrt{2}$$

$$\vec{b} + \vec{c} = (2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}$$

$$\therefore \quad |\vec{b} + \vec{c}| = \sqrt{(2+\lambda)^2 + 6^2 + (-2)^2}$$

$$= \sqrt{4 + \lambda^2 + 4\lambda + 36 + 4}$$

$$= \sqrt{\lambda^2 + 4\lambda + 44}$$

$$\Rightarrow \quad \vec{a} \times (\vec{b} + \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2+\lambda & 6 & -2 \end{vmatrix}$$

$$= (-2-6)\hat{i} - (-2-2-\lambda)\hat{j} + (6-2-\lambda)\hat{k}$$

$$= -8\hat{i} + (4+\lambda)\hat{j} + (4-\lambda)\hat{k}$$

Putting it in (i), we get

$$\left|\frac{-8\hat{i} + (4+\lambda)\hat{j} + (4-\lambda)\hat{k}}{\sqrt{\lambda^2 + 4\lambda + 44}}\right| = \sqrt{2} \quad \Rightarrow \quad \frac{\sqrt{(-8)^2 + (4+\lambda)^2 + (4-\lambda)^2}}{\sqrt{\lambda^2 + 4\lambda + 44}} = \sqrt{2}$$

Squaring both sides we get

$$\frac{64 + 16 + \lambda^2 + 8\lambda + 16 + \lambda^2 - 8\lambda}{\lambda^2 + 4\lambda + 44} = 2 \quad \Rightarrow \quad \frac{96 + 2\lambda^2}{\lambda^2 + 4\lambda + 44} = 2$$

$$\Rightarrow \quad 8\lambda = 8 \quad \Rightarrow \quad \lambda = 1$$

20. Let $I = \int \frac{x^2 + 1}{x^2 - 5x + 6}\,dx$

$$= \int \left(1 + \frac{5x - 5}{x^2 - 5x + 6}\right)dx$$

$$\begin{array}{r} 1 \\ x^2 - 5x + 6 \,\big)\, x^2 + 1 \\ x^2 - 5x + 6 \\ \hline 5x - 5 \end{array}$$

$$= \int dx + \int \frac{5x-5}{x^2-5x+6}\,dx = x + \int \frac{5x-5}{(x^2-3x-2x+6)}\,dx$$

$$= x + \int \frac{5x-5}{x(x-3)-2(x-3)}\,dx = x + \int \frac{5x-5}{(x-3)(x-2)}\,dx$$

$$I = x + I_1 \qquad \text{....}(i)$$

Now $I_1 = \int \frac{5x-5}{(x-3)(x-2)}\,dx$

Let $\frac{5x-5}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2} \quad \Rightarrow \quad 5x-5 = A(x-2) + B(x-3)$

If $x = 2 \Rightarrow 5 = -B \Rightarrow B = -5$

If $x = 3 \Rightarrow 10 = A \Rightarrow A = 10$

$$\therefore \quad \frac{5x-5}{(x-3)(x-2)} = \frac{10}{x-3} + \frac{-5}{x-2}$$

$$\therefore \quad I_1 = \int \left(\frac{10}{x-3} - \frac{5}{x-2}\right)dx$$

$$= 10\log|x-3| - 5\log|x-2| + C$$

$$\Rightarrow \quad I = x + 10\log|x-3| - 5\log|x-2| + C \quad (\text{using } (i))$$

21. Refer to Q 13, Page 437.

OR

Let the equation of required plane be

$$a(x-2) + b(y-1) + c(z+1) = 0$$

Q $\quad (i)$ passes through $(-1,3,4)$ also

$$a(-1-2) + b(3-1) + c(4+1) = 0$$

$$-3a + 2b + 5c = 0$$

Again, Qplane (i) is perpendicular to plane $x - 2y + 4z = 10$

$a - 2b + 4c = 0$

From (ii) and (iii)

$$\frac{a}{8+10} = \frac{b}{5+12} = \frac{c}{6-2}$$

$$\frac{a}{18} = \frac{b}{17} = \frac{c}{4} = \lambda$$

$\therefore \quad a = 18\lambda, b = 17\lambda, c$

$= 4\lambda$ Putting in (i) we get

$$18\lambda(x-2) + 17\lambda(y-1) + 4\lambda(z+1) = 0$$

$$\Rightarrow 18(x-2) + 17(y-1) + 4(z+1) = 0$$

$$\Rightarrow 18x + 17y + 4z - 36 - 17 + 4 = 0$$

$$\Rightarrow 18x + 17y + 4z = 49$$

22. Let X be no. of selected scouts who are well trained in first aid. Here random variable X may have value 0, 1, 2.

Now $P(X=0) = \dfrac{{}^{20}C_2}{{}^{50}C_2} = \dfrac{20 \times 19}{50 \times 49} = \dfrac{38}{245}$

$P(X=1) = \dfrac{{}^{20}C_1 \times {}^{30}C_1}{{}^{50}C_2} = \dfrac{20 \times 30 \times 2}{50 \times 49} = \dfrac{120}{245}$

$\text{P}(X=2) = \dfrac{{}^{30}C_2}{{}^{50}C_2} = \dfrac{30 \times 29}{50 \times 49} = \dfrac{87}{245}$

Now distribution table is as

X	0	1	2
P(x)	$\dfrac{38}{245}$	$\dfrac{120}{245}$	$\dfrac{87}{245}$

Now Mean $= \sum x_i p_i = 0 \times \dfrac{38}{245} + 1 \times \dfrac{120}{245} + 2 \times \dfrac{87}{245}$

$= \dfrac{120}{245} + \dfrac{174}{245} = \dfrac{294}{245}$

A well trained scout should be disciplined

23. Let no. of students in Ist, 2nd and 3rd group to x, y, z respectively.

From question

$$x + y + z = 10$$
$$2x + y = 13$$
$$x + y - 4z = 0$$

The above system of linear equations may be written in matrix form as

$AX = B$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 10 \\ 13 \\ 0 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{vmatrix} = 1(-4-0) - 1(-8-0) + 1(2-1) = -4 + 8 + 1 = 5 \neq 0$$

Now $A_{11} = (-1)^{1+1}\begin{vmatrix} 1 & 0 \\ 1 & -4 \end{vmatrix} = -4 - 0 = -4$

$A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 0 \\ 1 & -4 \end{vmatrix} = -(-8-0) = 8;$ $A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2 - 1 = 1$

$A_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 1 \\ 1 & -4 \end{vmatrix} = -(-4-1) = 5;$ $A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 1 & -4 \end{vmatrix} = -4 - 1 = -5$

$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = -(1-1) = 0;$ $A_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -(0 - 2) = 2; \qquad A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1 - 2 = -1$$

$$\therefore \quad \text{Adj } A = \begin{bmatrix} -4 & 8 & 1 \\ 5 & -5 & 0 \\ -1 & 2 & -1 \end{bmatrix}^T = \begin{bmatrix} -4 & 5 & -1 \\ 8 & -5 & 2 \\ 1 & 0 & -1 \end{bmatrix}$$

Now $AX = B \Rightarrow X = A^{-1} B$.

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -4 & 5 & -1 \\ 8 & -5 & 2 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 10 \\ 13 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -40 + 65 \\ 80 - 65 \\ 10 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 25 \\ 15 \\ 10 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix} \qquad \Rightarrow \qquad x = 5, y = 3, z = 2$$

Apart from these values, the school should consider "disciplined behaviour" for awards.

24. Refer to Q 3, Page 235.

OR

Given $\quad x = a\cos\theta + a\theta\sin\theta$

$\quad y = a\sin\theta - a\theta\cos\theta$

$$\therefore \quad \frac{dx}{d\theta} = -a\sin\theta + a(\theta\cos\theta + \sin\theta)$$

$$= -a\sin\theta + a\theta\cos\theta + a\sin\theta = a\theta\cos\theta$$

and $$\frac{dy}{d\theta} = a\cos\theta - a(-\theta\sin\theta + \cos\theta)$$

$$= a\cos\theta + a\theta\sin\theta - a\cos\theta = a\theta\sin\theta$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

$\therefore$ Slope of tangent at $\theta = \tan\theta$

$$\Rightarrow \quad \text{Slope of normal at } \theta = -\frac{1}{\tan\theta} = -\cot\theta$$

Hence equation of Normal at θ is

$$\frac{y - (a\sin\theta - a\theta\cos\theta)}{x - (a\cos\theta + a\theta\sin\theta)} = -\cot\theta$$

$$\Rightarrow \quad y - a\sin\theta + a\theta\cos\theta + x\cot\theta - \cot\theta\,(a\cos\theta + a\theta\sin\theta) = 0$$

$$\Rightarrow \quad y - a\sin\theta + a\theta\cos\theta + x\frac{\cos\theta}{\sin\theta} - a\frac{\cos^2\theta}{\sin\theta} - a\theta\cos\theta = 0$$

$$\Rightarrow \quad x\cos\theta + y\sin\theta - a = 0 \qquad \ldots(i)$$

$$\text{Distance from origin (0, 0) to } (i) = \left|\frac{0.\cos\theta + 0.\sin\theta - a}{\sqrt{\cos^2\theta + \sin^2\theta}}\right| = a$$

25. Refer to Q 23, Page 339.

dy $\quad y$

26. Given $x^2 \frac{\quad}{dx} - xy = 2\cos^2\left(\frac{\quad}{2x}\right)$, $x \neq 0$

$$\Rightarrow \quad \frac{x^2\dfrac{dy}{dx} - xy}{2\cos^2\left(\dfrac{y}{2x}\right)} = 1 \qquad\qquad \Rightarrow \quad \frac{\sec^2\left(\dfrac{y}{2x}\right)}{2}\left[x^2\frac{dy}{dx} - xy\right] = 1$$

Dividing both sides by x^3

$$\Rightarrow \quad \frac{\sec^2\left(\dfrac{y}{2x}\right)}{2}\cdot\left[\frac{1}{x}\cdot\frac{dy}{dx} - \frac{y}{x^2}\right] = \frac{1}{x^3} \qquad\qquad \Rightarrow \quad \frac{d}{dx}\left[\tan\left(\frac{y}{2x}\right)\right] = x^{-3}$$

Integrating both sides w.r.t. x we get.

$$\int \frac{d}{dx}\left[\tan\left(\frac{y}{2x}\right)\right]dx = \int x^{-3}dx$$

$$\Rightarrow \quad \tan\left(\frac{y}{2x}\right) = \frac{x^{-3+1}}{-3+1} + C \qquad\qquad \Rightarrow \quad \tan\left(\frac{y}{2x}\right) = -\frac{1}{2}x^{-2} + C$$

$$\Rightarrow \quad \tan\left(\frac{y}{2x}\right) = -\frac{1}{2x^2} + C$$

For particular solution when $x = 1$, $y = \dfrac{\pi}{2}$, we have

$$\tan\left(\frac{\pi}{4}\right) = -\frac{1}{2} + C$$

$$\Rightarrow \quad 1 + \frac{1}{2} = C \qquad\qquad \Rightarrow \quad C = \frac{3}{2}$$

Hence Particular Solution is

$$\tan\left(\frac{y}{2x}\right) = -\frac{1}{2x^2} + \frac{3}{2}$$

27. Refer to Q 6, Page 450.

OR

Since, the required plane is at $3\sqrt{3}$ unit distance from the origin and its normal is equally inclined to the coordinate axes.

$$\Rightarrow \quad d = 3\sqrt{3}$$

and Normal vector of required plane $= l\hat{i} + m\hat{j} + n\hat{k}$ where

$$l = \cos\frac{\pi}{4}, = \frac{1}{\sqrt{2}}; \qquad m = \cos\frac{\pi}{4}, = \frac{1}{\sqrt{2}}; \qquad n = \cos\frac{\pi}{4}, = \frac{1}{\sqrt{2}}$$

$\therefore \quad \vec{n}$ (normal unit vector of plane)

$$= \frac{\dfrac{1}{\sqrt{2}}\hat{i} + \dfrac{1}{\sqrt{2}}\hat{j} + \dfrac{1}{\sqrt{2}}\hat{k}}{\sqrt{\left(\dfrac{1}{\sqrt{2}}\right)^2 + \left(\dfrac{1}{\sqrt{2}}\right)^2 + \left(\dfrac{1}{\sqrt{2}}\right)^2}}$$

$$= \frac{\sqrt{2}}{\sqrt{3}}\left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}\right)$$

$$= \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$$

k

Hence equation of required plane

$$\vec{r} \cdot \vec{n} = d$$

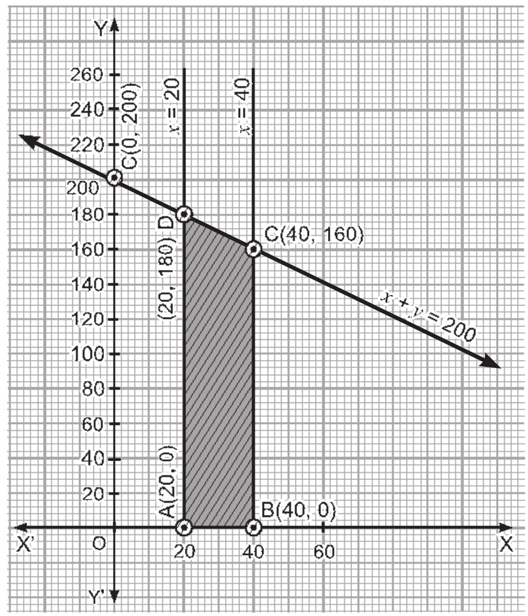
$$\vec{r} \cdot \left(\frac{1}{\sqrt{3}} \hat{i} + \frac{1}{\sqrt{3}} \hat{j} + \frac{1}{\sqrt{3}} \hat{k} \right) = 3\sqrt{3}$$

$$\Rightarrow \quad \vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 3\sqrt{3} \times \sqrt{3}$$

$$\Rightarrow \quad (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) = 9$$

$$\Rightarrow \quad x + y + z = 9$$

28. Let there be x tickets of executive class and y tickets of economy class. Let Z be net profit of the airline.



Here, we have to maximise z.

Now $\quad Z = 500x \times \frac{80}{100} + 400y \times \frac{75}{100}$

$Z = 400x + 300y \quad$ ....(*i*)

According to question

$x \geq 20 \quad$ ....(*ii*)

Also $\quad x + y \leq 200 \quad$ ....(*iii*)

$\Rightarrow \quad x + 4x \leq 200$

$\Rightarrow \quad 5x \leq 200$

$\Rightarrow \quad x \leq 40 \quad$ ....(*iv*)

Shaded region is feasible region having corner points A (20, 0), B (40,0) C (40, 160), D (20,180)

Now value of Z is calculated at corner point as

Corner points	$Z = 400x + 300y$
(20, 0)	8,000
(40, 0)	16,000
(40, 160)	64,000 ← Maximum
(20, 180)	60,000

Hence, 40 tickets of executive class and 160 tickets of economy class should be sold to maximise the net profit of the airlines.

Yes, more passengers would prefer to travel by such an airline, because some amount of profit is invested for welfare fund.

29. Let E_1, E_2 and E be three events such that

E_1 = six occurs

E_2 = six does not occurs

E = man reports that six occurs in the throwing of the dice.

1 5

Now $P\,(E_1) = \frac{}{6}$, $P\,(E_2) = \frac{}{6}$

$$P\left(\frac{E}{E_1}\right)=\frac{4}{5},\ P\left(\frac{E}{E_2}\right)=1-\frac{4}{5}=\frac{1}{5}$$

We have to find $P\left(\frac{E_1}{E}\right)$

$$P\left(\frac{E_1}{E}\right)=\frac{P(E_1).P\left(\frac{E}{E_1}\right)}{P(E_1).P\left(\frac{E}{E_1}\right)+P(E_2).P\left(\frac{E}{E_2}\right)}$$

$$=\frac{\frac{1}{6}\times\frac{4}{5}}{\frac{1}{6}\times\frac{4}{5}+\frac{5}{6}\times\frac{1}{5}}=\frac{4}{30}\times\frac{30}{4+5}=\frac{4}{9}$$

Everybody trust a truthful person, so he receives respect from everyone.

SET–II

9. Given Q $(\vec{x}-\vec{p}).(\vec{x}+\vec{p})=48$

$$\Rightarrow \quad \vec{x}.\vec{x}+\vec{x}.\vec{p}-\vec{p}.\vec{x}-\vec{p}.\vec{p}=48$$

$$\Rightarrow \quad |\vec{x}|^2-1=48 \quad \Rightarrow \quad |\vec{x}|^2=49$$

$$\Rightarrow \quad |\vec{x}|=7$$

10. $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)=\tan^{-1}\left(\tan\left(\pi+\frac{\pi}{6}\right)\right)$

$$=\tan^{-1}\left(\tan\frac{\pi}{6}\right)=\frac{\pi}{6}\left[\text{Q}\frac{\pi}{6}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right]$$

19. Let $u=(\sin x)^x$ and $v=(\cos x)^{\sin x}$

$\therefore$ Given differential equation becomes

$$y=u+v$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$$

Now $u=(\sin x)^x$

Taking log on both sides, we have

$$\log u=x\log\sin x$$

Differentiating w.r.t.x, we get

$$\frac{1}{u}\frac{du}{dx}=x\frac{1}{\sin x}.\cos x+\log\sin x$$

$$\Rightarrow \quad \frac{du}{dx}=u(x\cot x+\log\sin x)$$

$\}$

$\Rightarrow \quad \frac{du}{dx}=(\sin x)^x\{x\cot x+\log\sin x$

Again $v=(\cos x)^{\sin x}$

....(i)

....(ii)

Taking log on both sides we get

$$\log v = \sin x . \log \cos x$$

Differentiating both sides w.r.t.x, we get

$$\frac{1}{v}.\frac{dv}{dx} = \sin x . \frac{1}{\cos x}(-\sin x) + \log(\cos x).\cos x$$

$$\Rightarrow \quad \frac{dv}{dx} = v\left\{-\frac{\sin^2 x}{\cos x} + \cos x . \log \cos x\right\}$$

$$= (\cos x)^{\sin x}\left\{\cos x . \log(\cos x) - \frac{\sin^2 x}{\cos x}\right\}$$

$$\frac{dv}{dx} = (\cos x)^{1+\sin x}\{\log(\cos x) - \tan^2 x\} \qquad ....(iii)$$

From (i), (ii) and (iii)

$$\frac{dy}{dx} = (\sin x)^x\{x \cot x + \log \sin x\} + (\cos x)^{1+\sin x}\{\log(\cos x) - \tan^2 x\}$$

20. $\vec{a} + \vec{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$\vec{a} - \vec{b} = -\hat{j} - 2\hat{k}$

Now vector perpendicular to $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ is

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix} = (-6+4)\hat{i} - (-4-0)\hat{j} + (-2-0)\hat{k} = -2\hat{i} + 4\hat{j} - 2\hat{k}$$

$$\therefore \quad \text{Required vector} = \pm 6\frac{(-2\hat{i} + 4\hat{j} - 2\hat{k})}{\sqrt{(-2)^2 + 4^2 + (-2)^2}}$$

$$= \pm\frac{6}{\sqrt{24}}(-2\hat{i} + 4\hat{j} - 2\hat{k})$$

$$= \pm\frac{6}{2\sqrt{6}}(-2\hat{i} + 4\hat{j} - l\hat{k}) = \pm\sqrt{6}\,(-\hat{i} + 2\hat{j} - \hat{k})$$

21. We have the relation

$$R = \{(a, b) : |a - b| \text{ is divisible by } 3\}$$

We discuss the following properties of relation R on set A.

Reflexivity

For any $a \in A$ we have

$|a - a| = 0$ which is divisible by 3

$\Rightarrow \quad (a, a) \in R \ \forall a \in R$

So, R is reflexive

Symmetry

Let $(a, b) \in R$

$\Rightarrow \quad |a - b|$ is divisible by 3

$\Rightarrow \quad |a-b| = 3\text{k} \qquad$ [where $k \in n$]

$\Rightarrow \quad a - b = \pm 3k$

$\Rightarrow \quad b - a = \mp 3k$

$\Rightarrow \quad |b - a|$ is divisible by 3

$\Rightarrow \quad (b, a) \in R$

So, R is symmetric

Transitivity

Let $a, b, c \in A$ such that $(a, b) \in R$ and $(b, c) \in R$.

$\Rightarrow \quad |a - b|$ is divisible by 3 and $|b - c|$ is divisible by 3

$\Rightarrow \quad |a - b| = 3m$ and $\Rightarrow |b - c| = 3n \qquad m, n \in N$

$\Rightarrow \quad a - b = \pm 3m$ and $b - c = \pm 3n$

$\Rightarrow \quad (a - b) + (b - c) = \pm 3(m + n)$

$\Rightarrow \quad a - b + b - c = \pm 3(m + n)$

$\Rightarrow \quad |a - c| = \pm 3(m + n)$

$\Rightarrow \quad |a - c| = 3(m + n)$

$\Rightarrow \quad |a - c|$ is divisible by 3

$\Rightarrow \quad (a, c) \in R.$

So, R is transitive

Therefore, R is an equivalence relation.

22. Refer to Q. 20, Page 287.

28. Given curves are $y^2 = 2x$ ....(*i*)

and $x - y = 4$ ....(*ii*)

Obviously, curve (i) is right handed parabola having vertex at (0, 0) and axis along +ve direction of x-axis while curve (ii) is a straight line.

For intersection point of curve (*i*) and (*ii*)

$(x - 4)^2 = 2x$

$\Rightarrow \quad x^2 - 8x + 16 = 2x \quad \Rightarrow \quad x^2 - 10x + 16 = 0$

$\Rightarrow \quad x^2 - 8x - 2x + 16 = 0 \quad \Rightarrow \quad x(x - 8) - 2(x - 8) = 0$

$\Rightarrow \quad (x - 8)(x - 2) = 0 \quad \Rightarrow \quad x = 2, 8$

$\Rightarrow \quad y = -2, 4$

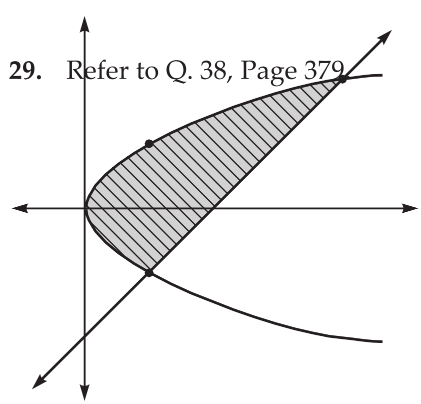
Intersection points are (2, –2), (8, 4)

Therefore, required Area = Area of shaded region

$$= \int_{-2}^{4} (y + 4)\,dy - \int_{-2}^{4} \frac{y^2}{2}\,dy$$

$$= \left[\frac{(y + 4)^2}{2}\right]_{-2}^{4} - \frac{1}{2}\left[\frac{y^3}{3}\right]_{-2}^{4}$$

$$= \frac{1}{2}.[64 - 4] - \frac{1}{6}[64 + 8] = 30 - \frac{72}{6} = 18 \text{ sq. unit}$$



29. Refer to Q. 38, Page 379.

Y

(8, 4)

X' O X

(2, –2)

Y'

SET–III

9. $\cot^{-1}\left(\dfrac{1}{\sqrt{x^2-1}}\right)$

Let $x = \sec\theta \Rightarrow \theta = \sec^{-1} x$

Now, $\cot^{-1}\left(\dfrac{1}{\sqrt{x^2-1}}\right) = \cot^{-1}\left(\dfrac{1}{\sqrt{\sec^2\theta - 1}}\right)$

$= \cot^{-1}\left(\dfrac{1}{\tan\theta}\right) = \cot^{-1}(\cot\theta) = \theta = \sec^{-1} x$

10. Given:

$\Rightarrow \quad (2\vec{x} - 3\vec{a})\cdot(2\vec{x} + 3\vec{a}) = 91$

$\Rightarrow \quad 4|\vec{x}|^2 + 6\vec{a}\cdot\vec{x} - 6\vec{x}\cdot\vec{a} - 9|\vec{a}|^2 = 91$

$4|\vec{x}|^2 - 9 = 91$

$\Rightarrow \quad |\vec{x}|^2 = \dfrac{100}{4} \quad \Rightarrow \quad |\vec{x}| = 5$

19. Let $I = \int \dfrac{2x^2+3}{x^2+5x+6}\,dx$

$= \int\left(2 - \dfrac{10x+9}{x^2+5x+6}\right)dx$

$= 2\int dx - \int \dfrac{10x+9}{x^2+5x+6}\,dx$

$= 2x - \int \dfrac{10x+9}{x^2+3x+2x+6}\,dx$

$= 2x - \int \dfrac{10x+9}{x(x+3)+2(x+3)}\,dx$

$= 2x - \int \dfrac{10x+9}{(x+3)(x+2)}\,dx$

$= 2x - \int\left(\dfrac{-11}{x+2} + \dfrac{21}{x+3}\right)dx$

$= 2x + 11\int \dfrac{dx}{x+2} - 21\int \dfrac{dx}{x+3}$

$= 2x + 11\log|x+2| - 21\log|x+3| + C$

20. Refer to Q 5, Page 101.

21. Given $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$

$\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$

$$2$$

$$[\, x^2 + 5x + 6 \quad 2x^2 + 3 \quad] - 2x^2 \pm 10x$$

$$\pm 12$$

$$-10x - 9$$

$$\left| \frac{10x+9}{(x+2)\,(x+3)} \quad \frac{A}{x+2} \quad \frac{B}{x+3} \right| \Rightarrow 10x + 9 = A$$

$$(x+3) + B\ (x+2)$$

$$\left| \textit{Putting } x = -3 \text{ we get } B = 21 \right| \left\lfloor \textit{Putting } x = -2 \text{ we get } A = -11 \right\rfloor$$

$$\therefore \quad \vec{a} + 2\vec{b} = (3\hat{i} + 2\hat{j} + 2\hat{k}) + (2\hat{i} + 4\hat{j} - 4\hat{k})$$

$$= 5\hat{i} + 6\hat{j} - 2\hat{k}$$

$$2\vec{a} + \vec{b} = (6\hat{i} + 4\hat{j} + 4\hat{k}) + (\hat{i} + 2\hat{j} - 2\hat{k})$$

$$= 7\hat{i} + 6\hat{j} + 2\hat{k}$$

Now, perpendicular vector of $(\vec{a} + 2\vec{b})$ and $(2\vec{a} + \vec{b})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 6 & -2 \\ 7 & 6 & 2 \end{vmatrix}$$

$$= (12 + 12)\hat{i} - (10 + 14)\hat{j} + (30 - 42)\hat{k}$$

$$= 24\hat{i} - 24\hat{j} - 12\hat{k} = 12(2\hat{i} - 2\hat{j} - \hat{k})$$

$$\therefore \text{Required unit vector} = \frac{\pm 12(2\hat{i} - 2\hat{j} - \hat{k})}{12\sqrt{2^2 + (-2)^2 + (-1)^2}} = \pm \frac{2\hat{i} - 2\hat{j} - \hat{k}}{3} = \pm\left(\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}\right)$$

22. Let $y = \tan^{-1}\left(\dfrac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right)$

$$= \tan^{-1}\left[\frac{\sqrt{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2} + \sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2} - \sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}}\right]$$

$$= \tan^{-1}\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right) = \tan^{-1}\left(\cot\frac{x}{2}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{2} - \frac{x}{2}\right)\right)$$

$$\because 0 < x < \frac{\pi}{2}$$
$$\Rightarrow 0 < \frac{x}{2} < \frac{\pi}{4}$$
$$\Rightarrow 0 > -\frac{x}{2} > -\frac{\pi}{4}$$
$$\Rightarrow \frac{\pi}{2} > \left(\frac{\pi}{2} - \frac{x}{2}\right) > \frac{\pi}{2} - \frac{\pi}{4}$$
$$\Rightarrow \frac{\pi}{2} > \left(\frac{\pi}{2} - \frac{x}{2}\right) > \frac{\pi}{4}$$
$$\Rightarrow \left(\frac{\pi}{2} - \frac{x}{2}\right) \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$y = \frac{\pi}{2} - \frac{x}{2} \quad \Rightarrow \quad \frac{dy}{dx} = 0 - \frac{1}{2} = -\frac{1}{2}$$

28. Refer to Q 8, Page 329.

29. Refer to Q. 26 CBSE (Delhi) SeI-I.

zzz

CBSE Examination Paper, Delhi-2014

Time allowed: 3 hours **Maximum marks: 100**

○ ***General Instructions:***

1. ***All*** *questions are compulsory.*
2. *The question paper consists of* ***29*** *questions divided into three Sections* ***A, B*** *and* ***C.*** *Section A comprises of* ***10*** *questions of* ***one*** *mark each; Section B comprises of* ***12*** *questions of* ***four*** *marks each; and Section C comprises of* ***7*** *questions of* ***six*** *marks each.*
3. ***All*** *questions in Section A are to be answered in* ***one*** *word,* ***one*** *sentence or as per the exact requirement of the question.*
4. *There is no overall choice. However, internal choice has been provided in* ***4*** *questions of* ***four*** *marks each and* ***2*** *questions of* ***six*** *marks each. You have to attempt only* ***one*** *of the alternatives in all such questions.*
5. *Use of calculator is* ***not*** *permitted. You may ask for logarithmic tables, if required.*

SET–I

SECTION–A

Question numbers 1 to 10 carry 1 mark each.

1. Let * *be* a binary operation, on the set of all non-zero real numbers, given by $a * b = \frac{ab}{5}$ for all a, $b \in R - \{0\}$. Find the value of x, given that $2 * (x * 5) = 10$.
2. If $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1} x\right) = 1$, then find the value of x.
3. If $2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$, find $(x - y)$.
4. Solve the following matrix equation for $x : [x \;\; 1]\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = O$.
5. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, write the value of x.
6. Write the antiderivative of $\left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right)$.

7. Evaluate: $\int_0^3 \frac{dx}{9+x^2}$.

8. Find the projection of the vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $2\hat{i} - 3\hat{j} + 6\hat{k}$.

9. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a} + \vec{b}$ is also a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$.

10. Write the vector equation of the plane, passing through the point (a, b, c) and parallel to the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$.

SECTION–B

Question numbers 11 to 22 carry 4 marks each.

11. Let $A = \{1, 2, 3, \ldots, 9\}$ and R be the relation in $A \times A$ defined by $(a, b)\ R\ (c, d)$ if $a + d = b + c$ for $(a, b), (c, d)$ in $A \times A$. Prove that R is an equivalence relation. Also obtain the equivalence class $[(2, 5)]$.

12. Prove that $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2};\ x \in \left(0, \frac{\pi}{4}\right)$.

OR

Prove that $2\tan^{-1}\left(\frac{1}{5}\right) + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right) + 2\tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$.

13. Using properties of determinants, prove that

$$\begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix} = (x+y+z)^3.$$

14. Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ with respect to $\cos^{-1}(2x\sqrt{1-x^2})$, when $x \neq 0$.

15. If $y = x^x$, prove that $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$.

16. Find the intervals in which the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is

(*a*) strictly increasing

(*b*) strictly decreasing

OR

Find the equations of the tangent and normal to the curve $x = a\sin^3\theta$ and $y = a\cos^3\theta$ at $\theta = \frac{\pi}{4}$.

17. Evaluate: $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x}\, dx$

OR

Evaluate: $\int (x-3)\sqrt{x^2+3x-18}\,dx$

18. Find the particular solution of the differential equation $e^x \sqrt{1-y^2}\, dx + \frac{y}{x} dy = 0$, given that $y = 1$ when $x = 0$.

19. Solve the following differential equation:

$$(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{2}{x^2 - 1}$$

20. Prove that, for any three vectors $\vec{a}, \vec{b}, \vec{c}$

$[\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}] = 2[\vec{a}, \vec{b}, \vec{c}]$

OR

Vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $|\vec{a}| = 3, |\vec{b}| = 5$ and $|\vec{c}| = 7$. Find the angle between $\vec{a}$ and $\vec{b}$.

21. Show that the lines $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$ intersect. Also find their point of intersection.

22. Assume that each born child is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls given that

(i) the youngest is a girl?

(ii) atleast one is a girl?

SECTION–C

Question numbers 23 to 29 carry 6 marks each.

23. ***Two schools P and Q want to award their selected students on the values of Discipline, Politeness and Punctuality. The school P wants to award ` x each, ` y each and ` z each for the three respective values to its 3, 2 and 1 students with a total award money of ` 1,000. School Q wants to spend ` 1,500 to award its 4, 1, and 3 students on the respective values (by giving the same award money for the three values as before). If the total amount of awards for one prize on each value is ` 600, using matrices find the award money for each value. Apart from the above three values, suggest one more value for awards.***

24. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\cos^{-1} \frac{1}{\sqrt{3}}$.

25. Evaluate: $\int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\cot x}}$

26. Find the area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$.

27. Find the distance between the point (7, 2, 4) and the plane determine by the points $A(2, 5, -3)$, $B(-2, -3, 5)$ and $C(5, 3, -3)$.

OR

Find the distance of the point (–1, – 5, – 10) from the point of intersection of the line

$\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r}\cdot(\hat{i} - \hat{j} + \hat{k}) = 5$.

28. A dealer in rural area wishes to purchase a number of sewing machines. He has only ` 5,760 to invest and has space for at most 20 items for storage. An electronic sewing machine cost him ` 360 and a manually operated sewing machine ` 240. He can sell an electronic sewing machine at a profit of ` 22 and a manually operated sewing machine at a profit of ` 18. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximise his profit? Make it as a LPP and solve it graphically.

29. A card from a pack of 52 playing cards is lost. From the remaining cards of the pack three cards are drawn at random (without replacement) and are found to be all spades. Find the probability of the lost card being a spade.

OR

From a lot of 15 bulbs which include 5 defectives, a sample of 4 bulbs is drawn one by one with replacement. Find the probability distribution of number of defective bulbs. Hence find the mean of the distribution.

SET–II

Only those questions, not included in Set I, are given.

9. Evaluate: $\int \cos^{-1}(\sin x)dx.$

10. If vectors $\vec{a}$ and $\vec{b}$ are such that, $|\vec{a}| = 3, |\vec{b}| = \frac{2}{3}$ and $\vec{a} \times \vec{b}$ is a unit vector, then write the angle between $\vec{a}$ and $\vec{b}$.

19. Prove the following using properties of determinants:

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$

20. Differentiate $\tan^{-1}\frac{x}{\sqrt{1-x^2}}$ with respect to $\sin^{-1}(2x\sqrt{1-x^2})$.

21. Solve the following differential equation:

$\operatorname{cosec} x \log y \frac{dy}{dx} + x^2y^2 = 0.$

22. Show that the lines $\frac{5-x}{-4} = \frac{y-7}{4} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = \frac{2y-8}{2} = \frac{z-5}{3}$ are coplanar.

28. Evaluate: $\int_0^{\pi} \frac{x \tan x}{\sec x . \operatorname{cosec} x} dx.$

29. Prove that the semi-vertical angle of the right circular cone of given volume and least curved surface area is $\cot^{-1}\sqrt{2}$.

SET–III

Only those questions, not included in Set I and Set II, are given.

9. Evaluate: $\int_0^{\pi/2} e^x(\sin x - \cos x)dx.$

10. Write a unit vector in the direction of the sum of the vectors $\vec{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 7\hat{k}$.

19. Using properties of determinants, prove the following:

$$\begin{vmatrix} x^2+1 & xy & xz \\ xy & y^2+1 & yz \\ xz & yz & z^2+1 \end{vmatrix} = 1 + x^2 + y^2 + z^2.$$

20. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, when $x \neq 0$.

21. Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$ given that $y = \frac{\pi}{2}$ when $x = 1$.

22. Show that lines $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$ and $\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$ intersect. Also find their point of intersection.

28. Evaluate: $\displaystyle\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x}\, dx.$

29. Of all the closed right circular cylindrical cans of volume $128\,\pi$ cm^3, find the dimensions of the can which has minimum surface area.

Solutions

SET–I

SECTION–A

1. Given $2 * (x * 5) = 10$

$\Rightarrow \quad 2 * \frac{x \times 5}{5} = 10 \qquad \Rightarrow \qquad 2 * x = 10$

$\Rightarrow \quad \frac{2 \times x}{5} = 10 \qquad x = \frac{10 \times 5}{2} \qquad \Rightarrow \qquad x = 25.$

2. Given $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1} x\right) = 1 \Rightarrow \sin^{-1}\frac{1}{5} + \cos^{-1} x = \sin^{-1} 1$

$\Rightarrow \quad \sin^{-1}\frac{1}{5} + \cos^{-1} x = \frac{\pi}{2} \Rightarrow \sin^{-1}\frac{1}{5} = \frac{\pi}{2} - \cos^{-1} x$

$\Rightarrow \quad \sin^{-1}\frac{1}{5} = \sin^{-1} x \qquad \Rightarrow \qquad x = \frac{1}{5}.$

3. Given $2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 7 & 8+y \\ 10 & 2x+1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} 6 & 8 \\ 10 & 2x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$

Equating we get $8 + y = 0$ and $2x + 1 = 5$

$\Rightarrow \quad y = -8$ and $x = 2 \qquad \Rightarrow \qquad x - y = 2 + 8 = 10$

4. Given $[x \quad 1]\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = 0$

$\Rightarrow \quad [x - 2 \quad 0] = [0 \quad 0]$

$\Rightarrow \quad x - 2 = 0 \qquad \Rightarrow \qquad x = 2$

5. Given $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$

$\Rightarrow \quad 2x^2 - 40 = 18 - (-14) \qquad \Rightarrow \qquad 2x^2 - 40 = 32$

$\Rightarrow \quad 2x^2 = 72 \qquad \Rightarrow \qquad x^2 = 36 \qquad \Rightarrow \qquad x = \pm 6$

6. Antiderivative of $\left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right) = \int\left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right)dx$

$$= 3\int \sqrt{x}\,dx + \int \frac{1}{\sqrt{x}}\,dx = 3\int x^{1/2}\,dx + \int x^{-1/2}\,dx$$

$$= 3.\frac{x^{1/2+1}}{1} + \frac{x^{-1/2+1}}{1} + c$$

$$\frac{}{2}+1 \qquad -\frac{}{2}+1$$

$$= 3 \times \frac{2}{3} x^{3/2} + 2x^{1/2} + c$$

$$= 2x^{3/2} + 2\sqrt{x} + c$$

7. Let $I = \int_0^3 \frac{dx}{9+x^2}$

$$= \int_0^3 \frac{dx}{3^2 + x^2} = \frac{1}{3}\left[\tan^{-1}\frac{x}{3}\right]_0^3$$

$$= \frac{1}{3}[\tan^{-1}(1) - \tan^{-1}(0)] = \frac{1}{3}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{12}$$

8. Let $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$

$\vec{b} = 2\hat{i} - 3\hat{j} + 6\hat{k}$

Now projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

$$= \frac{(\hat{i} + 3\hat{j} + 7\hat{k}).(2\hat{i} - 3\hat{j} + 6\hat{k})}{|2\hat{i} - 3\hat{j} + 6\hat{k}|}$$

$$= \frac{2 - 9 + 42}{\sqrt{2^2 + (-3)^2 + 6^2}} = \frac{35}{\sqrt{49}} = \frac{35}{7} = 5.$$

9. $|\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b}).(\vec{a} + \vec{b})$

$\Rightarrow \quad 1^2 = |\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2 \qquad [\because |\vec{a} + \vec{b}| = 1]$

$\Rightarrow \quad 1 = 1 + 2\vec{a}.\vec{b} + 1 \qquad [\because \vec{a}$ and $\vec{b}$ are unit vector, hence $|\vec{a}| = 1$ and $|\vec{b}| = 1]$

$\Rightarrow \quad 1 = 2\vec{a}.\vec{b} + 2 \qquad \Rightarrow \qquad \vec{a}.\vec{b} = -\frac{1}{2}$

$\Rightarrow \quad |\vec{a}|.|\vec{b}|\cos\theta = -\frac{1}{2},$ where θ is angle between $\vec{a}$ and $\vec{b}$

$\Rightarrow \quad 1.1 \cos\theta = -\frac{1}{2} \qquad [\because |\vec{a}| = |\vec{b}| = 1]$

$\Rightarrow \quad \cos\theta = -\frac{1}{2}$

$\Rightarrow \quad \cos\theta = -\cos\frac{\pi}{3} \qquad \Rightarrow$

$\cos\theta = \cos\left(\pi - \frac{\pi}{3}\right) \Rightarrow \qquad \cos\theta = \cos\frac{2\pi}{3}$

$\Rightarrow \qquad \theta = \frac{2\pi}{3}.$

10. Since, the required plane is parallel to plane $\vec{r}.(\hat{i} + \hat{j} + \hat{k}) = 2$

$\therefore$ Normal of required plane is normal of given plane.

$\Rightarrow$ Normal of required plane $= \hat{i} + \hat{j} + \hat{k}$.

$\therefore$ Required vector equation of plane

$$\{ r - (a\hat{i} + b\hat{j} + c\hat{k})\}.(\hat{i} + \hat{j} + \hat{k}) = 0$$

SECTION–B

11. Given, R is a relation in $A \times A$ defined by

$$(a, b)R(c, d) \Leftrightarrow a + d = b + c$$

(*i*) **Reflexivity:** $\forall a, b \in A$

$$\because \quad a + b = b + a \Rightarrow (a, b)R(a, b)$$

So, R in reflexive.

(*ii*) **Symmetry:** Let $(a, b)\ R\ (c, d)$

$$\because \quad (a, b)R(c, d) \Rightarrow a + d = b + c$$

$\Rightarrow b + c = d + a$ [$\because a, b, c, d \in N$ and N is commutative under addition]

$$\Rightarrow c + b = d + a$$

$$\Rightarrow (c, d)R(a, b)$$

So, R is symmetric.

(*iii*) **Transitivity:** Let $(a, b)R(c, d)$ and $(c, d)R(e, f)$

Now, $(a, b)R(c, d)$ and $(c, d)R(e, f)$ $\Rightarrow$ $a + d = b + c$ and $c + f = d + e$

$$\Rightarrow a + d + c + f = b + c + d + e$$

$$\Rightarrow a + f = b + e$$

$$\Rightarrow (a, b)R(e, f).$$

$\Rightarrow$ R is transitive.

Hence, R is an equivalence relation.

2nd Part: **Equivalence class:** $[(2, 5)] = \{(a, b) \in A \times A : (a, b)R(2, 5)\}$

$$= \{(a, b) \in A \times A : a + 5 = b + 2\}$$

$$= \{(a, b) \in A \times A : b - a = 3\}$$

$$= \{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$$

12. $$\text{LHS} = \cot^{-1}\left(\frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}\right), \quad x \in \left(0, \frac{\pi}{4}\right)$$

$$= \cot^{-1}\left(\frac{\sqrt{(\cos x/2 + \sin x/2)^2} + \sqrt{(\cos x/2 - \sin x/2)^2}}{\sqrt{(\cos x/2 + \sin x/2)^2} - \sqrt{(\cos x/2 - \sin x/2)^2}}\right)$$

$$= \cot^{-1}\left(\frac{\cos x/2 + \sin x/2 + \cos x/2 - \sin x/2}{\cos x/2 + \sin x/2 - \cos x/2 + \sin x/2}\right)$$

$$= \cot^{-1}\left(\frac{\cos x/2}{\sin x/2}\right) = \cot^{-1}(\cot x/2) = \frac{x}{2} = \text{R.H.S.}$$

OR

L.H.S.

$$= 2\tan^{-1}\left(\frac{1}{5}\right) + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right) + 2\tan^{-1}\left(\frac{1}{8}\right)$$

$$= 2\left\{\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right)\right\} + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)$$

Giv
en $\Rightarrow x \in \left(0, \frac{\pi}{}\right) \subset (0, \pi)$

$\overline{2}$ $\overline{8}$ $\overline{4}$

0
<
x
<
π

$\Rightarrow$

0
<
x
<
π

$$= 2\tan^{-1}\left\{\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}.\frac{1}{8}}\right\} + \tan^{-1}\sqrt{\left(\frac{5\sqrt{2}}{7}\right)^2 - 1} \qquad [\because \sec^{-1} x = \tan^{-1}\sqrt{x^2-1}]$$

$$= 2\tan^{-1}\frac{\frac{13}{40}}{\frac{39}{40}} + \tan^{-1}\sqrt{\frac{50}{49}-1} = 2\tan^{-1}\frac{13}{40}\times\frac{40}{39} + \tan^{-1}\sqrt{\frac{1}{49}}$$

$$= 2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right) + \tan^{-1}\left(\frac{1}{7}\right)\left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \tan^{-1}\left(\frac{\frac{2}{3}}{\frac{8}{9}}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{2}{3}\times\frac{9}{8}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right) = \tan^{-1}\left(\frac{25}{28}\times\frac{28}{25}\right) = \tan^{-1}(1) = \frac{\pi}{4} = \text{R.H.S.}$$

13. L.H.S. $\Delta = \begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$

Applying $R_2 \leftrightarrow R_3$ then $R_1 \leftrightarrow R_2$, we have

$$\Delta = \begin{vmatrix} x-y-z & 2x & 2x \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we have

$$\Delta = \begin{vmatrix} x+y+z & y+z+x & z+x+y \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix}$$

Taking out $(x+y+z)$ from first row, we have

$$\Delta = (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$, we have

$$\Delta = (x+y+z)\begin{vmatrix} 0 & 0 & 1 \\ y+z+x & -(y+z+x) & 2y \\ 0 & z+x+y & z-x-y \end{vmatrix}$$

Expanding along first row, we have

$$\Delta = (x + y + z)\,(x + y + z)^2 = (a + b + c)^3 = \text{R.H.S.}$$

14. Let $u = \tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$ and $v = \cos^{-1}(2x\sqrt{1-x^2})$

We have to determine $\dfrac{du}{dv}$

Let $x = \sin\theta \Rightarrow \theta = \sin^{-1} x$

Now, $u = \tan^{-1}\left(\dfrac{\sqrt{1-\sin^2\theta}}{\sin\theta}\right)$

$\Rightarrow \quad u = \tan^{-1}\left(\dfrac{\cos\theta}{\sin\theta}\right) \qquad \Rightarrow \quad u = \tan^{-1}(\cot\theta)$

$\Rightarrow \quad u = \tan^{-1}\left[\tan\left(\dfrac{\pi}{2} - \theta\right)\right] \qquad \Rightarrow \quad u = \dfrac{\pi}{2} - \theta$

$\Rightarrow \quad u = \dfrac{\pi}{2} - \sin^{-1} x \qquad \Rightarrow \quad \dfrac{du}{dx} = 0 - \dfrac{1}{\sqrt{1-x^2}}$

$\Rightarrow \quad \dfrac{du}{dx} = -\dfrac{1}{\sqrt{1-x^2}}$

Again, $v = \cos^{-1}(2x\sqrt{1-x^2})$

$\because \quad x = \sin\theta$

$\therefore \quad v = \cos^{-1}(2\sin\theta\sqrt{1-\sin^2\theta})$

$\Rightarrow \quad v = \cos^{-1}(2\sin\theta.\cos\theta)$

$\Rightarrow \quad v = \cos^{-1}(\sin 2\theta)$

$\Rightarrow \quad v = \cos^{-1}\left(\cos\left(\dfrac{\pi}{2} - 2\theta\right)\right)$

$\Rightarrow \quad v = \dfrac{\pi}{2} - 2\theta$

$\Rightarrow \quad v = \dfrac{\pi}{2} - 2\sin^{-1} x$

$\Rightarrow \quad \dfrac{dv}{dx} = 0 - \dfrac{2}{\sqrt{1-x^2}} \qquad \Rightarrow \quad \dfrac{dv}{dx} = -\dfrac{2}{\sqrt{1-x^2}}$

$\therefore \quad \dfrac{du}{dv} = \dfrac{\frac{du}{dx}}{\frac{dv}{dx}} = \dfrac{-\frac{1}{\sqrt{1-x^2}}}{-\frac{2}{\sqrt{1-x^2}}} = \dfrac{1}{2}.$

$\left[\begin{array}{l} \because -\dfrac{1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}} \Rightarrow \sin\left(-\dfrac{\pi}{4}\right) < \sin\theta < \sin\dfrac{\pi}{4} \\ \Rightarrow -\dfrac{\pi}{4} < \theta < \dfrac{\pi}{4} \\ \Rightarrow -\dfrac{\pi}{2} < 2\theta < \dfrac{\pi}{2} \\ \Rightarrow \dfrac{\pi}{2} > -2\theta > -\dfrac{\pi}{2} \\ \Rightarrow \pi > \left(\dfrac{\pi}{2} - 2\theta\right) > 0 \\ \Rightarrow \left(\dfrac{\pi}{2} - 2\theta\right) \in (0, \pi) \subset [0, \pi] \end{array}\right]$

$dx \qquad \sqrt{1-x^2}$

[Note: Here the range of x is taken as $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$]

15. Given $y = x^x$

Taking logarithm of both sides, we get

$$\log y = x.\log x$$

Differentiating both sides, we get

$\Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = x.\frac{1}{x} + \log x \qquad \Rightarrow \quad \frac{dy}{dx} = y(1 + \log x) \qquad \ldots(i)$

Again differentiating both sides, we get

$$\frac{d^2y}{dx^2} = y.\frac{1}{x} + (1 + \log x).\frac{dy}{dx}$$

$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{y}{x} + \frac{1}{y}.\frac{dy}{dx}.\frac{dy}{dx}$ [From (*i*)]

$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{y}{x} + \frac{1}{y}\left(\frac{dy}{dx}\right)^2 \qquad \Rightarrow \quad \frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$

16. Given $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

$\Rightarrow \quad f'(x) = 12x^3 - 12x^2 - 24x \qquad \Rightarrow \quad f'(x) = 12x(x^2 - x - 2)$

$\Rightarrow \quad f'(x) = 12x\{x^2 - 2x + x - 2\} \qquad \Rightarrow \quad f'(x) = 12x\{x(x-2) + 1(x-2)\}$

$\Rightarrow \quad f'(x) = 12x(x-2)(x+1) \qquad \ldots(i)$

For critical points

$f'(x) = 0 \quad \Rightarrow \quad 12x(x-2)(x+1) = 0$

$\Rightarrow \quad x = 0, -1, 2$ (critical points)

These critical points divide the real number line into 4 disjoint intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 2)$ and $(2, \infty)$.

For $(-\infty, -1)$

$f'(x) = +\text{ve} \times -\text{ve} \times -\text{ve} \times -\text{ve} = -\text{ve}$ [From (*i*)]

$\Rightarrow \quad f(x)$ is decreasing in $(-\infty, -1)$

For $(-1, 0)$

$f'(x) = +\text{ve} \times -\text{ve} \times -\text{ve} \times +\text{ve} = +\text{ve}$

$\Rightarrow \quad f(x)$ is increasing in $(-1, 0)$

For $(0, 2)$

$f'(x) = +\text{ve} \times +\text{ve} \times -\text{ve} \times +\text{ve} = -\text{ve}$

$\Rightarrow \quad f(x)$ is decreasing in $(0, 2)$.

For $(2, \infty)$

$f'(x) = +\text{ve} \times +\text{ve} \times +\text{ve} \times +\text{ve} = +\text{ve}$

$\Rightarrow \quad f(x)$ is increasing in $(2, \infty)$.

Hence, $f(x)$ is strictly increasing in $(-1, 0) \cup (2, \infty)$

and $f(x)$ is strictly decreasing in $(-\infty, -1) \cup (0, 2)$.

OR

Q $\quad x = a\sin^3\theta$ and $y = a\cos^3\theta$

$$\Rightarrow \qquad \frac{dx}{d\theta} = 3a\sin^2\theta.\cos\theta \text{ and } \frac{dy}{d\theta} = -3a\cos^2\theta\sin\theta$$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \dfrac{-3a\cos^2\theta.\sin\theta}{3a\sin^2\theta.\cos\theta} = -\cot\theta$

$\Rightarrow \quad \dfrac{dy}{dx} = -\cot\theta$

$\Rightarrow$ Slope of tangent to the given curve at $\theta = \dfrac{\pi}{4} = \left[\dfrac{dy}{dx}\right]_{\theta=\frac{\pi}{4}} = -\cot\dfrac{\pi}{4} = -1.$

Since for $\theta = \dfrac{\pi}{4}$, $x = a\sin^3\dfrac{\pi}{4}$ and $y = a\cos^3\dfrac{\pi}{4}$

$\Rightarrow \quad x = a\left(\dfrac{1}{\sqrt{2}}\right)^3$ and $y = a\left(\dfrac{1}{\sqrt{2}}\right)^3 \quad \Rightarrow \quad x = \dfrac{a}{2\sqrt{2}}$ and $y = \dfrac{a}{2\sqrt{2}}$

i.e., co-ordinates of the point of contact $= \left(\dfrac{a}{2\sqrt{2}}, \dfrac{a}{2\sqrt{2}}\right)$

$\therefore$ Equation of tangent is

$\left(y - \dfrac{a}{2\sqrt{2}}\right) = (-1).\left(x - \dfrac{a}{2\sqrt{2}}\right) \quad \Rightarrow \quad y - \dfrac{a}{2\sqrt{2}} = -x + \dfrac{a}{2\sqrt{2}}$

$\Rightarrow \quad x + y = \dfrac{a}{\sqrt{2}}$

Also slope of normal (at $\theta = \dfrac{\pi}{4}$) $= -\dfrac{1}{\text{slope of tangent}} = -\dfrac{1}{-1} = 1$

$\therefore$ Equation of normal is

$\left(y - \dfrac{a}{2\sqrt{2}}\right) = (1).\left(x - \dfrac{a}{2\sqrt{2}}\right)$

$\Rightarrow \quad y - \dfrac{a}{2\sqrt{2}} = x - \dfrac{a}{2\sqrt{2}} \quad \Rightarrow \quad y - x = 0$

17. Let $I = \int \dots dx$

$\Rightarrow$

$\Rightarrow$ $\sin^2 x.\cos^2 x$

$\Rightarrow$

$\Rightarrow$

s i n 6 x + c o s 6 x s i n 2 x . c o s 2 x (s i n 2 x) 3 + (c o

$$\text{s}^2\, x)^3 \sin^2 x.\cos^2 x$$

$$I = \int \frac{(\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x.\cos^2 x + \cos^4 x)}{\sin^2 x.\cos^2 x} dx$$

$$I = \int \frac{\sin^4 x - \sin^2 x.\cos^2 x + \cos^4 x}{} dx = \int \tan^2 x\, dx - \int dx + \int \cot^2 x dx$$

$$I = \int (\sec^2 x - 1)dx - x + \int (\mathrm{cosec}^2 x - 1)dx$$

$$I = \int \sec^2 x\, dx + \int \mathrm{cosec}^2 x\, dx - x - x - x + c = \tan x - \cot x - 3x + c$$

$$I = \int \frac{}{} dx$$

OR

Let $I = \int (x-3)\sqrt{x^2+3x-18}\,dx$ … (*i*)

Let $x-3 = A\frac{d}{dx}(x^2+3x-18)+B \quad \Rightarrow \quad x-3 = A(2x+3)+B$ … (*ii*)

$\Rightarrow \quad x-3 = 2Ax+(3A+B)$

Equating the co-efficient, we get

$2A = 1$ and $3A+B = -3 \quad \Rightarrow \quad A = \frac{1}{2}$ and $3\times\frac{1}{2}+B = -3$

$\Rightarrow \quad A = \frac{1}{2}$ and $B = -3-\frac{3}{2} = -\frac{9}{2}$

$\therefore \quad I = \int\left(\frac{1}{2}(2x+3)-\frac{9}{2}\right)\int\sqrt{x^2+3x-18}\,dx$ [From (*i*) and (*ii*)]

$I = \frac{1}{2}\int(2x+3)\sqrt{x^2+3x-18}\,dx - \frac{9}{2}\int\sqrt{x^2+3x-18}\,dx$

$\Rightarrow \quad I = \frac{1}{2}I_1 - \frac{9}{2}I_2$ … (*iii*) where $I_1 = \int(2x+3)\sqrt{x^2+3x-18}\,dx$

and $I_2 = \int\sqrt{x^2+3x-18}\,dx$

Now $I_1 = \int(2x+3)\sqrt{x^2+3x-18}\,dx$

Let $x^2+3x-18 = z$

$\Rightarrow \quad (2x+3) = dz$

$\therefore \quad I_1 = \int\sqrt{z}\,dz = \frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1}+c_1 = \frac{2}{3}(z)^{\frac{3}{2}}+c_1$

$\Rightarrow \quad I_1 = \frac{2}{3}(x^2+3x-18)^{\frac{3}{2}}+c_1$ … (*iv*)

Again $I_2 = \int\sqrt{x^2+3x-18}\,dx$

$= \int\sqrt{x^2+2.x.\frac{3}{2}+\left(\frac{3}{2}\right)^2-\frac{9}{4}-18}\,dx = \int\sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{9}{2}\right)^2}$

$I_2 = \frac{1}{2}\left(x+\frac{3}{2}\right)\sqrt{x^2+3x-18} - \frac{81}{4\times 2}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x-18}\right|$

$\Rightarrow \quad I_2 = \frac{1}{2}\left(x+\frac{3}{2}\right)\sqrt{x^2+3x-18} - \frac{81}{8}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x-18}\right|+c_2$ … (*v*)

Putting the value of I_1 and I_2 in (*iii*), we get

$$I = \frac{1}{3}(x^2+3x-18)^{\frac{3}{2}} - \frac{9}{4}\left(x+\frac{3}{2}\right)\sqrt{x^2+3x-18}+\frac{729}{16}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x-18}\right|+c$$

$\left[\text{where } c = \frac{c_1}{2}-\frac{9}{2}c_2\right]$

18. We have,

$$e^x\sqrt{1-y^2}\,dx + \frac{y}{x}\,dy = 0$$

$$\Rightarrow \quad e^x\sqrt{1-y^2}\,dx = -\frac{y}{x}\,dy \qquad \Rightarrow \qquad xe^x\,dx = -\frac{y}{\sqrt{1-y^2}}\,dy$$

$$\Rightarrow \quad \int \underset{I}{x}\,\underset{II}{e^x}\,dx = -\int \frac{y}{\sqrt{1-y^2}}\,dy$$

$$\Rightarrow \quad xe^x - \int e^x\,dx = \frac{1}{2}\int\frac{dt}{\sqrt{t}}, \text{ where } t = 1-y^2 \qquad \text{(Using I LATE on LHS)}$$

$$\Rightarrow \quad xe^x - e^x = \frac{1}{2}\left(\frac{t^{1/2}}{1/2}\right) + C \qquad \Rightarrow \qquad xe^x - e^x = \sqrt{t} + C$$

$$\Rightarrow \quad xe^x - e^x = \sqrt{1-y^2} + C, \text{ where } x \in R \text{ is the required solution.}$$

Putting $y = 1$ and $x = 0$

$$0e^0 - e^0 = \sqrt{1-1^2} + C \quad \Rightarrow \quad C = -1$$

Therefore required particular solution is $xe^x - e^x = \sqrt{1-y^2} - 1$.

19. The given differential equation is

$$(x^2-1)\frac{dy}{dx} + 2xy = \frac{2}{x^2-1}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2x}{x^2-1}\,y = \frac{2}{(x^2-1)^2} \qquad \ldots(i)$$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$,

where $\quad P = \frac{2x}{x^2-1}$ and $Q = \frac{2}{(x^2-1)^2}$

$$\therefore \quad I.F. = e^{\int P\,dx} = e^{\int 2x/(x^2-1)\,dx} = e^{\log(x^2-1)} = (x^2-1)$$

$$y(x^2-1) = \int \frac{2}{(x^2-1)^2}\times(x^2-1)\,dx + C \qquad [\text{Using: } y(I.F.) = \int Q.(I.F.)\,dx + C]$$

$$\therefore \quad y(x^2-1) = \int \frac{2}{x^2-1}\,dx + C$$

$$\Rightarrow \quad y(x^2-1) = 2\times\frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C \qquad \Rightarrow \qquad y(x^2-1) = \log\left|\frac{x-1}{x+1}\right| + C$$

This is the required solution.

20. L.H.S. $= [\vec{a}+\vec{b},\ \vec{b}+\vec{c},\ \vec{c}+\vec{a}] = (\vec{a}+\vec{b}).\{(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\}$

$$= (\vec{a}+\vec{b}).\{\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{c} + \vec{c}\times\vec{a}\}$$

$$= (\vec{a}+\vec{b}).\{\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{a}\} \qquad [\because \vec{c}\times\vec{c} = \vec{0}]$$

$$= \vec{a}.(\vec{b}\times\vec{c}) + \vec{a}.(\vec{b}\times\vec{a}) + \vec{a}.(\vec{c}\times\vec{a}) + \vec{b}.(\vec{b}\times\vec{c}) + \vec{b}.(\vec{b}\times\vec{a}) + \vec{b}.(\vec{c}\times\vec{a})$$

$$= [\vec{a}, \vec{b}, \vec{c}] + [\vec{a}, \vec{b}, \vec{a}] + [\vec{a}, \vec{c}, \vec{a}] + [\vec{b}, \vec{b}, \vec{c}] + [\vec{b}, \vec{b}, \vec{a}] + [\vec{b}, \vec{c}, \vec{a}]$$

$$= [\vec{a}, \vec{b}, \vec{c}] + 0 + 0 + 0 + 0 + [\vec{b}, \vec{c}, \vec{a}] \qquad \text{[By property of scalar triple product]}$$

$$= [\vec{a}, \vec{b}, \vec{c}] + [\vec{b}, \vec{c}, \vec{a}]$$

$$= [\vec{a}, \vec{b}, \vec{c}] + [\vec{a}, \vec{b}, \vec{c}] \qquad \text{[By property of circularly rotation]}$$

$$= 2[\vec{a}, \vec{b}, \vec{c}]$$

OR

$$\vec{a} + \vec{b} + \vec{c} = 0 \quad \Rightarrow \quad (\vec{a} + \vec{b})^2 = (-\vec{c})^2$$

$$\Rightarrow \quad (\vec{a} + \vec{b}).(\vec{a} + \vec{b}) = \vec{c}.\vec{c}$$

$$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + 2\,\vec{a}.\vec{b} = |\vec{c}|^2 \quad \Rightarrow \quad 9 + 25 + 2\,\vec{a}.\vec{b} = 49$$

$$\Rightarrow \quad 2\,\vec{a}.\vec{b} = 49 - 25 - 9$$

$$\Rightarrow \quad 2|\vec{a}||\vec{b}|\cos\theta = 15 \quad \Rightarrow \quad 30\cos\theta = 15$$

$$\Rightarrow \quad \cos\theta = \frac{1}{2} = \cos 60^\circ \qquad \theta = 60^\circ$$

21. Given lines are

$$\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7} \qquad \ldots(i)$$

$$\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5} \qquad \ldots(ii)$$

Let two lines (i) and (ii) intersect at a point $P(\alpha, \beta, \gamma)$.

$\Rightarrow \quad (\alpha, \beta, \gamma)$ satisfy line (i)

$$\Rightarrow \quad \frac{\alpha+1}{3} = \frac{\beta+3}{5} = \frac{\gamma+5}{7} = \lambda \qquad \text{(say)}$$

$$\Rightarrow \quad \alpha = 3\lambda - 1, \quad \beta = 5\lambda - 3, \qquad \gamma = 7\lambda - 5 \qquad \ldots(iii)$$

Again (α, β, γ) also lie on (ii)

$$\therefore \quad \frac{\alpha-2}{1} = \frac{\beta-4}{3} = \frac{\gamma-6}{5}$$

$$\Rightarrow \quad \frac{3\lambda-1-2}{1} = \frac{5\lambda-3-4}{3} = \frac{7\lambda-5-6}{5}$$

$$\Rightarrow \quad \underset{\text{I}}{\frac{3\lambda-3}{1}} = \underset{\text{II}}{\frac{5\lambda-7}{3}} = \underset{\text{III}}{\frac{7\lambda-11}{5}}$$

From I and II

$$\frac{3\lambda - 3}{1} = \frac{5\lambda - 7}{3}$$

$\Rightarrow \quad 9\lambda - 9 = 5\lambda - 7$

$\Rightarrow \quad 4\lambda = 2$

$\Rightarrow \quad \lambda = \frac{1}{2}$

= From II and III

$$\frac{5\lambda - 7}{3} = \frac{7\lambda - 11}{5}$$

$\Rightarrow \quad 25\lambda - 35 = 21\lambda - 33$

$\Rightarrow \quad 4\lambda = 2$

$\Rightarrow \quad \lambda = \frac{1}{2}$

Since, the value of λ in both the cases is same

$\Rightarrow$ Both lines intersect each other at a point.

$\therefore$ Intersecting point $= (\alpha, \beta, \gamma) = \left(\frac{3}{2} - 1, \frac{5}{2} - 3, \frac{7}{2} - 5\right)$ [From (*iii*)]

$$= \left(\frac{1}{2}, -\frac{1}{2}, \frac{-3}{2}\right)$$

22. A family has 2 children,

then Sample space $= S = \{BB, BG, GB, GG\}$, where B stands for Boy and G for Girl.

(*i*) Let A and B be two event such that

$A =$ Both are girls $= \{GG\}$

$B =$ the youngest is a girl $= \{BG, GG\}$

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} \qquad [\because A \cap B = \{GG\}]$$

$$P\left(\frac{A}{B}\right) = \frac{\frac{1}{4}}{\frac{2}{4}} = \frac{1}{2}$$

(*ii*) Let C be event such that

$C =$ at least one is a girl $= \{BG, GB, GG\}$

$$\text{Now } P(A/C) = \frac{P(A \cap C)}{P(C)} \qquad [\because A \cap C = \{GG\}]$$

$$= \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$

SECTION–C

23. According to question

$$3x + 2y + z = 1000$$
$$4x + y + 3z = 1500$$
$$x + y + z = 600$$

The given system of linear equations may be written in matrix form as $AX = B$ where

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 1000 \\ 1500 \\ 600 \end{bmatrix}$$

$\because \quad AX = B \quad \Rightarrow \quad X = A^{-1}B \quad \ldots (i)$

Now for A^{-1}

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 3(1-3) - 2(4-3) + 1(4-1)$$

$$= -6 - 2 + 3 = -8 + 3 = -5 \neq 0$$

Hence, A^{-1} exists.

Also, $A_{11} = \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = 1 - 3 = -2$ $\qquad A_{12} = -\begin{vmatrix} 4 & 3 \\ 1 & 1 \end{vmatrix} = -(4-3) = -1$

$A_{13} = \begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} = 4 - 1 = 3$ $\qquad A_{21} = -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -(2-1) = -1$

$A_{22} = \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = 3 - 1 = 2$ $\qquad A_{23} = -\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = -(3-2) = -1$

$A_{31} = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 6 - 1 = 5$ $\qquad A_{32} = -\begin{vmatrix} 3 & 1 \\ 4 & 3 \end{vmatrix} = -(9-4) = -5$

$A_{33} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 3 - 8 = -5$

$$\therefore \quad \text{Adj } A = \begin{bmatrix} -2 & -1 & 3 \\ -1 & 2 & -1 \\ 5 & -5 & -5 \end{bmatrix}^T = \begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{adj(A)}{|A|} = \frac{1}{-5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

Putting the value of X, A^{-1}, B in (i), we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix} \cdot \begin{bmatrix} 1000 \\ 1500 \\ 600 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -2000 - 1500 + 3000 \\ -1000 + 3000 - 3000 \\ 3000 - 1500 - 3000 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -500 \\ -1000 \\ -1500 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix} \qquad \Rightarrow \qquad x = \text{` } 100,\ y = \text{` } 200,\ z = \text{` } 300$$

i.e., ` 100 for discipline

` 200 for politeness and

` 300 for punctuality

One more value like sincerity or truthfulness can be awarded.

24. Let ABC be cone having slant height l and semi-vertical angle θ.

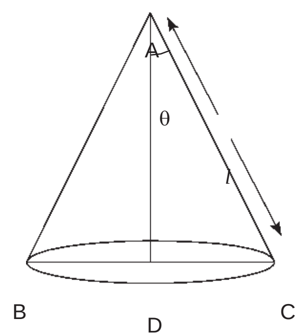
If V be the volume of cone then.

$$V = \frac{1}{3}.\pi \times DC^2 \times AD = \frac{\pi}{3} \times l^2 \sin^2$$

$$\overline{\quad\ } \\ 3$$

$$\theta \times l \cos\theta \Rightarrow \qquad V = \pi l^3 \sin^2\theta \cos\theta$$

$$\Rightarrow \quad \frac{dV}{d\theta} = \frac{\pi l^3}{3}[-\sin^3\theta + 2\sin\theta.\cos^2\theta]$$



For maximum value of V.

$$\frac{dV}{d\theta} = 0$$

$$\Rightarrow \quad \frac{\pi l^3}{3}[-\sin^3\theta + 2\sin\theta.\cos^2\theta] = 0$$

$$\Rightarrow \quad -\sin^3\theta + 2\sin\theta.\cos^2\theta = 0$$

$$\Rightarrow \quad -\sin\theta(\sin^2\theta - 2\cos^2\theta) = 0$$

$$\Rightarrow \quad \sin\theta = 0 \quad \text{or} \quad 1 - \cos^2\theta - 2\cos^2\theta = 0$$

$$\Rightarrow \quad \theta = 0 \quad \text{or} \quad 1 - 3\cos^2\theta = 0$$

$$\Rightarrow \quad \theta = 0 \quad \text{or} \quad \cos\theta = \frac{1}{\sqrt{3}}$$

Now $\dfrac{d^2V}{d\theta^2} = \dfrac{\pi l^3}{3}\{-3\sin^2\theta.\cos\theta - 4\sin^2\theta.\cos\theta + 2\cos^3\theta\}$

$$\Rightarrow \quad \frac{d^2V}{d\theta^2} = \frac{\pi l^3}{3}\{-7\sin^2\theta\cos\theta + 2\cos^3\theta\}$$

$$\Rightarrow \quad \left.\frac{d^2V}{d\theta^2}\right]_{\theta=0} = +\text{ve}$$

and $\left.\dfrac{d^2V}{d\theta^2}\right]_{\cos\theta=\frac{1}{\sqrt{3}}} = -\text{ve}$ $\quad$ $\left[\text{Putting } \cos\theta = \dfrac{1}{\sqrt{3}} \text{ and } \sin\theta = \sqrt{1-\left(\dfrac{1}{\sqrt{3}}\right)^2} = \dfrac{\sqrt{2}}{\sqrt{3}}\right]$

Hence for $\cos\theta = \dfrac{1}{\sqrt{3}}$ or $\theta = \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$, V is maximum.

25. Let $I = \displaystyle\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}} = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\frac{\sqrt{\cos x}}{\sqrt{\sin x}}}$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)} + \sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\,dx$$

$$\frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} \quad \ldots (i)$$

$b \quad b$

$$\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$\ldots (ii)$

Adding (*i*) and (*ii*), we get

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx = \int_{\pi/6}^{\pi/3} dx$$

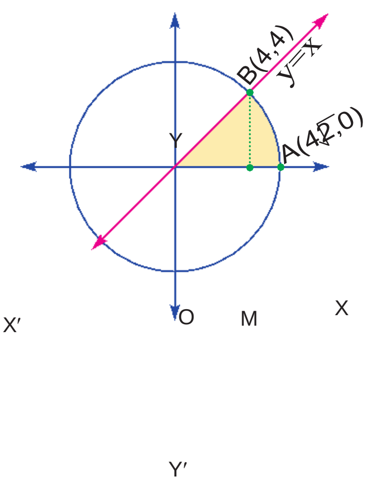
$$2I = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{2\pi - \pi}{6} = \frac{\pi}{6}$$

$\Rightarrow \qquad I = \dfrac{\pi}{12}.$

26. The given equations are

$$y = x \qquad \qquad ...(i)$$

and $\quad x^2 + y^2 = 32 \qquad \qquad ...(ii)$



Solving (*i*) and (*ii*), we find that the line and the circle meet at $B(4, 4)$ in the first quadrant. Draw perpendicular BM to the x-axis.

Therefore, the required area = area of the region $OBMO$ + area of the region $BMAB$.

Now, the area of the region $OBMO$

$$= \int_0^4 y\, dx = \int_0^4 x\, dx = \frac{1}{2}[x^2]_0^4 = 8 \qquad ...(iii)$$

Again, the area of the region $BMAB$

$$= \int_4^{4\sqrt{2}} y\, dx = \int_4^{4\sqrt{2}} \sqrt{32 - x^2}\, dx = \left[\frac{1}{2} x\sqrt{32 - x^2} + \frac{1}{2} \times 32 \times \sin^{-1}\frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$$

$$= \left(\frac{1}{2} 4\sqrt{2} \times 0 + \frac{1}{2} \times 32 \times \sin^{-1} 1\right) - \left(\frac{4}{2}\sqrt{32 - 16} + \frac{1}{2} \times 32 \times \sin^{-1}\frac{1}{\sqrt{2}}\right)$$

$$= 8\pi - (8 + 4\pi) = 4\pi - 8. \qquad ...(iv)$$

Adding (*iii*) and (*iv*), we get the required area = 4π sq units.

27. The equation of plane determined by the points $A(2, 5, -3)$, $B(-2, -3, 5)$ and $C(5, 3, -3)$ is

$$\begin{vmatrix} x-2 & y-5 & z+3 \\ -2-2 & -3-5 & 5+3 \\ 5-2 & 3-5 & -3+3 \end{vmatrix} = 0 \quad \Rightarrow \quad \begin{vmatrix} x-2 & y-5 & z+3 \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix} = 0$$

$\Rightarrow \quad (x-2)\{0+16\} - (y-5)\{0-24\} + (z+3)\{+8+24\} = 0$

$\Rightarrow \quad 16x - 32 + 24y - 120 + 32z + 96 = 0$

$\Rightarrow \quad 16x + 24y + 32z - 56 = 0$

$$\Rightarrow \qquad 2x + 3y + 4z - 7 = 0 \qquad \ldots (i)$$

Now the distance of point (7, 2, 4) to plane (*i*) is

$$\left|\frac{2 \times 7 + 3 \times 2 + 4 \times 4 - 7}{2^2 + 3^2 + 4^2}\right| = \frac{14 + 6 + 16 - 7}{\sqrt{29}} = \frac{29}{\sqrt{29}} \quad \sqrt{29} \text{ unit.}$$

OR

Given line and plane are

$$\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k}) \qquad \ldots(i)$$

$$\vec{r}\,.(\hat{i} - \hat{j} + \hat{k}) = 5 \qquad \ldots(ii)$$

For intersection point, we solve equations (*i*) and (*ii*) by putting the value of $\vec{r}$ from (*i*) in (*ii*).

$$[(2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})].(\hat{i} - \hat{j} + \hat{k}) = 5$$

$\Rightarrow \qquad (2 + 1 + 2) + \lambda(3 - 4 + 2) = 5 \quad \Rightarrow 5 + \lambda = 5 \quad \Rightarrow \lambda = 0$

Hence, position vector of intersecting point is $2\hat{i} - \hat{j} + 2\hat{k}$.

i.e., coordinates of intersection of line and plane is (2, –1, 2).

Hence, Required distance $= \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2} = \sqrt{9 + 16 + 144} = \sqrt{169} = 13$ units

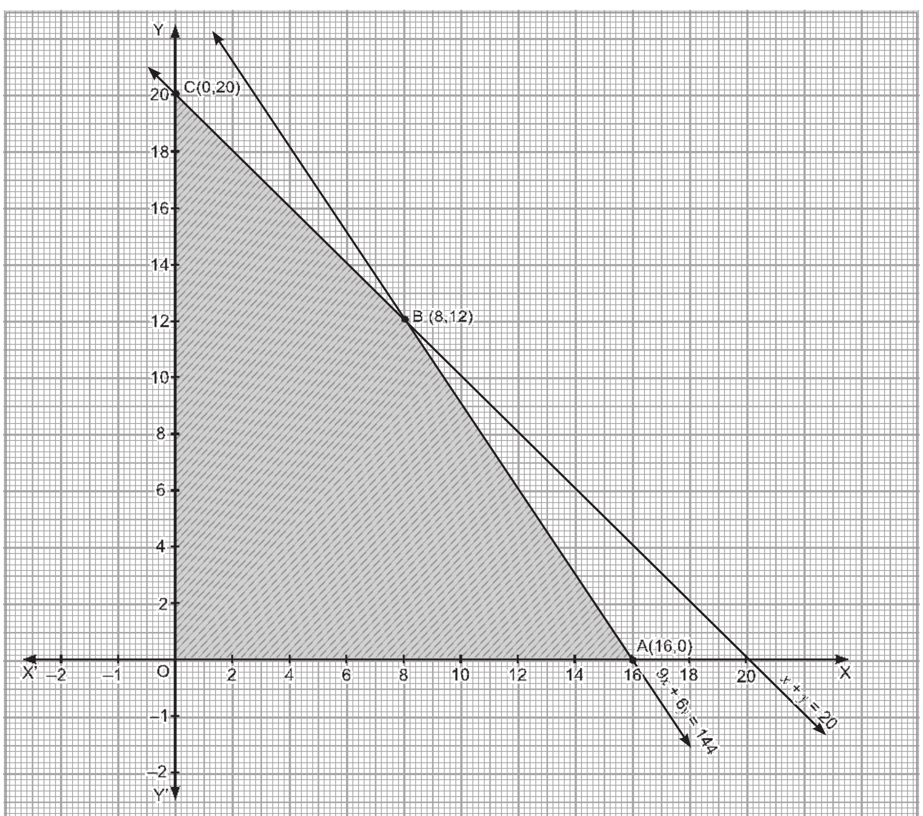
28. Suppose dealer purchase x electronic sewing machines and y manually operated sewing machines. If Z denotes the total profit. Then according to question

(Objective function) $Z = 22x + 18y$

Also, $x + y \le 20$

$360x + 240y \le 5760 \quad \Rightarrow \qquad 9x + 6y \le 144$

$x \ge 0,\ y \ge 0.$



We have to maximise Z subject to above constraint.

To solve graphically, at first we draw the graph of line corresponding to given inequations and shade the feasible region $OABC$.

The corner points of the feasible region $OABC$ are $O(0, 0)$, $A(16, 0)$, $B(8, 12)$ and $C(0, 20)$.

Now the value of objective function Z at corner points are obtained in table as

Corner points	$\mathbf{Z = 22x + 18y}$	
$O(0, 0)$	$Z = 0$	
$A(16, 0)$	$Z = 22 \times 16 + 18 \times 0 = 352$	
$B(8, 12)$	$Z = 22 \times 8 + 18 \times 12 = 392$	Maximum
$C(0, 20)$	$Z = 22 \times 0 + 18 \times 20 = 360$	

From table, it is obvious that Z is maximum when $x = 8$ and $y = 12$.

Hence, dealer should purchase 8 electronic sewing machines and 12 manually operated sewing machines to obtain the maximum profit ` 392 under given condition.

29. Let E_1, E_2, E_3, E_4 and A be event defined as

$E_1 =$ the lost card is a spade card.

$E_2 =$ the lost card is a heart card.

$E_3 =$ the lost card is a club card.

$E_4 =$ the lost card is diamond card.

and $A =$ Drawing three spade cards from the remaining cards.

$$P(E_1) = P(E_2) = P(E_3) = P(E_4) = \frac{13}{52} = \frac{1}{4}$$

$$P\left(\frac{A}{E_1}\right) = \frac{{}^{12}C_3}{{}^{51}C_3} = \frac{220}{20825}, \qquad P\left(\frac{A}{E_2}\right) = \frac{{}^{13}C_3}{{}^{51}C_3} = \frac{286}{20825}$$

$$P\left(\frac{A}{E_3}\right) = \frac{{}^{13}C_3}{{}^{51}C_3} = \frac{286}{20825}, \qquad P\left(\frac{A}{E_4}\right) = \frac{{}^{13}C_3}{{}^{51}C_3} = \frac{286}{20825}$$

Now, required probability $= P\left(\frac{E_1}{A}\right)$

$$P\left(\frac{E_1}{A}\right) = \frac{P(E_1).P\left(\frac{A}{E_1}\right)}{P(E_1).P\left(\frac{A}{E_1}\right) + P(E_2).P\left(\frac{A}{E_2}\right) + P(E_3).P\left(\frac{A}{E_3}\right) + P(E_4).P\left(\frac{A}{E_4}\right)}$$

$$= \frac{\frac{1}{4} \times \frac{220}{20825}}{\frac{1}{4} \times \frac{220}{20825} + \frac{1}{4} \times \frac{286}{20825} + \frac{1}{4} \times \frac{286}{20825} + \frac{1}{4} \times \frac{286}{20825}}$$

$$= \frac{220}{220 + 286 + 286 + 286}$$

$= \quad =$

$$\frac{220}{1078} \quad \frac{10}{49}$$

OR

Let the number of defective bulbs be represented by a random variable X. X may have value 0, 1, 2, 3, 4.

If p is the probability of getting defective bulb in a single draw then

$$p = \frac{5}{15} = \frac{1}{3}$$

$\therefore$ q = Probability of getting non defective bulb $= 1 - \frac{1}{3} = \frac{2}{3}$.

Since each trial in this problem is Bernaulli trials, therefore we can apply binomial distribution as

$$P(X = r) = {}^nC_r \cdot p^r \cdot q^{n-r} \text{ when } n = 4$$

$$P(X = 0) = {}^4C_0\left(\frac{1}{3}\right)^0 \cdot \left(\frac{2}{3}\right)^4 = \frac{16}{81}$$

Now $P(X = 1) = {}^4C_1\left(\frac{1}{3}\right)^1 \cdot \left(\frac{2}{3}\right)^3 = 4 \times \frac{1}{3} \times \frac{8}{27} = \frac{32}{81}$

$$P(X = 2) = {}^4C_2\left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^2 = 6 \times \frac{1}{9} \times \frac{4}{9} = \frac{24}{81}$$

$$P(X = 3) = {}^4C_3\left(\frac{1}{3}\right)^3 \cdot \left(\frac{2}{3}\right)^1 = 4 \times \frac{1}{27} \times \frac{2}{3} = \frac{8}{81}$$

$$P(X = 4) = {}^4C_4\left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^0 = \frac{1}{81}$$

Now probability distribution table is

X	0	1	2	3	4
$P(X)$	$\frac{16}{81}$	$\frac{32}{81}$	$\frac{24}{81}$	$\frac{8}{81}$	$\frac{1}{81}$

Now mean $E(X) = \Sigma p_i x_i$

$$= 0 \times \frac{16}{81} + 1 \times \frac{32}{81} + 2 \times \frac{24}{81} + 3 \times \frac{8}{81} + 4 \times \frac{1}{81}$$

$$\text{Mean} = \frac{32}{81} + \frac{48}{81} + \frac{24}{81} + \frac{4}{81} = \frac{108}{81} = \frac{4}{3}.$$

SET–II

9. Let $I = \int \cos^{-1}(\sin x)dx$

$$= \int \cos^{-1}\left(\cos\left(\frac{\pi}{2} - x\right)\right)dx = \int\left(\frac{\pi}{2} - x\right)dx$$

$\therefore$ $I = \frac{\pi}{2}\int dx - \int x dx$

$$= \frac{\pi}{2}x - \frac{x^2}{2} + c$$

10. $\vec{a} \times \vec{b} = |\vec{a}|\, |\vec{b}| \sin\theta \hat{n}$

$\Rightarrow \quad |\vec{a} \times \vec{b}| = ||\vec{a}|.|\vec{b}| \sin\theta \hat{n}|$

$\Rightarrow \quad 1 = \left| 3 \times \frac{2}{3} \sin\theta\, n \right| \qquad 1 = 2\sin\theta\, |n|$

$\Rightarrow \Rightarrow \quad 1 = 2\sin\theta \qquad [\because |\hat{n}| = 1]$

$\Rightarrow \quad \sin\theta = \frac{1}{2} \qquad \Rightarrow \qquad \theta = 30°.$

19. L.H.S. $\quad \Delta = \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$

Applying $R_1 \to R_1 - R_2;\ R_2 \to R_2 - R_3$, we get

$$\Delta = \begin{vmatrix} a+b+c & -(a+b+c) & 0 \\ 0 & a+b+c & -(a+b+c) \\ c & a & c+a+2b \end{vmatrix}$$

Taking $(a+b+c)$ common along R_1 and R_2, we get

$$\Delta = (a+b+c)^2 \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ c & a & c+a+2b \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ c & a+c & c+a+2b \end{vmatrix} \qquad [\text{Applying } C_2 \to C_2 + C_1]$$

Again applying $C_3 \to C_3 + C_2$, we get

$$\Delta = (a+b+c)^2 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & a+c & 2(c+a+b) \end{vmatrix}$$

$= (a+b+c)^2 \cdot 2(a+b+c)$ ($\because$ determinant of triangular matrix is product of its diagonal elements)

$= 2(a+b+c)^3$

$\Delta =$ R.H.S.

20. Let $u = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ and $v = \sin^{-1}(2x\sqrt{1-x^2})$

We have to determine $\frac{du}{dv}$

Let $x = \sin\theta \qquad \Rightarrow \qquad \theta = \sin^{-1} x$

Now, $\qquad u = \tan^{-1}\Big($

$\sin\theta \mid 1 - \sin^2\theta)$

$$\Rightarrow \quad u = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right) \quad \Rightarrow \quad u = \theta$$

$$\Rightarrow \quad u = \sin^{-1} x \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Again, $v = \sin^{-1}(2x\sqrt{1-x^2})$

$$\Rightarrow \quad v = \sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta})$$

$$\Rightarrow \quad v = \sin^{-1}(2\sin\theta\cos\theta)$$

$$\Rightarrow \quad v = \sin^{-1}(\sin 2\theta) \quad \Rightarrow \quad v = 2\theta$$

$$\Rightarrow \quad v = 2\sin^{-1} x \quad \Rightarrow \quad \frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}}$$

$$\therefore \quad \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{1}{\sqrt{1-x^2}}}{\frac{2}{\sqrt{1-x^2}}} = \frac{1}{2}$$

$$\left[\because \; -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}} \;\Rightarrow\; \sin\left(-\frac{\pi}{4}\right) < \sin\theta < \sin\left(\frac{\pi}{4}\right) \;\Rightarrow\; -\frac{\pi}{4} < \theta < \frac{\pi}{4} \;\Rightarrow\; -\frac{\pi}{2} < 2\theta < \frac{\pi}{2} \;\Rightarrow\; 2\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$$

[**Note:** Here the range of x is taken as $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$]

21. $\operatorname{cosec} x \log y \dfrac{dy}{dx} + x^2 y^2 = 0$

$$\operatorname{cosec} x . \log y \frac{dy}{dx} = -x^2 y^2$$

$$\Rightarrow \quad \frac{\log y . dy}{y^2} = -\frac{x^2 dx}{\operatorname{cosec} x}$$

$$\Rightarrow \quad \int y^{-2} . \log y \, dy = -\int x^2 \sin x \, dx$$

$$\Rightarrow \quad \log y . \frac{y^{-2+1}}{-2+1} - \int \frac{1}{y} . \frac{y^{-2+1}}{-2+1} dy = -[x^2(-\cos x) - \int 2x(-\cos x) \, dx]$$

$$\Rightarrow \quad -\frac{1}{y}\log y + \int y^{-2} dy = x^2 \cos x - 2\int x \cos x \, dx$$

$$\Rightarrow \quad = -\frac{1}{y}\log y + \frac{y^{-2+1}}{-2+1} = x^2 \cos x - 2\,[x \sin x - \int \sin x \, dx]$$

$$\Rightarrow \quad = -\frac{1}{y}\log y - \frac{1}{y} = x^2 \cos x - 2x \sin x + 2(-\cos x) + c$$

$$\Rightarrow \quad = -\frac{1}{y}(\log y + 1) = x^2 \cos x - 2x \sin x - 2\cos x + c.$$

$\Rightarrow$

22. Given lines are

$$\frac{5-x}{-4}=\frac{y-7}{4}=\frac{z+3}{-5} \quad\Rightarrow\quad \frac{x-5}{4}=\frac{y-7}{4}=\frac{z+3}{-5} \qquad \ldots (i)$$

and $$\frac{x-8}{7}=\frac{2y-8}{2}-\frac{z-5}{3} \Rightarrow \frac{x-8}{7}=\frac{y-4}{1}=\frac{z-5}{3} \qquad \ldots (ii)$$

We know that, $\dfrac{x-x_1}{a_1}=\dfrac{y-y_1}{b_1}=\dfrac{z-z_1}{c_1}$ and $\dfrac{x-x_2}{a_2}=\dfrac{y-y_2}{b_2}=\dfrac{z-z_2}{c_2}$ are coplanar iff

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$

Now $$\begin{vmatrix} 8-5 & 4-7 & 5-(-3) \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix}=\begin{vmatrix} 3 & -3 & 8 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix}$$

$$= 3(12+5)+3(12+35)+8(4-28)$$
$$= 51+141-192 = 192-192 = 0$$

Hence lines (i) and (ii) are coplanar.

28. Let $$I=\int_0^{\pi}\frac{x\tan x}{\sec x\,.\,\operatorname{cosec} x}\,dx=\int_0^{\pi}\frac{x\,.\,\dfrac{\sin x}{\cos x}}{\dfrac{1}{\cos x}\,.\,\dfrac{1}{\sin x}}\,dx$$

$$I=\int_0^{\pi} x\sin^2 x\,dx$$

$$=\int_0^{\pi}(\pi-x)\sin^2(\pi-x)\,dx \qquad \left[\because \int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right]$$

$$I=\int_0^{\pi}\pi\sin^2 x\,dx-\int_0^{\pi}x\sin^2 x\,dx \quad\Rightarrow\quad 2I=\frac{\pi}{2}\int_0^{\pi}2\sin^2 x\,dx$$

$$=\frac{\pi}{2}\int_0^{\pi}(1-\cos 2x)\,dx=\frac{\pi}{2}\int_0^{\pi}dx-\frac{\pi}{2}\int_0^{\pi}\cos 2x\,dx$$

$$=\frac{\pi}{2}[x]_0^{\pi}-\frac{\pi}{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi}$$

$$\Rightarrow\quad 2I=\frac{\pi}{2}(\pi-0)-\frac{\pi}{4}(\sin 2\pi-\sin 0)$$

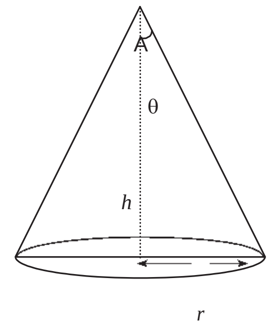
$$\Rightarrow\quad 2I=\frac{\pi^2}{2}-0 \quad\Rightarrow\quad I=\frac{\pi^2}{4}.$$

29. Let r, h, θ be radius, height and semi-vertical angle of cone having volume V.

If S be the surface area of cone then

$$S=\pi r\sqrt{h^2+r^2} \quad\Rightarrow\quad S^2=\pi^2 r^2(h^2+r^2)$$

$$\Rightarrow\quad S^2=\pi^2 r^2\left(\frac{9V^2}{\pi^2 r^4}+r^2\right) \qquad \left[\because V=\frac{1}{3}\pi r^2 h \;\Rightarrow\; h=\frac{3V}{\pi r^2}\right]$$



$\Rightarrow \quad S^2 = \dfrac{9V^2}{r^2} + \pi^2 r^4$

$\Rightarrow \quad \dfrac{d(S^2)}{dr} = -\dfrac{18V^2}{r^3} + 4\pi^2 r^3$

For extremum value of S or S^2.

$$\frac{d(S^2)}{dr} = 0$$

$\Rightarrow \quad -\dfrac{18V^2}{r^3} + 4\pi^2 r^3 = 0 \qquad \Rightarrow \quad \dfrac{18V^2}{r^3} = 4\pi^2 r^3$

$\Rightarrow \quad r^6 = \dfrac{18V^2}{4\pi^2} = \dfrac{9V^2}{2\pi^2} \qquad \Rightarrow \quad r^3 = \dfrac{3V}{\sqrt{2}\pi}$

Again $\quad \dfrac{d^2(S^2)}{dr^2} = \dfrac{54V^2}{r^4} + 12\pi^2 r^2$

$\Rightarrow \quad \left[\dfrac{d^2(S^2)}{dr^2}\right]_{r^3 = \frac{3V}{\sqrt{2}\pi}} > 0$

i.e., For $r^3 = \dfrac{3V}{\sqrt{2}\pi}$, S^2 or S is minimum.

Hence for minimum curve surface area

$$r^3 = \frac{3}{\sqrt{2}\pi}\left(\frac{1}{3}\pi r^2 h\right)$$

$\Rightarrow \quad r^3 = \dfrac{r^2 h}{\sqrt{2}} \qquad \Rightarrow \quad \dfrac{r}{h} = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \quad \tan\theta = \dfrac{1}{\sqrt{2}} \qquad \cot\theta = \sqrt{2}$

$\Rightarrow \quad \Rightarrow \theta = \cot^{-1}(\sqrt{2}).$

SET–III

9. Let $I = \displaystyle\int_0^{\frac{\pi}{2}} e^x (\sin x - \cos x)\,dx$

$$I = \int_0^{\frac{\pi}{2}} -e^x(\cos x - \sin x)\,dx = -\int_0^{\frac{\pi}{2}} e^x(\cos x + (-\sin x))\,dx$$

$$= -[e^x \cos x]_0^{\pi} \qquad \left.\because \int e^x (f(x) + f'(x))dx = e^x . f(x) + c\right]$$

$$= -[e^{\frac{\pi}{2}} . \cos\frac{\pi}{2} - e^0 . \cos 0]$$

$$= -[0-1] = 1.$$

10. $\vec{a} + \vec{b} = (2\hat{i} + 2\hat{j} - 5\hat{k}) + (2\hat{i} + \hat{j} - 7\hat{k}) = 4\hat{i} + 3\hat{j} - 12\hat{k}$

$\therefore$ Required vector in the direction of $4\hat{i} + 3\hat{j} - 12\hat{k}$

$$= \frac{4\hat{i} + 3\hat{j} - 12\hat{k}}{\sqrt{4^2 + 3^2 + (-12)^2}} = \frac{4\hat{i} + 3\hat{j} - 12\hat{k}}{\sqrt{169}} = \frac{4\hat{i} + 3\hat{j} - 12\hat{k}}{13}$$

$$= \frac{4}{13}\hat{i} + \frac{3}{13}\hat{j} - \frac{12}{13}\hat{k}.$$

19. L.H.S. $\Delta = \begin{vmatrix} x^2+1 & xy & xz \\ xy & y^2+1 & yz \\ zx & zy & z^2+1 \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we have

$$\Delta = \begin{vmatrix} 1 + x(x+y+z) & xy & xz \\ 1 + y(x+y+z) & y^2+1 & yz \\ 1 + z(x+y+z) & zy & z^2+1 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 1 & xy & xz \\ 1 & y^2+1 & yz \\ 1 & zy & z^2+1 \end{vmatrix} + (x+y+z)\begin{vmatrix} x & xy & xz \\ y & y^2+1 & yz \\ z & zy & z^2+1 \end{vmatrix}$$

Changing row into column, we have

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ xy & y^2+1 & zy \\ xz & yz & z^2+1 \end{vmatrix} + (x+y+z)\begin{vmatrix} x & y & z \\ xy & y^2+1 & zy \\ xz & yz & z^2+1 \end{vmatrix}$$

For I determinant we apply, $C_1 \to C_1 - C_2, C_2 \to C_2 - C_3$

For II determinant we take out a from 1st column, we have

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ xy - y^2 - 1 & y^2 + 1 - zy & zy \\ xz - yz & yz - z^2 - 1 & z^2+1 \end{vmatrix} + x(x+y+z)\begin{vmatrix} 1 & y & z \\ y & y^2+1 & zy \\ z & yz & z^2+1 \end{vmatrix}$$

Expanding along first row, we have

$$\Delta = 1\,[(xy - y^2 - 1)(yz - z^2 - 1) - (xz - yz)(y^2 + 1 - zy)]$$

$$+ x(x+y+z)\,[\{1(y^2+1)(z^2+1) - y^2z^2\} - y(yz^2 + y - z^2y) + z(y^2z - y^2z - z)]$$

On solving, we have

$$\Delta = 1 + x^2 + y^2 + z^2 = \text{R.H.S.}$$

20. Let $u = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ and $v = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

Now $u = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$

Let $\quad x = \tan\theta \quad \Rightarrow \quad \theta = \tan^{-1} x$

$\therefore \quad u = \tan^{-1}\left(\dfrac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$

$\Rightarrow \quad u = \tan^{-1}\left(\dfrac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\dfrac{\dfrac{1}{\cos\theta}-1}{\dfrac{\sin\theta}{\cos\theta}}\right) = \tan^{-1}\left(\dfrac{1-\cos\theta}{\sin\theta}\right)$

$\Rightarrow \quad u = \tan^{-1}\left(\dfrac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}\right) = \tan^{-1}\left(\dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right) = \tan^{-1}\left(\tan\dfrac{\theta}{2}\right) \Rightarrow \quad u = \dfrac{\theta}{2}$

$\Rightarrow \quad u = \dfrac{1}{2}\tan^{-1} x$

$\therefore \quad \dfrac{du}{dx} = \dfrac{1}{2(1+x^2)} \qquad \ldots (i)$

Again $v = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

Let $x = \tan\theta \quad \Rightarrow \quad \theta = \tan^{-1} x$

$\therefore \quad v = \sin^{-1}\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right)$

$\Rightarrow \quad v = \sin^{-1}(\sin 2\theta) \qquad \left[\because \sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}\right]$

$\Rightarrow \quad v = 2\theta \quad \Rightarrow \quad v = 2\tan^{-1} x$

$\therefore \quad \dfrac{dv}{dx} = \dfrac{2}{1+x^2} \qquad \ldots (ii)$

Now $\dfrac{du}{dv} = \dfrac{\dfrac{du}{dx}}{\dfrac{dv}{dx}} \qquad$ [From (*i*) and (*ii*)]

$= \dfrac{\dfrac{1}{2(1+x^2)}}{\dfrac{2}{1+x^2}} = \dfrac{1}{2(1+x^2)} \times \dfrac{1+x^2}{2} = \dfrac{1}{4}.$

$2 \qquad 2(1 + x^2) \qquad 2 \qquad 4$

$1 + x$

21. Given differential equation is

$$\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$$

$\Rightarrow \quad (\sin y + y\cos y)dy = x(2\log x + 1)dx$

$\Rightarrow \quad \int \sin y\, dy + \int y\cos y\, dy = 2\int x\log x\, dx + \int x\, dx$

$\Rightarrow \quad \int \sin y\, dy + [y\sin y - \int \sin y\, dy] = 2\left[\log x\frac{x^2}{2} - \int \frac{1}{x}.\frac{x^2}{2}\, dx\right] + \int x\, dx$

$\Rightarrow \quad \int \sin y\, dy + y\sin y - \int \sin y\, dy = x^2 \log x - \int x\, dx + \int x\, dx + c$

$\Rightarrow \quad y\sin y = x^2 \log x + c \qquad \ldots (i)$

It is general solution.

For particular solution we put $y = \frac{\pi}{2}$ when $x = 1$

(i) becomes $\frac{\pi}{2}\sin\frac{\pi}{2} = 1.\log 1 + c$

$\frac{\pi}{2} = c \qquad [\because \log 1 = 0]$

Putting the value of c in (i), we get the required particular solution.

$$y\sin y = x^2 \log x + \frac{\pi}{2}.$$

22. Given lines are

$$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$$

$$\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$$

Given lines also may be written in cartesian form as

$$\frac{x-1}{3} = \frac{y-1}{-1} = \frac{z+1}{0} \qquad \ldots(i)$$

and $\quad \frac{x-4}{2} = \frac{y-0}{0} = \frac{z+1}{3} \qquad \ldots(ii)$

Let given lines (i) and (ii) intersect at point (α, β, γ).

$\Rightarrow \quad$ Point (α, β, γ) satisfy equation (i)

$\Rightarrow \quad \frac{\alpha - 1}{3} = \frac{\beta - 1}{-1} = \frac{\gamma + 1}{0} = \lambda$ (say)

$\Rightarrow \quad \alpha = 3\lambda + 1, \beta = -\lambda + 1, \gamma = -1$

Also, point (α, β, γ) satisfy equation (ii)

$\Rightarrow$

$\therefore$

α $+1\ 2$ 0

3

$-$

4

β

$-$

0

γ

$+$

1

2

0

3

3

$\rule{4em}{0.4pt} = \rule{3em}{0.4pt} = \rule{3em}{0.4pt}$

λ

$+$

1

$-$

4

$-$

λ

$+$

1

$-$

1

$$\Rightarrow \quad \frac{3\lambda - 3}{2} = \frac{-\lambda + 1}{0} = 0$$

I II III

From I and III

$$\frac{3(\lambda - 1)}{2} = 0$$

$$3\lambda - 3 = 0$$

$$\lambda = \frac{3}{3} = 1$$

From II and III

$$-\lambda + 1 = 0$$

$$\lambda = 1$$

The value of λ in both cases are same. Hence both lines (*i*) and (*ii*) intersect at a point.

The co-ordinate of intersecting point is (4, 0, –1).

28. Let $I = \int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) . \sin\left(\frac{\pi}{2} - x\right) . \cos\left(\frac{\pi}{2} - x\right)}{\sin^4\left(\frac{\pi}{2} - x\right) + \cos^4\left(\frac{\pi}{2} - x\right)} dx \qquad \left[\begin{array}{l}\text{By Property} \\ \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\end{array}\right.$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) \cos x . \sin x}{\cos^4 x + \sin^4 x} dx$$

$$\Rightarrow \quad I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\cos x . \sin x}{\sin^4 x + \cos^4 x} dx - \int_0^{\pi/2} \frac{x \sin x . \cos x}{\sin^4 x + \cos^4 x} dx$$

$$\Rightarrow \quad I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sin x . \cos x\, dx}{\sin^4 x + \cos^4 x} - I$$

$$\Rightarrow \quad 2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sin x . \cos x\, dx}{\sin^4 x + \cos^4 x} = \frac{\pi}{2} \int_0^{\pi/2} \frac{\frac{\sin x . \cos x}{\cos^4 x} dx}{\tan^4 x + 1}$$

[Dividing numerator and denominator by $\cos^4 x$]

$$= \frac{\pi}{2 \times 2} \int_0^{\pi/2} \frac{2 \tan x . \sec^2 x\, dx}{1 + (\tan^2 x)^2}$$

Let $\tan^2 x = z$; $2 \tan x . \sec^2 x\, dx = dz$

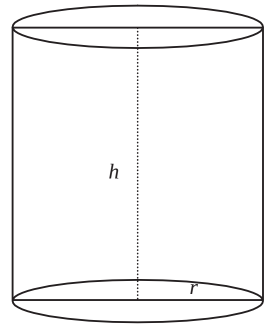
If $x = 0, z = 0$; $x = \frac{\pi}{2}, z = \infty$

$$\therefore \quad 2I = \frac{\pi}{4} \int_0^{\infty} \frac{dz}{1 + z^2} = \frac{\pi}{4} [\tan^{-1} z]_0^{\infty} = \frac{\pi}{4} (\tan^{-1} \infty - \tan^{-1} 0)$$

$\therefore \quad 2I = \frac{\pi}{4}\left(\frac{\pi}{2} - 0\right) \quad \Rightarrow \quad I = \frac{\pi^2}{16}$

29. Let r, h be radius and height of closed right circular cylinder having volume 128π cm^3.

If S be the surface area then



$S = 2\pi rh + 2\pi r^2$

$S = 2\pi(rh + r^2)$

$$S = 2\pi\left(r.\frac{128}{r^2} + r^2\right) \qquad \left[\begin{array}{l} \because V = \pi r^2 h \\ \Rightarrow 128\pi = \pi r^2 h \\ \therefore h = \frac{128}{r^2} \end{array}\right]$$

$$S = 2\pi\left(\frac{128}{r} + r^2\right)$$

$$\Rightarrow \quad \frac{dS}{dr} = 2\pi\left(-\frac{128}{r^2} + 2r\right)$$

For extreme value of S

$$\frac{dS}{dr} = 0$$

$$\Rightarrow \quad 2\pi\left(-\frac{128}{r^2} + 2r\right) = 0$$

$$\Rightarrow \quad -\frac{128}{r^2} + 2r = 0$$

$$\Rightarrow \quad 2r = \frac{128}{r^2} \quad \Rightarrow \quad r^3 = \frac{128}{2}$$

$$\Rightarrow \quad r^3 = 64 \quad \Rightarrow \quad r = 4$$

Again $\frac{d^2S}{dr^2} = 2\pi\left(\frac{128 \times 2}{r^3} + 2\right)$

$$\Rightarrow \quad \left.\frac{d^2S}{dr^2}\right]_{r=4} = +\text{ve}$$

Hence, for $r = 4$ cm, S(surface area) is minimum.

Therefore, dimensions for minimum surface area of cylindrical can are

radius $r = 4$ cm and $h = \frac{128}{r^2} = \frac{128}{16} = 8$ cm.

z z z

CBSE Examination Paper, All India-2014

Time allowed: 3 hours | **Maximum marks: 100**

General Instructions: *As per given in CBSE Examination Paper Delhi-2014.*

SET–I

SECTION–A

Question numbers 1 to 10 carry 1 mark each.

1. If $R = \{(x, y): x + 2y = 8\}$ is a relation on N, write the range of R.

2. If $\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4}$, $xy < 1$, then write the value of $x + y + xy$.

3. If A is a square matrix such that $A^2 = A$, then write the value of $7A - (I + A)^3$, where I is an identity matrix.

4. If $\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$, find the value of $x + y$.

5. If $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$, find the value of x.

6. If $f(x) = \int_0^x t \sin t \, dt$, then write the value of $f'(x)$.

7. Evaluate $\int_2^4 \frac{x}{x^2 + 1} dx$.

8. Find the value of 'p' for which the vectors $3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\hat{i} - 2p\hat{j} + 3\hat{k}$ are parallel.

9. Find $\vec{a}.(\vec{b} \times \vec{c})$, if $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$.

10. If the cartesian equations of a line are $\frac{3-x}{5} = \frac{y+4}{7} = \frac{2z-6}{4}$, write the vector equation for the line.

SECTION–B

Question numbers 11 to 22 carry 4 marks each.

11. If the function $f: R \to R$ be given by $f(x) = x^2 + 2$ and $g: R \to R$ be given by $g(x) = \frac{x}{x-1}$, $x \neq 1$, find fog and gof and hence find fog (2) and gof (– 3).

12. Prove that $\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right] = \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x, \frac{-1}{\sqrt{2}} \leq x \leq 1$

OR

If $\tan^{-1}\left(\frac{x-2}{x-4}\right) + \tan^{-1}\left(\frac{x+2}{x+4}\right) = \frac{\pi}{4}$, find the value of x.

13. Using properties of determinants, prove that

$$\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3$$

14. Find the value of $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$, if $x = ae^{\theta}(\sin\theta - \cos\theta)$ and $y = ae^{\theta}(\sin\theta + \cos\theta)$.

15. If $y = Pe^{ax} + Qe^{bx}$, show that

$$\frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby = 0.$$

16. Find the value(s) of x for which $y = [x(x-2)]^2$ is an increasing function.

OR

Find the equations of the tangent and normal to the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(\sqrt{2}a, b)$.

17. Evaluate: $\int_0^{\pi} \frac{4x \sin x}{1+\cos^2 x} dx$

OR

Evaluate: $\int \frac{x+2}{\sqrt{x^2+5x+6}} dx$

18. Find the particular solution of the differential equation $\frac{dy}{dx} = 1 + x + y + xy$, given that $y = 0$ when $x = 1$.

19. Solve the differential equation $(1+x^2)\frac{dy}{dx} + y = e^{\tan^{-1} x}$.

20. Show that the four points A, B, C and D with position vectors $4\hat{i} + 5\hat{j} + \hat{k}, -\hat{j} - \hat{k}, 3\hat{i} + 9\hat{j} + 4\hat{k}$ and $4(-\hat{i} + \hat{j} + \hat{k})$ respectively are coplanar.

OR

The scalar product of the vector $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of vectors $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{c} = \lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one. Find the value of λ and hence find the unit vector along $\vec{b} + \vec{c}$.

21. A line passes through (2, –1, 3) and is perpendicular to the lines

$\vec{r} = \hat{i} + \hat{j} - \hat{k} + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$ and

$\vec{r} = (2\hat{i} - \hat{j} - 3\hat{k}) + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$. Obtain its equation in vector and cartesian form.

22. An experiment succeeds thrice as often as it fails. Find the probability that in the next five trials, there will be at least 3 successes.

SECTION–C

Question numbers 23 to 29 carry 6 marks each.

23. ***Two schools A and B want to award their selected students on the values of sincerity, truthfulness and helpfulness. The school A wants to award ₹ x each, ₹ y each and ₹ z each for the three respective values to 3, 2 and 1 students respectively with a total award money of ₹ 1,600. School B wants to spend ₹ 2,300 to award its 4, 1 and 3 students on the respective values (by giving the same award money to the three values as before). If the total amount of award for one prize on each value is ₹ 900, using matrices, find the award money for each value. Apart from these three values, suggest one more value which should be considered for award.***

24. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$. Also show that the maximum volume of the cone is $\frac{8}{27}$ of the volume of the sphere.

25. Evaluate: $\int \frac{1}{\cos^4 x + \sin^4 x}\, dx$

26. Using integration, find the area of the region bounded by the triangle whose vertices are (–1, 2), (1, 5) and (3, 4).

27. Find the equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$. Also find the distance of the plane obtained above, from the origin.

OR

Find the distance of the point (2, 12, 5) from the point of intersection of the line $\vec{r} = 2\hat{i} - 4\hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r}\cdot(\hat{i} - 2\hat{j} + \hat{k}) = 0$.

28. A manufacturing company makes two types of teaching aids A and B of Mathematics for class XII. Each type of A requires 9 labour hours of fabricating and 1 labour hour for finishing. Each type of B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available per week are 180 and 30 respectively. The company makes a profit of ₹ 80 on each piece of type A and ₹ 120 on each piece of type B. How many pieces of type A and type B should be manufactured per week to get a maximum profit? Make it as an LPP and solve graphically. What is the maximum profit per week?

29. There are three coins. One is a two-headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the times and third is also a biased coin that comes up tails 40% of the times. One of the three coins is chosen at random and tossed, and it shows heads. What is the probability that it was the two-headed coin?

OR

Two numbers are selected at random (without replacement) from the first six positive integers. Let X denote the larger of the two numbers obtained. Find the probability distribution of the random variable X, and hence find the mean of the distribution.

SET–II

Only those questions, not included in Set I, are given.

9. Evaluate: $\int_e^{e^2} \frac{dx}{x \log x}$

10. Find a vector $\vec{a}$ of magnitude $5\sqrt{2}$, making an angle of $\frac{\pi}{4}$ with x-axis., $\frac{\pi}{2}$ with y-axis and an acute angle θ with z-axis.

19. Using properties of determinants, prove that

$$\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

20. If $x = a \sin 2t(1 + \cos 2t)$ and $y = b \cos 2t(1 - \cos 2t)$, show that at $t = \frac{\pi}{4}, \left(\frac{dy}{dx}\right) = \frac{b}{a}$.

21. Find the particular solution of the differential equation $x(1 + y^2)dx - y(1 + x^2)dy = 0$, given that $y = 1$ when $x = 0$.

22. Find the vector and cartesian equations of the line passing through the point (2, 1, 3) and perpendicular to the lines $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ and $\frac{x}{-3} = \frac{y}{2} = \frac{z}{5}$.

28. Evaluate: $\int (\sqrt{\cot x} + \sqrt{\tan x})\, dx$

29. Prove that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

SET–III

Only those questions, not included in Set I and Set II, are given.

9. If $\int_0^a \frac{1}{4 + x^2}\, dx = \frac{\pi}{8}$, find the value of a.

10. If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, $|\vec{a} + \vec{b}| = 13$ and $|\vec{a}| = 5$, find the value of $|\vec{b}|$.

19. Using properties of determinants, prove that:

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc + bc + ca + ab.$$

20. If $x = \cos t(3 - 2\cos^2 t)$ and $y = \sin t(3 - 2\sin^2 t)$, find the value of $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$.

21. Find the particular solution of the differential equation $\log\left(\frac{dy}{dx}\right) = 3x + 4y$, given that $y = 0$ when $x = 0$.

22. Find the value of p, so that the lines $l_1 = \frac{1-x}{3} = \frac{7y-14}{p} = \frac{z-3}{2}$ and $l_2 : \frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$ are perpendicular to each other. Also find the equations of a line passing through a point (3, 2, –4) and parallel to line l_1.

28. If the sum of the lengths of the hypotenuse and a side of a right triangle is given, show that the area of the triangle is maximum, when the angle between them is 60°.

29. Evaluate:

$$\int \frac{1}{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x} dx$$

Solutions

SET–I

SECTION–A

1. Given:

$R = \{(x, y) : x + 2y = 8\}$

$\because \quad x + 2y = 8$

$\Rightarrow \quad y = \dfrac{8 - x}{2} \quad \Rightarrow \quad$ when $x = 6, y = 1; \quad x = 4, y = 2; \; x = 2, y = 3.$

$\therefore \quad$ Range $= \{1, 2, 3\}$

2. Given

$$\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{x + y}{1 - xy}\right] = \frac{\pi}{4} \qquad [\because xy < 1]$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{x + y}{1 - xy}\right] = \tan^{-1} 1$$

$$\Rightarrow \quad \frac{x + y}{1 - xy} = 1 \qquad \Rightarrow \qquad x + y = 1 - xy$$

$$\Rightarrow \quad x + y + xy = 1$$

3. $7A - (I + A)^3 = 7A - \{I^3 + 3I^2A + 3I.A^2 + A^3\}$

$$= 7A - \{I + 3A + 3A + A^2A\} \qquad [\because I^3 = I^2 = I, A^2 = A]$$

$$= 7A - \{I + 6A + A^2\} = 7A - \{I + 6A + A\}$$

$$= 7A - \{I + 7A\} = 7A - I - 7A = -I$$

4. Given $\begin{bmatrix} x - y & z \\ 2x - y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$

Equating, we get

$x - y = -1 \qquad \ldots (i)$

$2x - y = 0 \qquad \ldots (ii)$

$z = 4, \qquad w = 5$

$(ii) - (i) \quad \Rightarrow \quad 2x - y - x + y = 0 + 1$

$\Rightarrow \quad x = 1$ and $y = 2$

$\therefore \quad x + y = 2 + 1 = 3.$

5. Given $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$

$\Rightarrow \quad 12x + 14 = 32 - 42 \qquad \Rightarrow \qquad 12x = -10 - 14$

$\Rightarrow \quad 12x = -24 \qquad \Rightarrow \qquad x = -2$

6. Given $f(x) = \int_0^x t \sin t \, dt$

According to Leibnitz' Rule

$$\frac{d}{dx}\left(\int_{g(x)}^{h(x)} f(t)dt\right) = f(h(x)).\frac{d}{dx}(h(x)) - f(g(x)).\frac{d}{dx}(g(x))$$

Here $g(x) = 0$, $h(x) = x$. $f(t) = t \sin t$

$\therefore \quad f'(x) = f(x).\frac{d}{dx}(x) - f(0).\frac{d}{dx}(0)$

$\qquad = x. \sin x \,.1 - 0 = x \sin x.$

7. Let, $I = \int_2^4 \frac{x}{x^2+1} dx$

Let $x^2 + 1 = z$

$\Rightarrow \quad 2xdx = dz \qquad \Rightarrow \quad xdx = \frac{dz}{2}$

Also $x = 2 \Rightarrow z = 5$ and $x = 4 \Rightarrow z = 17$

$\therefore \quad I = \frac{1}{2}\int_5^{17} \frac{dz}{z}$

$\qquad = \frac{1}{2}\left[\log z\right]_5^{17} = \frac{1}{2}[\log 17 - \log 5] = \frac{1}{2}\log\frac{17}{5}.$

8. $\because$ Given two vectors are parallel

$\Rightarrow \quad \frac{3}{1} = \frac{2}{-2p} = \frac{9}{3} \qquad \Rightarrow \quad \frac{3}{1} = \frac{2}{-2p}$

$\Rightarrow \quad -6p = 2 \qquad \Rightarrow \quad p = -\frac{1}{3}.$

9. Given

$\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}, \quad \vec{b} = -\hat{i} + 2\hat{j} + \hat{k}, \quad \vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$

$$\therefore \quad \vec{a}.(\vec{b} \times \vec{c}) = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

$= 2(4-1) - 1(-2-3) + 3(-1-6)$

$= 2 \times 3 - 1 \times (-5) + 3 \times (-7) = 6 + 5 - 21 = -10$

10. Given cartesian equation of a line is

$$\frac{3-x}{5} = \frac{y+4}{7} = \frac{2z-6}{4}$$

$$\Rightarrow \quad \frac{x-3}{-5} = \frac{y-(-4)}{7} = \frac{z-3}{2}$$

Hence its vector form is

$\rightarrow$

$$r = (3\hat{i} - 4\hat{j} + 3\hat{k}) + \lambda(-5\hat{i} + 7\hat{j} + 2\hat{k})$$

SECTION–B

11. Given, $f(x) = x^2 + 2,\ g(x) = \dfrac{x}{x-1}$

$\therefore \quad fog(x) = f(g(x))$

$= f\left(\dfrac{x}{x-1}\right) \qquad \left[\because g(x) = \dfrac{x}{x-1}\right]$

$= \left(\dfrac{x}{x-1}\right)^2 + 2 \qquad [\because f(x) = x^2 + 2]$

$$= \frac{x^2}{(x-1)^2} + 2 = \frac{x^2 + 2(x-1)^2}{(x-1)^2} = \frac{x^2 + 2x^2 - 4x + 2}{(x-1)^2} = \frac{3x^2 - 4x + 2}{(x-1)^2}$$

Again $gof(x) = g(f(x))$

$= g(x^2 + 2) \qquad [\because f(x) = x^2 + 2]$

$= \dfrac{x^2 + 2}{x^2 + 2 - 1} \qquad \left[\because g(x) = \dfrac{x}{x-1}\right]$

$= \dfrac{x^2 + 2}{x^2 + 1}$

$\therefore \quad fog(2) = \dfrac{3 \times 2^2 - 4 \times 2 + 2}{(2-1)^2} = \dfrac{12 - 8 + 2}{1} = 6$

and $gof(-3) = \dfrac{(-3)^2 + 2}{(-3)^2 + 1} = \dfrac{9 + 2}{9 + 1} = \dfrac{11}{10}$

12. L.H.S. $= \tan^{-1}\left(\dfrac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right)$

$= \tan^{-1}\left(\dfrac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \times \dfrac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} - \sqrt{1-x}}\right) \qquad$ [Rationalize]

$= \tan^{-1}\left(\dfrac{2 - 2\sqrt{1-x^2}}{1 + x - 1 + x}\right) = \tan^{-1}\left(\dfrac{1 - \sqrt{1-x^2}}{x}\right)$

Putting $x = \sin\theta \Rightarrow \theta = \sin^{-1} x$

$= \tan^{-1}\left(\dfrac{1 - \cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\dfrac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2} \cdot \cos\frac{\theta}{2}}\right)$

$\sqrt{2}$ ... $\tan$ ... $\dfrac{\pi}{4}$ ... $\sin^{-1} x = \dfrac{\pi}{2} - \cos^{-1} x$

$$= \frac{\ }{4} - \frac{\ }{2} \cos^{-1} x$$

$$\left[\begin{array}{l} \text{Q} \quad -\frac{1}{\ } \le x \le 1 \\ \Rightarrow \quad \sin\left(-\frac{\pi}{\ }\right) \le \sin\theta \le \sin\frac{\pi}{\ } \\ \Rightarrow \quad -\frac{\pi}{\ } \le \theta \le \frac{\pi}{\ } \quad \Rightarrow \quad -\frac{\pi}{\ } \le \frac{\theta}{\ } \le \frac{\pi}{\ } \\ \Rightarrow \quad \frac{\theta}{2} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{array}\right]$$

OR

Given $\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right)=\frac{\pi}{4}$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\frac{x-2}{x-4}+\frac{x+2}{x+4}}{1-\frac{x-2}{x-4}\cdot\frac{x+2}{x+4}}\right]=\frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{(x-2)(x+4)+(x+2)(x-4)}{(x-4)(x+4)-(x-2)(x+2)}\right]=\frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{x^2+4x-2x-8+x^2-4x+2x-8}{(x^2-16)-(x^2-4)}\right]=\frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{2x^2-16}{-12}\right]=\frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2x^2-16}{-12}\right)=\frac{\pi}{4} \quad \Rightarrow \quad \frac{x^2-8}{-6}=\tan\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \quad \frac{x^2-8}{-6}=1 \quad \Rightarrow \quad x^2-8=-6 \quad \Rightarrow \quad x^2=-6+8 \quad \Rightarrow \quad x^2=2$$

$$\Rightarrow \quad x=\pm\sqrt{2}$$

13. L.H.S. $=\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix}$

$=x^2\begin{vmatrix} x+y & 1 & 1 \\ 5x+4y & 4 & 2 \\ 10x+8y & 8 & 3 \end{vmatrix}$ [Taking out x from C_2 and C_3]

$=x^2\begin{vmatrix} x+y & 1 & 1 \\ 3x+2y & 2 & 0 \\ 7x+5y & 5 & 0 \end{vmatrix}$ [Applying $R_2 \to R_2-2R_1$ and $R_3 \to R_3-3R_1$]

Expanding along C_3, we get

$x^2\,[1\{(3x+2y)5-2(7x+5y)\}-0+0]$

$=x^2(15x+10y-14x-10y)=x^2(x)=x^3=$ R.H.S.

14. Given $x=ae^{\theta}(\sin\theta-\cos\theta)$

$$y = ae^{\theta}(\sin\theta + \cos\theta)$$

Q $$x = ae^{\theta}(\sin\theta - \cos\theta)$$

Differentiating w.r.t. θ, we get

$$\frac{dx}{d\theta} = ae^{\theta}(\cos\theta + \sin\theta) + a(\sin\theta - \cos\theta).\,e^{\theta}$$

$$= ae^{\theta}(\cos\theta + \sin\theta + \sin\theta - \cos\theta)$$

$$= 2ae^{\theta}\sin\theta \qquad \ldots (i)$$

Again $\because$ $y = ae^{\theta}(\sin\theta + \cos\theta) \quad \Rightarrow \quad \underset{I}{e^{\theta}}\,.\,\underset{II}{a}(\sin\theta + \cos\theta)$

$$\frac{dy}{d\theta} = ae^{\theta}(\cos\theta - \sin\theta)$$

$$+ a(\sin\theta + \cos\theta).e^{\theta} = ae^{\theta}(\cos\theta - \sin\theta$$

$$+ \sin\theta + \cos\theta)$$

$$= 2ae^{\theta}.\cos\theta \qquad \ldots (ii)$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2ae^{\theta}.\cos\theta}{2ae^{\theta}.\sin\theta} \qquad \text{[From (i) and (ii)]}$$

$$\Rightarrow \quad \frac{dy}{dx} = \cot\theta$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right]_{\theta=\frac{\pi}{4}} = \cot\frac{\pi}{4} = 1$$

15. $y = Pe^{ax} + Qe^{bx}$ $\ldots (i)$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = Pae^{ax} + Qbe^{bx}$$

Again differentiating both sides w.r.t. x we get

$$\frac{d^2y}{dx^2} = Pa^2e^{ax} + Qb^2e^{bx}$$

L.H.S. $= \dfrac{d^2y}{dx^2} - (a+b)\dfrac{dy}{dx} + aby = 0$

$$= Pa^2e^{ax} + Qb^2e^{bx} - (a+b)\{Pae^{ax} + Qbe^{bx}\} + aby$$

$$= Pa^2e^{ax} + Qb^2e^{bx} - Pa^2e^{ax} - Qabe^{bx} - Pabe^{ax} - Qb^2e^{bx} + aby$$

$$= -ab(Pe^{ax} + Qe^{bx}) + aby$$

$$= -aby + aby \qquad \text{[From (i)]}$$

$$= 0$$

16. Given, $y = [x(x-2)]^2$

$$\therefore \quad \frac{dy}{dx} = 2[x(x-2)] \times (2x-2)$$

$$\frac{dy}{dx} = 4x(x-1)(x-2)$$

For increasing function, $\dfrac{dy}{dx} > 0$

$4x(x$

$$-1)(x-2)>0$$

–ve +ve –ve +ve

0 1 2

Sign rule

$$\Rightarrow \qquad x(x-1)(x-2)>0$$

From sign rule,

For $\quad \dfrac{dy}{dx} > 0$ value of $x = 0 < x < 1$ and $x > 2$

Therefore, y is increasing $\forall \; x \in (0, 1) \cup (2, \infty)$

OR

Given curve is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \ldots(i)$$

Differentiating both sides w.r.t. x we get

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0 \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{2x}{2y} \times \frac{b^2}{a^2}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{b^2}{a^2}\frac{x}{y}$$

Now, slope of tangent at $(\sqrt{2}a, b)$ to the curve (i)

$$= \left.\frac{dy}{dx}\right]_{(\sqrt{2}a,\, b)} = \frac{b^2}{a^2} \cdot \frac{\sqrt{2}a}{b} = \frac{\sqrt{2}b}{a}$$

Also slope of normal at $(\sqrt{2}a, b)$ to curve $(i) = -\dfrac{a}{\sqrt{2}b}$.

$\therefore \quad$ Equation of tangent is

$$(y - b) = \frac{\sqrt{2}b}{a}(x - \sqrt{2}a)$$

And Equation of normal is

$$(y - b) = \frac{-a}{\sqrt{2}\,b}(x - \sqrt{2}a)$$

17. Let $I = \displaystyle\int_0^{\pi} \frac{4x \sin x}{1 + \cos^2 x}\,dx \qquad \ldots(i)$

$$= \int_0^{\pi} \frac{4(\pi - x) . \sin(\pi - x)}{1 + \cos^2(\pi - x)}\,dx$$

$$I = \int_0^{\pi} \frac{4(\pi - x) . \sin x}{1 + \cos^2 x}\,dx \qquad \ldots(ii)$$

Adding (i) and (ii) we get

$$2I = \int_0^{\pi} \frac{4(x + \pi - x) \sin x}{1 + \cos^2 x}\,dx \qquad \Rightarrow \qquad 2I = 4\int_0^{\pi} \frac{\pi \sin x}{1 + \cos^2 x}\,dx$$

$$\Rightarrow \qquad I = 2\pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x}\,dx$$

Let $\cos x = z \;\Rightarrow\; -\sin x\, dx = dz \;\Rightarrow\; \sin x\, dx = -dz$

Also $\quad x = 0, \qquad z = 1$

$x = \pi, \qquad z = -1$

$\therefore \qquad I = 2\pi \int_{1}^{-1} \frac{-dz}{1+z^2} = 2\pi[\tan^{-1} z]_{-1}^{1}$

$= 2\pi[\tan^{-1} 1 - \tan^{-1}(-1)] = 2\pi\left[\frac{\pi}{4} + \frac{\pi}{4}\right] = 2\pi \times \frac{\pi}{2}$

$\Rightarrow \qquad I = \pi^2.$

OR

Let $\quad I = \int \frac{x+2}{\sqrt{x^2+5x+6}}\, dx$

Let $\quad x + 2 = A\frac{d}{dx}(x^2+5x+6) + B$

$x + 2 = A(2x+5) + B \qquad \Rightarrow \qquad x + 2 = 2Ax + (5A+B)$

Equating both sides, we get

$2A = 1, 5A + B = 2 \qquad \Rightarrow \qquad A = \frac{1}{2}, \quad B = 2 - \frac{5}{2} = -\frac{1}{2}$

$\therefore \qquad x + 2 = \frac{1}{2}(2x+5) - \frac{1}{2}$

Hence, $I = \int \frac{\frac{1}{2}(2x+5) - \frac{1}{2}}{\sqrt{x^2+5x+6}}\, dx = \frac{1}{2}\int \frac{2x+5}{\sqrt{x^2+5x+6}}\, dx - \frac{1}{2}\int \frac{dx}{\sqrt{x^2+5x+6}}$

$I = \frac{1}{2}.I_1 - \frac{1}{2}I_2 \qquad \ldots (i)$

where, $I_1 = \int \frac{2x+5}{\sqrt{x^2+5x+6}}\, dx, I_2 = \int \frac{dx}{\sqrt{x^2+5x+6}}$

Now, $I_1 = \int \frac{2x+5}{\sqrt{x^2+5x+6}}\, dx$

Let $x^2 + 5x + 6 = z \quad \Rightarrow \quad (2x+5)dx = dz$

$\therefore \qquad I_1 = \int \frac{dz}{\sqrt{z}} = \int z^{\frac{-1}{2}}\, dz = \frac{z^{-\frac{1}{2}+1}}{\frac{-1}{2}+1} + c_1 = 2\sqrt{z} + c_1$

$= 2\sqrt{x^2+5x+6} + c_1$

Again $I_2 = \int \frac{dx}{\sqrt{x2+5x+6}}$

$= \int \frac{dx}{\sqrt{x^2 + 2 \times x \times \frac{5}{2} + \left(\frac{5}{2}\right)^2 - \frac{25}{4} + 6}}$

$\left(= \int \frac{5}{} \right) \quad dx \frac{1}{2}$ $\qquad = \int \left(x + \frac{}{2}\right) - \frac{}{4}$

dx

$($

$\sqrt{|\, x + - | \quad - | - |}$

5

$)$

2

$($

1

$)$

2

$($

2

$)$

$($

2

$)$

$$= \log \left| \left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6} \right| + c_2$$

Putting the value of I_1 and I_2 in (*i*)

$$I = \frac{1}{2}\left\{2\sqrt{x^2+5x+6}+c_1\right\} - \frac{1}{2}\left\{\log\left|\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right|+c_2\right\}$$

$$\Rightarrow \quad I = \sqrt{x^2+5x+6} - \frac{1}{2}\log\left|\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right| + \frac{1}{2}c_1 - \frac{1}{2}c_2$$

$$= \sqrt{x^2+5x+6} - \frac{1}{2}\log\left|\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right| + c \qquad \left[\text{where } c = \frac{1}{2}c_1 - \frac{1}{2}c_2\right]$$

18. Given differential equation is

$$\frac{dy}{dx} = 1 + x + y + xy$$

$$\Rightarrow \quad \frac{dy}{dx} = (1+x) + y(1+x) \qquad \Rightarrow \qquad \frac{dy}{dx} = (1+x)(1+y)$$

$$\Rightarrow \quad \frac{dy}{1+y} = (1+x)dx$$

Integrating both sides, we get

$$\log|1+y| = \int (1+x)dx$$

$\Rightarrow \quad \log|1+y| = x + \frac{x^2}{2} + c$, it is general solution.

Putting $x = 1, y = 0$, we get

$$\log 1 = 1 + \frac{1}{2} + c \quad \Rightarrow \quad 0 = \frac{3}{2} + c \qquad c = \frac{-3}{2}$$

Hence particular solution is $\log|1+y| = x + \frac{x^2}{2} - \frac{3}{2}$.

19. Given differential equation is

$$(1+x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{1}{1+x^2}y = \frac{e^{\tan^{-1}x}}{1+x^2} \qquad \ldots (i)$$

Equation (*i*) is of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = \frac{1}{1+x^2}, Q = \frac{e^{\tan^{-1}x}}{1+x^2}$$

$$\therefore \quad I.F. = e^{\int P dx} = e^{\int \frac{1}{1+x^2}dx} = e^{\tan^{-1}x}.$$

Therefore, General solution of required differential equation is

$$y.e^{\tan^{-1}x} = \int e^{\tan^{-1}x}.\frac{e^{\tan^{-1}x}}{1+x^2}dx + c$$

$$y.e^{\tan^{-1}x} = \int \frac{e^{2\tan^{-1}x}}{1+x^2}\,dx + c \qquad \dots (i)$$

Let $\tan^{-1} x = z \quad \Rightarrow \quad \frac{1}{1+x^2}\,dx = dz.$

(i) becomes

$$y.e^{\tan^{-1}x} = \int e^{2z}\,dz + c \qquad \Rightarrow \qquad y.e^{\tan^{-1}x} = \frac{e^{2z}}{2} + c$$

$$\Rightarrow \quad y.e^{\tan^{-1}x} = \frac{e^{2\tan^{-1}x}}{2} + c \qquad [\text{Putting } z = \tan^{-1} x]$$

$$\Rightarrow \quad y = \frac{e^{\tan^{-1}x}}{2} + c.e^{-\tan^{-1}x} \qquad [\text{Dividing both sides by } e^{\tan^{-1}x}]$$

It is required solution.

20. Position vectors of A, B, C and D are

Position vector of $A \equiv 4\hat{i} + 5\hat{j} + \hat{k}$

Position vector of $B \equiv -\hat{j} - \hat{k}$

Position vector of $C \equiv 3\hat{i} + 9\hat{j} + 4\hat{k}$

Position vector of $D \equiv -4\hat{i} + 4\hat{j} + 4\hat{k}$

$\therefore \quad \vec{AB} = -4\hat{i} - 6\hat{j} - 2\hat{k}, \quad \vec{AC} = -\hat{i} + 4\hat{j} + 3\hat{k}, \quad \vec{AD} = -8\hat{i} - \hat{j} + 3\hat{k}$

$$\text{Now } \vec{AB}.(\vec{AC} \times \vec{AD}) = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$$

$$= -4(12+3) + 6(-3+24) - 2(1+32) = -60 + 126 - 66 = 0$$

i.e., $\vec{AB}.(\vec{AC} \times \vec{AD}) = 0$

Hence, $\vec{AB}$, $\vec{AC}$ and $\vec{AD}$ are coplanar i.e. points A, B, C, D are coplanar.

[Note. Three vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar if the scalar triple product of these three vectors is zero.]

OR

Let $\vec{d} = \vec{b} + \vec{c}$

$\therefore \quad \vec{d} = (2\hat{i} + 4\hat{j} - 5\hat{k}) + (\lambda\hat{i} + 2\hat{j} + 3\hat{k})$

$\Rightarrow \quad \vec{d} = (2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}$

$\Rightarrow \quad |\vec{d}| = |(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}| = \sqrt{(2+\lambda)^2 + 6^2 + (-2)^2} = \sqrt{(2+\lambda)^2 + 40}$

$$\therefore \quad \text{Unit vector along } \vec{d} = \hat{d} = \frac{1}{|\vec{d}|}\vec{d} = \frac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(2+\lambda)^2 + 40}} \qquad \dots (i)$$

Now, from question,

$$\vec{a}.\hat{d} = 1$$

$\Rightarrow \quad (\hat{i} + \hat{j} + \hat{k}).\dfrac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(2+\lambda))^2 + 40}} = 1$

$\Rightarrow \quad (\hat{i} + \hat{j} + \hat{k}).\{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}\} = \sqrt{(2+\lambda))^2 + 40}$

$\Rightarrow \quad (2+\lambda) + 6 - 2 = \sqrt{(2+\lambda)^2 + 40} \qquad \Rightarrow \quad (\lambda + 6)^2 = (2+\lambda)^2 + 40$

$\Rightarrow \quad \lambda^2 + 12\lambda + 36 = \lambda^2 + 4\lambda + 4 + 40$

$\Rightarrow \quad 8\lambda + 36 = 44 \qquad \Rightarrow \quad 8\lambda = 8 \quad \Rightarrow \quad \lambda = 1$

Putting the value $\lambda = 1$ in (*i*), we get

Required unit vector along $\vec{d} = \hat{d} = \dfrac{3\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{3^2 + 40}} = \dfrac{3\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{49}}$

$$= \frac{3\hat{i} + 6\hat{j} - 2\hat{k}}{7} = \frac{3}{7}\hat{i} + \frac{6}{7}\hat{j} - \frac{2}{7}\hat{k}$$

21. Let $\vec{b}$ be parallel vector of required line.

$\Rightarrow \quad \vec{b}$ is perpendicular to both given line.

$\Rightarrow \quad \vec{b} = (2\hat{i} - 2\hat{j} + \hat{k}) \times (\hat{i} + 2\hat{j} + 2\hat{k})$

$$= \begin{vmatrix} i & j & k \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{vmatrix}$$

$$= (-4-2)\hat{i} - (4-1)\hat{j} + (4+2)\hat{k} = -6\hat{i} - 3\hat{j} + 6\hat{k}.$$

Hence, the equation of line in vector form is

$$\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(-6\hat{i} - 3\hat{j} + 6\hat{k})$$

$$\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) - 3\lambda(2\hat{i} + \hat{j} - 2\hat{k})$$

$$\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \mu(2\hat{i} + \hat{j} - 2\hat{k}) \qquad [\mu = -3\lambda]$$

Equation in cartesian form is

$$\frac{x-2}{2} = \frac{y+1}{1} = \frac{z-3}{-2}$$

22. An experiment succeeds thrice as often as it fails.

$\Rightarrow \quad p = P(\text{getting success}) = \dfrac{3}{4}$ and $q = P(\text{getting failure}) = \dfrac{1}{4}$.

Here, number of trials $= n = 5$

By binomial distribution, we have

$$P(x = r) = {}^nC_r\, p^r . q^{n-r}$$

Now , $P(\text{getting at least 3 success}) = P(X = 3) + P(X = 4) + P(X = 5)$

$$= {}^5C_3\left(\frac{3}{4}\right)^3\cdot\left(\frac{1}{4}\right)^2 + {}^5C_4\left(\frac{3}{4}\right)^4\cdot\left(\frac{1}{4}\right)^1 + {}^5C_5\left(\frac{3}{4}\right)^5\cdot\left(\frac{1}{4}\right)^0$$

$$= \left(\frac{3}{4}\right)^3\left[{}^5C_3\times\frac{1}{16} + {}^5C_4\times\frac{3}{4}\times\frac{1}{4} + {}^5C_5\left(\frac{3}{4}\right)^2\right]$$

$$= \frac{27}{64}\left[\frac{10}{16}+\frac{15}{16}+\frac{9}{16}\right] = \frac{27}{64}\times\frac{34}{16} = \frac{459}{512}.$$

23. According to question

$$x + y + z = 900$$
$$3x + 2y + z = 1600$$
$$4x + y + 3z = 2300$$

The given system of linear equation may be written in matrix form as $AX = B$

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 4 & 1 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 900 \\ 1600 \\ 2300 \end{bmatrix}$

$$AX = B \Rightarrow X = A^{-1}B \qquad \dots (i)$$

Now, $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 4 & 1 & 3 \end{vmatrix} = 1(6-1) - 1(9-4) + 1(3-8) = 5 - 5 - 5 = -5$

Also, $A_{11} = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 6 - 1 = 5$ $\qquad A_{12} = -\begin{vmatrix} 3 & 1 \\ 4 & 3 \end{vmatrix} = -(9-4) = -5$

$A_{13} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 3 - 8 = -5$ $\qquad A_{21} = -\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -(3-1) = -2$

$A_{22} = \begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} = 3 - 4 = -1$ $\qquad A_{23} = -\begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} = -(1-4) = 3$

$A_{31} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1 - 2 = -1$ $\qquad A_{32} = -\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} = -(1-3) = 2$

$A_{33} = \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = 2 - 3 = -1$

$\therefore$ $\quad \text{adj}(A) = \begin{bmatrix} 5 & -5 & -5 \\ -2 & -1 & 3 \\ -1 & 2 & -1 \end{bmatrix}^T = \begin{bmatrix} 5 & -2 & -1 \\ -5 & -1 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

$\therefore$ $\quad A^{-1} = \dfrac{adj(A)}{|A|} = -\dfrac{1}{5}\begin{bmatrix} 5 & -2 & -1 \\ -5 & -1 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

From equation (i)

$$X = A^{-1}B$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} 5 & -2 & -1 \\ -5 & -1 & 2 \\ -5 & 3 & -1 \end{bmatrix}\begin{bmatrix} 900 \\ 1600 \\ 2300 \end{bmatrix}$$

$$= -\frac{1}{5}\begin{bmatrix} 4500 - 3200 - 2300 \\ -4500 - 1600 + 4600 \\ -4500 + 4800 - 2300 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -1000 \\ -1500 \\ -2000 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 200 \\ 300 \\ 400 \end{bmatrix}$$

$\Rightarrow$ $x =$ ₹ 200, $y =$ ₹ 300, $z =$ ₹ 400.

i.e., ₹ 200 for sincerity,

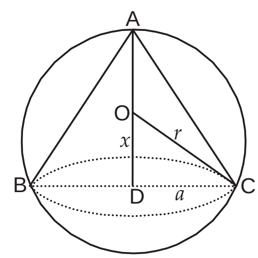
₹ 300 for truthfulness and ₹ 400 for helpfulness.

One more value like honesty, kindness etc. can be awarded.

SECTION–C

24. Consider a sphere of radius r with centre at O such that

$OD = x$ and $DC = a$.



Let h be the height of the cone.

Then $h = AD = AO + OD = r + x$ ...(*i*)

($OA = OC =$ radius)

In the right angled $\Delta\, ODC$,

$r^2 = a^2 + x^2$ (by Pythagoras theorem) ...(*ii*)

Let V be the volume the cone, then $V = \frac{1}{3}\pi r^2 h$

$$\Rightarrow \quad V(x) = \frac{1}{3}\pi(r^2 - x^2)(r + x) \quad \text{[From (1) and (2)]}$$

$$\Rightarrow \quad V'(x) = \frac{1}{3}\pi\left[(r^2 - x^2)\frac{d}{dx}(r + x) + (r + x)\frac{d}{dx}(r^2 - x^2)\right]$$

$$\Rightarrow \quad = \frac{1}{3}\pi[(r^2 - x^2)(1) + (r + x)(-2x)] = \frac{1}{3}\pi[(r + x)(r - x - 2x)] = \frac{1}{3}\pi(r + x)(r - 3x)$$

Also, $$V''(x) = \frac{1}{3}\pi\left[(r + x)\frac{d}{dx}(r - 3x) + (r - 3x)\frac{d}{dx}(r + x)\right]$$

$$\Rightarrow \quad V''(x) = \frac{1}{3}\pi[(r + x)(-3) + (r - 3x)(1)]$$

For maximum or minimum value, we have $V'(x) = 0$

$\Rightarrow$ $\frac{1}{3}\pi(r + x)(r - 3x) = 0$ $\Rightarrow$ $x = -r$ or $x = \frac{r}{3}$

Neglecting $x = -r$ $[\because x > 0]$

$$V''\left(\frac{r}{3}\right) = \frac{1}{3}\pi\left[\left(r + \frac{r}{3}\right)(-3) + \left(r - 3\left(\frac{r}{3}\right)\right)\right] = \frac{-4\pi r}{3} < 0$$

$\therefore$ Volume is maximum when $x = \dfrac{r}{3}$.

Putting $x = \dfrac{r}{3}$ in equation (i) and (ii) we get $h = r + \dfrac{r}{3} = \dfrac{4r}{3}$

and $a^2 = r^2 - \dfrac{r^2}{9} = \dfrac{8r^2}{9}$

Now, Volume of cone $= \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi\left(\dfrac{8r^2}{9}\right)\left(\dfrac{4r}{3}\right) = \dfrac{8}{27}\left(\dfrac{4}{3}\pi r^3\right)$

Thus, Volume of the cone $= \dfrac{8}{27}$ (volume of the sphere).

25. Let $I = \displaystyle\int \frac{dx}{\cos^4 x + \sin^4 x}$

$$= \int \frac{\sec^4 x\, dx}{1 + \tan^4 x} \qquad \text{[Dividing } N^r \text{ and } D^r \text{ by } \cos^4 x]$$

$$= \int \frac{\sec^2 x.\sec^2 x\, dx}{1 + \tan^4 x} = \int \left(\frac{1 + \tan^2 x}{1 + \tan^4 x}\right).\sec^2 x dx$$

Let $\tan x = z \Rightarrow \sec^2 x\, dx = dz$

$$\therefore \quad I = \int \left(\frac{1 + z^2}{1 + z^4}\right) dz$$

$$= \int \frac{\left(\frac{1}{z^2} + 1\right)}{\left(\frac{1}{z^2} + z^2\right)} dz \qquad \text{[Dividing } N^r \text{ and } D^r \text{ by } z^2]$$

$$= \int \frac{\left(1 + \frac{1}{z^2}\right) dz}{\left(z - \frac{1}{z}\right)^2 + 2}$$

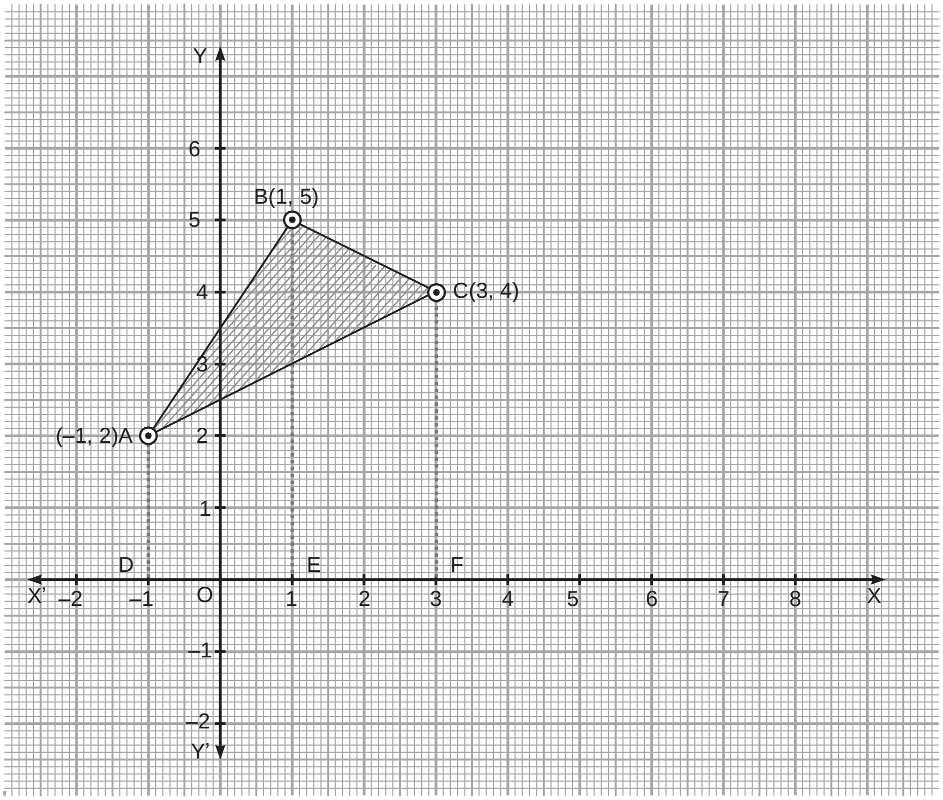
Let $z - \dfrac{1}{z} = t \Rightarrow \left(1 + \dfrac{1}{z^2}\right) dz = dt$

$$\therefore \quad I = \int \frac{dt}{t^2 + (\sqrt{2})^2} = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + c$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{z - \frac{1}{z}}{\sqrt{2}}\right) + c \qquad \left[\text{Putting } t = z - \frac{1}{z}\right]$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{z^2 - 1}{\sqrt{2}z}\right) + c = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan^2 x - 1}{\sqrt{2} \tan x}\right) + c.$$

26. Triangle ΔABC having vertices $A(-1, 2)$, $B(1, 5)$ and $C(3, 4)$ is drawn and shaded like as figure.



Equation of AB is

$$y - 2 = \frac{5-2}{1+1}(x+1) \qquad \Rightarrow \qquad y - 2 = \frac{3}{2}(x+1)$$

$$\Rightarrow \qquad 2y - 4 = 3x + 3 \qquad \Rightarrow \qquad 2y = 3x + 7$$

$$\Rightarrow \qquad y = \frac{3}{2}x + \frac{7}{2} \qquad \ldots(i)$$

Equation of BC is

$$y = \frac{}{2} + \frac{}{2}$$

$$(y - 5) = \frac{4-5}{3-1}(x-1)$$

$$\Rightarrow \qquad y - 5 = \frac{-1}{2}(x-1) \qquad \Rightarrow$$

$$\Rightarrow \qquad y = -\frac{x}{2} + \frac{11}{2}$$

Equation of AC is

$$y - 2 = \frac{4-2}{3+1}(x+1) \qquad \Rightarrow$$

$$y = -\frac{x}{2} + \frac{1}{2} + 5 \quad \ldots(ii)$$

$$y - 2 = \frac{2}{4}(x+1) \;\Rightarrow\; y = \frac{x}{2} + \frac{1}{2} + 2 \quad \ldots(iii)$$

Now area of required region = $ar(\Delta ABC)$

$$= ar(Trap.\,ABED) + ar(Trap.\,BCFE) - a(Trap.\,ACFD)$$

$$= \int_{-1}^{1}\left(\frac{3}{2}x+\frac{7}{2}\right)dx + \int_{-1}^{3}\left(-\frac{x}{2}+\frac{11}{2}\right)dx - \int_{-1}^{3}\left(\frac{x}{2}+\frac{5}{2}\right)dx$$

$$= \frac{3}{2}\left[\frac{x^2}{2}\right]_{-1}^{1} + \frac{7}{2}[x]_{-1}^{1} - \frac{1}{2}\left[\frac{x^2}{2}\right]_{1}^{3} + \frac{11}{2}[x]_{1}^{3} - \frac{1}{2}\left[\frac{x^2}{2}\right]_{-1}^{3} - \frac{5}{2}[x]_{-1}^{3}$$

$$= \frac{3}{4}(1-1) + \frac{7}{2}(1+1) - \frac{1}{4}(9-1) + \frac{11}{2}(3-1) - \frac{1}{4}(9-1) - \frac{5}{2}(3+1)$$

$= 7 - 2 + 11 - 2 - 10 = 4$ square unit.

27. The equation of a plane passing through the intersection of the given planes is

$$(x+y+z-1) + \lambda(2x+3y+4z-5) = 0$$

$\Rightarrow$ $(1+2\lambda)x + (1+3\lambda)y + (1+4\lambda)z - (1+5\lambda) = 0$ … (i)

Since, (i) is perpendicular to $x - y + z = 0$

$\Rightarrow$ $(1+2\lambda).1 + (1+3\lambda).(-1) + (1+4\lambda).1 = 0$

$\Rightarrow$ $1 + 2\lambda - 1 - 3\lambda + 1 + 4\lambda = 0$ $\Rightarrow$ $3\lambda + 1 = 0$

$\Rightarrow$ $\lambda = -\frac{1}{3}.$

Putting the value of λ in (i) we get

$$\left(1-\frac{2}{3}\right)x + (1-1)y + \left(1-\frac{4}{3}\right)z - \left(1-\frac{5}{3}\right) = 0$$

$\Rightarrow$ $\frac{x}{3} - \frac{z}{3} + \frac{2}{3} = 0$

$\Rightarrow$ $x - z + 2 = 0$, it is required plane.

Let d be the distance of this plane from origin.

$\therefore$ $d = \left|\frac{0.x + 0.y + 0.(-z) + 2}{\sqrt{1^2 + 0^2 + (-1)^2}}\right| = \left|\frac{2}{\sqrt{2}}\right| = \sqrt{2}$ units.

[**Note:** The distance of the point (α, β, γ) to the plane $ax + by + cz + d = 0$ is given by $\left|\frac{a\alpha + b\beta + c\gamma + d}{\sqrt{a^2 + b^2 + c^2}}\right|$.

OR

Given line and plane are

$\vec{r} = (2\hat{i} - 4\hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ … (i)

and $\vec{r}\,(\hat{i} - 2\hat{j} + \hat{k}) = 0$ … (ii)

For intersection point, we solve equations (i) and (ii) by putting the value of $\vec{r}$ from (i) in (ii)

$$[(2\hat{i} - 4\hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})].(\hat{i} - 2\hat{j} + \hat{k}) = 0$$

$$\Rightarrow \qquad [(2 + 3\lambda)\hat{i} - (4 - 4\lambda)\hat{j} + (2 + 2\lambda)\hat{k}].(\hat{i} - 2\hat{j} + \hat{k}) = 0$$

$\Rightarrow \quad (2 + 3\lambda) + 2(4 - 4\lambda) + (2 + 2\lambda) = 0$

$\Rightarrow \quad 2 + 3\lambda + 8 - 8\lambda + 2 + 2\lambda = 0 \quad \Rightarrow \quad 12 - 3\lambda = 0 \quad \Rightarrow \quad \lambda = 4$

Hence position vector of intersecting point is $14\hat{i} + 12\hat{j} + 10\hat{k}$

$\therefore$ Co-ordinate of intersecting point $\equiv (14, 12, 10)$

$\therefore$ Required distance $= \sqrt{(14-2)^2 + (12-12)^2 + (10-5)^2}$

$= \sqrt{144 + 25} = \sqrt{169}$ units $= 13$ units.

28. Let x and y be the number of pieces of type A and B manufactured per week respectively. If Z be the profit then,

Objective function, $Z = 80x + 120y$

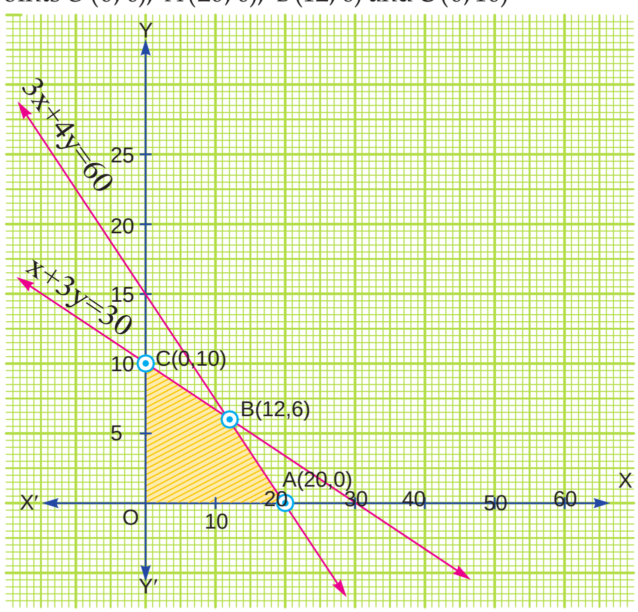
We have to maximize Z, subject to the constraints

$9x + 12y \le 180 \quad \Rightarrow \quad 3x + 4y \le 60 \qquad ...(i)$

$x + 3y \le 30 \qquad ...(ii)$

$x \ge 0, \quad y \ge 0 \qquad ...(iii)$

The graph of constraints are drawn and feasible region $OABC$ is obtained, which is bounded having corner points $O(0, 0)$, $A(20, 0)$, $B(12, 6)$ and $C(0, 10)$



Now the value of objective function is obtained at corner points as

Corner point	$Z = 80x + 120y$
O (0, 0)	0
A (20, 0)	1600
B (12, 6)	1680
C (0, 10)	1200

Maximum

Hence, the company will get the maximum profit of ₹1,680 by making 12 pieces of type A and 6 pieces of type B of teaching aid.

Yes, teaching aid is necessary for teaching learning process as

(*i*) it makes learning very easy.

(*ii*) it provides active learning.

(*iii*) students are able to grasp and understand concept more easily and in active manner.

29. Let E_1, E_2, E_3 and A be events defined as

E_1 = selection of two-headed coin

E_2 = selection of biased coin that comes up head 75% of the times.

E_3 = selection of biased coin that comes up tail 40% of the times.

A = getting head.

$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

$P\left(\frac{A}{E_1}\right) = 1, \quad P\left(\frac{A}{E_2}\right) = \frac{75}{100} = \frac{3}{4}, \quad P\left(\frac{A}{E_3}\right) = \frac{60}{100} = \frac{3}{5}$

$$\text{Now } P\left(\frac{E_1}{A}\right) = \frac{P(E_1).P\left(\frac{A}{E_1}\right)}{P(E_1).P\left(\frac{A}{E_1}\right) + P(E_2).P\left(\frac{A}{E_2}\right) + P(E_3).P\left(\frac{A}{E_3}\right)}$$

$$= \frac{\frac{1}{3} \times 1}{\frac{1}{3} \times 1 + \frac{1}{3} \times \frac{3}{4} + \frac{1}{3} \times \frac{3}{5}} = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{4} + \frac{1}{5}}$$

$$= \frac{\frac{1}{3}}{\frac{20 + 15 + 12}{60}} = \frac{1}{3} \times \frac{60}{47} = \frac{20}{47}.$$

OR

First six positive integers are 1, 2, 3, 4, 5, 6

If two numbers are selected at random from above six numbers then sample space S is given by

$S = \{(1, 2)\ (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 4), (3, 5),$
$(3, 6), (4, 1), (4, 2), (4, 3), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 6), (6, 1), (6, 2), (6, 3)$
$(6, 4)\ (6, 5)\}$

$n(s) = 30.$

Here, X is random variable, which may have value 2, 3, 4, 5 or 6.

Therefore, required probability distribution is given as

2

$P(X = 2) =$ Probability of event getting (1, 2), (2, 1) $= \frac{}{30}$

$P(X = 3)$ = Probability of event getting (1, 3), (2, 3), (3, 1), (3, 2) $= \dfrac{4}{30}$

$P(X = 4)$ = Probability of event getting (1, 4), (2, 4), (3, 4), (4, 1), (4, 2), (4, 3) $= \dfrac{6}{30}$

$P(X = 5)$ = Probability of event getting (1, 5), (2, 5), (3, 5), (4, 5), (5, 1), (5, 2), (5, 3), (5, 4) $= \dfrac{8}{30}$

$P(X = 6)$ = Probability of event getting (1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5) $= \dfrac{10}{30}$

It is represented in tabular form as

X	2	3	4	5	6
$P(X)$	$\frac{2}{30}$	$\frac{4}{30}$	$\frac{6}{30}$	$\frac{8}{30}$	$\frac{10}{30}$

$\therefore$ Required mean $= E(x) = \Sigma p_i x_i$

$$= 2 \times \frac{2}{30} + 3 \times \frac{4}{30} + 4 \times \frac{6}{30} + 5 \times \frac{8}{30} + 6 \times \frac{10}{30}$$

$$= \frac{4 + 12 + 24 + 40 + 60}{30}$$

$$= \frac{140}{30} = \frac{14}{3} = 4\frac{2}{3}.$$

SET–II

9. Let $I = \displaystyle\int_e^{e^2} \frac{dx}{x \log x}$

Let $\log x = z$ $\Rightarrow$ $\dfrac{1}{x} dx = dz$

For limit $x = e$ $\Rightarrow$ $z = \log e = 1$; $x = e^2$ $\Rightarrow$ $z = \log e^2 = 2.$

$\therefore$ $I = \displaystyle\int_1^2 \frac{dz}{z} = [\log z]_1^2 = \log 2 - \log 1 = \log 2.$ $[\because \log 1 = 0]$

10. Direction cosines of required vector $\vec{a}$ are

$l = \cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$

$m = \cos\dfrac{\pi}{2} = 0$ and $n = \cos\theta$

$\because$ $l^2 + m^2 + n^2 = 1$

$\Rightarrow$ $\left(\dfrac{1}{\sqrt{2}}\right)^2 + 0 + \cos^2\theta = 1$ $\Rightarrow$ $\cos^2\theta = 1 - \dfrac{1}{2}$

$\Rightarrow$ $\cos\theta = \dfrac{1}{\sqrt{2}}$ $\Rightarrow$ $n = \dfrac{1}{\sqrt{2}}$

$\therefore$ Unit vector in the direction of $\vec{a} = \dfrac{1}{\sqrt{2}}\hat{i} + 0\hat{j} + \dfrac{1}{\sqrt{2}}\hat{k}$

$\therefore$ $\vec{a} = 5\sqrt{2}\left(\dfrac{1}{\sqrt{2}}\hat{i} + \dfrac{1}{\sqrt{2}}\hat{k}\right) = 5\hat{i} + 5\hat{k}$

19. L.H.S. $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$

Applying, $R_1 \leftrightarrow R_3$ and $R_3 \leftrightarrow R_2$, we get

$$= \begin{vmatrix} a+b & b+c & c+a \\ p+q & q+r & r+p \\ x+y & y+z & z+x \end{vmatrix}$$

Applying, $R_1 \to R_1 + R_2 + R_3$, we get

$$\Delta = \begin{vmatrix} 2(a+b+c) & b+c & c+a \\ 2(p+q+r) & q+r & r+p \\ 2(x+y+z) & y+z & z+x \end{vmatrix}$$

$$= 2\begin{vmatrix} a+b+c & b+c & c+a \\ p+q+r & q+r & r+p \\ x+y+z & y+z & z+x \end{vmatrix}$$

$$= 2\begin{vmatrix} a & b+c & c+a \\ p & q+r & r+p \\ x & y+z & z+x \end{vmatrix} \qquad \text{[Applying } R_1 \to R_1 - R_2]$$

$$= 2\begin{vmatrix} a & b+c & c \\ p & q+r & r \\ x & y+z & z \end{vmatrix} \qquad \text{[Applying } R_3 \to R_3 - R_1]$$

Again applying $R_2 \to R_2 - R_3$, we get

$$\Delta = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = \text{RHS}$$

20. Given, $x = a\sin 2t\,(1+\cos 2t)$ and $y = b\cos 2t\,(1-\cos 2t)$

We have

$$\frac{dx}{dt} = a\,[\sin 2t \times (-2\sin 2t) + (1+\cos 2t) \times 2\cos 2t]$$

$$= a\,[-2\sin^2 2t + 2\cos 2t + 2\cos^2 2t]$$

$$= a\,(2\cos 4t + 2\cos 2t) = 2a\,(\cos 4t + \cos 2t)$$

Again, $\dfrac{dy}{dt} = b\,[\cos 2t \times 2\sin 2t + (1-\cos 2t) \times -2\sin 2t]$

$$= b\,[\sin 4t - 2\sin 2t + \sin 4t] = b\,[2\sin 4t - 2\sin 2t]$$

$$= 2b\,(\sin 4t - \sin 2t)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2b\,(\sin 4t - \sin 2t)}{2a\,(\cos 4t + \cos 2t)} = -\frac{b}{a}\left[\frac{\sin 4t - \sin 2t}{\cos 4t + \cos 2t}\right]$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{\text{at } t=\frac{\pi}{4}} = \frac{b}{a}\left(\frac{\sin \pi - \sin \frac{\pi}{2}}{\cos \pi + \cos \frac{\pi}{2}}\right) = \frac{b}{a}\cdot\frac{-1}{-1}$$

Hence, $\left.\dfrac{dy}{dx}\right|_{\text{at } t=\frac{\pi}{4}} = \dfrac{b}{a}$

21. Given differential equation is

$$x(1+y^2)dx - y(1+x^2)dy = 0 \quad \Rightarrow \quad y(1+x^2)dy = x(1+y^2)dx$$

$$\Rightarrow \quad \frac{y}{1+y^2}\,dy = \frac{x}{1+x^2}\,dx$$

Integrating both sides, we get

$$\Rightarrow \quad \frac{1}{2}\int \frac{2y}{1+y^2}\,dy = \frac{1}{2}\int \frac{2x}{1+x^2}\,dx \quad \Rightarrow \quad \frac{1}{2}\log|1+y^2| = \frac{1}{2}\log|1+x^2| + \log C$$

$$\Rightarrow \quad \log\sqrt{1+y^2} - \log\sqrt{1+x^2} = \log C \quad \Rightarrow \quad \log\sqrt{\frac{1+y^2}{1+x^2}} = \log C$$

$$\Rightarrow \quad \sqrt{\frac{1+y^2}{1+x^2}} = C \quad \Rightarrow \quad 1+y^2 = C^2(1+x^2)$$

$$\Rightarrow \quad y^2 = C^2(1+x^2) - 1 \qquad \ldots(i)$$

Now for particular solution, we put $y = 1$, $x = 0$ in (i) we get

$$1 = C^2(1+0) - 1$$

$$1 = C^2 - 1 \quad \Rightarrow \quad C^2 = 2 \quad \Rightarrow \quad C = \sqrt{2}.$$

Putting $C = \sqrt{2}$ in (i) we get particular solution as

$$y^2 = 2(1+x^2) - 1 \quad \Rightarrow \quad y^2 = 2 + 2x^2 - 1 \quad \Rightarrow \quad y^2 = 2x^2 + 1$$

22. Let the cartesian equation of the line passing through (2, 1, 3) be

$$\frac{x-2}{a} = \frac{y-1}{b} = \frac{z-3}{c} \qquad \ldots(i)$$

Since, line (i) is perpendicular to given line

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3} \qquad \ldots(ii)$$

and $\dfrac{x}{-3} = \dfrac{y}{2} = \dfrac{z}{5}$ $\qquad \ldots(iii)$

$$\therefore \quad a + 2b + 3c = 0$$

$$-3a + 2b + 5c = 0$$

$$\frac{a}{10-6} = \frac{b}{-9-5} = \frac{c}{2+6}$$

From equation (iv) and (v).

…(iv)

$$\Rightarrow \quad \frac{a}{4} = \frac{b}{-14} = \frac{c}{8} = \lambda \text{ (say)} \qquad \ldots(v)$$

$$\Rightarrow \quad a = 4\lambda, b = -14\lambda, c = 8\lambda$$

Putting the value of a, b and c in (i) we get

$$\frac{x-2}{4\lambda}=\frac{y-1}{-14\lambda}=\frac{z-3}{8\lambda} \qquad \Rightarrow \qquad \frac{x-2}{4}=\frac{y-1}{-14}=\frac{z-3}{8}$$

$\Rightarrow \quad \dfrac{x-2}{2}=\dfrac{y-1}{-7}=\dfrac{z-3}{4}$, which is the cartesian form

The vector form is $\vec{r} = (2\hat{i}+\hat{j}+3\hat{k})+\lambda(2\hat{i}-7\hat{j}+4\hat{k})$

SECTION C

28. Let $\quad I=\int(\sqrt{\cot x}+\sqrt{\tan x})\,dx$

$$I=\int\left(\frac{\sqrt{\cos x}}{\sqrt{\sin x}}+\frac{\sqrt{\sin x}}{\sqrt{\cos x}}\right)dx=\int\frac{(\cos x+\sin x)}{\sqrt{\sin x.\cos x}}\,dx$$

Let $\quad \sin x-\cos x=t \qquad (\cos x+\sin x)dx=dt$

$\Rightarrow$ Also $\because$

$\sin x-\cos x=t$

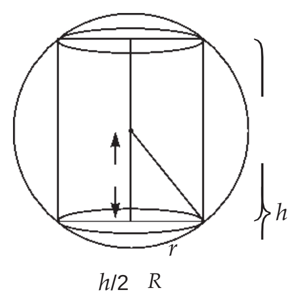
$\Rightarrow \quad (\sin x-\cos x)^2=t^2 \;\Rightarrow\; \sin^2 x+\cos^2 x-2\sin x.\cos x=t^2$

$\Rightarrow \quad 1-2\sin x.\cos x=t^2 \;\Rightarrow\; \sin x.\cos x=\dfrac{1-t^2}{2}$

Therefore, $\quad I=\displaystyle\int\frac{dt}{\sqrt{\dfrac{1-t^2}{2}}}=\sqrt{2}\int\frac{dt}{\sqrt{1-t^2}}$

$$=\sqrt{2}\sin^{-1}t+c=\sqrt{2}\sin^{-1}(\sin x-\cos x)+c$$

29. Let R, h be the radius and height of inscribed cylinder respectively. If V be the volume of cylinder then



$$V=\pi R^2 h \qquad \left[\because R^2+\left(\frac{h}{2}\right)^2=r^2\right.$$

$$V=\pi\left(r^2-\frac{h^2}{4}\right)h \qquad \left.R^2=r^2-\frac{h^2}{4}\right]$$

$$V=\pi\left(r^2h-\frac{h^3}{4}\right)$$

Differentiating w.r.t. h, we get

$$\frac{dV}{dh}=\pi\left(r^2-\frac{3h^2}{4}\right) \qquad \ldots(i)$$

For maxima or minima

$$\frac{dV}{dh}=0$$

$\Rightarrow \quad \pi\left(r^2-\dfrac{3h^2}{4}\right)=0 \qquad \Rightarrow \qquad r^2-\dfrac{3h^2}{4}=0$

$$\Rightarrow \qquad r = \frac{3h}{2} \qquad \Rightarrow \qquad h = \frac{2r}{}$$

Differentiating (i) again w.r.t. h, we get

$$\frac{d^2V}{dh^2} = -\frac{\pi 6h}{4}$$

$$\left.\frac{d^2V}{dh^2}\right]_{h=\frac{2r}{\sqrt{3}}} = -\frac{3\pi}{2} \cdot \frac{2r}{\sqrt{3}} < 0$$

Hence, V is maximum when $h = \dfrac{2r}{\sqrt{3}}$.

$\therefore$ Maximum volume $= \pi\left(r^2 \cdot \dfrac{2r}{\sqrt{3}} - \dfrac{8r^3}{4 \times 3\sqrt{3}}\right)$

$$= \pi\left(\frac{24r^3 - 8r^3}{12\sqrt{3}}\right) = \pi\,\frac{16r^3}{12\sqrt{3}} = \frac{4\pi r^3}{3\sqrt{3}}$$

SET–III

9. Given $\displaystyle\int_0^a \frac{dx}{4+x^2} = \frac{\pi}{8}$

$\Rightarrow \displaystyle\int_0^a \frac{dx}{2^2+x^2} = \frac{\pi}{8}$ $\quad\Rightarrow\quad$ $\dfrac{1}{2}\left[\tan^{-1}\left(\dfrac{x}{2}\right)\right]_0^a = \dfrac{\pi}{8}$

$\Rightarrow \dfrac{1}{2}\left[\tan^{-1}\dfrac{a}{2} - \tan^{-1} 0\right] = \dfrac{\pi}{8}$ $\quad$ [$\because \tan^{-1} 0 = 0$]

$\Rightarrow \tan^{-1}\dfrac{a}{2} = \dfrac{\pi}{4}$

$\Rightarrow \dfrac{a}{2} = \tan\dfrac{\pi}{4}$ $\quad\Rightarrow\quad$ $\dfrac{a}{2} = 1$ $\quad\Rightarrow\quad$ $a = 2$.

10. Given $|\vec{a} + \vec{b}| = 13$

$\Rightarrow |\vec{a} + \vec{b}|^2 = 169$ $\quad\Rightarrow\quad$ $(\vec{a} + \vec{b}).(\vec{a} + \vec{b}) = 169$

$\Rightarrow |\vec{a}|^2 + 2\,\vec{a}.\vec{b} + |\vec{b}|^2 = 169$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 = 169$ $\quad$ [$\because \vec{a} \perp \vec{b} \Rightarrow \vec{a}.\vec{b} = 0$]

$\Rightarrow |\vec{b}|^2 = 169 - |\vec{a}|^2$

$\Rightarrow |\vec{b}|^2 = 169 - 25$

$\Rightarrow |\vec{b}|^2 = 144$ $\quad\Rightarrow\quad$ $|\vec{b}| = 12$.

19. LHS $= \Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

Taking out *a, b, c* common from I, II, and III row respectively, we get

$$\Delta = abc\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$

$$\Delta = abc\begin{vmatrix} \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1 & \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1 & \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$$= abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$\Delta = abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1\right)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$$

$$= abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1\right) \times (1 \times 1 \times 1)$$

($\because$ the determinant of a triangular matrix is the product of its diagonal elements.)

$$= abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1\right)$$

$$= abc\left(\frac{bc+ac+ab+abc}{abc}\right) = ab+bc+ca+abc = \text{R.H.S.}$$

20. Given $x = \cos t(3 - 2\cos^2 t)$

Differentiating both sides w.r.t. t, we get

$$\frac{dx}{dt} = \cos t\,\{0 + 4\cos t.\sin t\} + (3 - 2\cos^2 t).(-\sin t)$$

$$= 4\sin t.\cos^2 t - 3\sin t + 2\cos^2 t.\sin t$$

$$= 6\sin t\cos^2 t - 3\sin t$$

$$= 3\sin t\,(2\cos^2 t - 1) = 3\sin t.\cos 2t$$

Again $\because$ $y = \sin t(3 - 2\sin^2 t)$

Differentiating both sides w.r.t. t, we get

dy

$$\frac{}{dt} = \sin t.\{0 - 4\sin t\cos t\} + (3 - 2\sin^2 t).\cos t$$

$$= -4\sin^2 t.\cos t + 3\cos t - 2\sin^2 t.\cos t = 3\cos t - 6\sin^2 t.\cos t$$
$$= 3\cos t(1 - 2\sin^2 t) = 3\cos t.\cos 2t$$

Now $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{3\cos t.\cos 2t}{3\sin t.\cos 2t}$

$$\frac{dy}{dx} = \cot t$$

$\therefore \quad \left.\dfrac{dy}{dx}\right]_{t=\frac{\pi}{4}} = \cot\dfrac{\pi}{4} = 1$

21. Given differential equation is

$$\log\left(\frac{dy}{dx}\right) = 3x + 4y$$

$\Rightarrow \quad \dfrac{dy}{dx} = e^{3x+4y} \qquad \Rightarrow \qquad \dfrac{dy}{dx} = e^{3x}.e^{4y}$

$\Rightarrow \quad \dfrac{dy}{e^{4y}} = e^{3x}.dx \qquad \Rightarrow \qquad e^{-4y}dy = e^{3x}dx$

Integrating both sides, we get

$$\int e^{-4y}dy = \int e^{3x}dx$$

$\dfrac{e^{-4y}}{-4} = \dfrac{e^{3x}}{3} + c_1 \qquad \Rightarrow \qquad -3e^{-4y} = 4e^{3x} + 12c_1$

$$4e^{3x} + 3e^{-4y} = -12c_1$$

$4e^{3x} + 3e^{-4y} = c \qquad \dots (i)$

It is general solution.

Now for particular solution we put $x = 0$ and $y = 0$ in (i)

$4 + 3 = c \qquad \Rightarrow \qquad c = 7.$

Putting $c = 7$ in (i), we get

$$4e^{3x} + 3e^{-4y} = 7$$

It is required particular solution.

22. Given line l_1 and l_2 are

$l_1 \equiv \dfrac{1-x}{3} = \dfrac{7y-14}{p} = \dfrac{z-3}{2} \qquad \Rightarrow \qquad \dfrac{x-1}{-3} = \dfrac{y-2}{\frac{p}{7}} = \dfrac{z-3}{2}$

$l_2 \equiv \dfrac{7-7x}{3p} = \dfrac{y-5}{1} = \dfrac{6-z}{5} \qquad \Rightarrow \qquad \dfrac{x-1}{\frac{-3p}{7}} = \dfrac{y-5}{1} = \dfrac{z-6}{-5}$

Since $l_1 \perp l_2$

$\Rightarrow \quad (-3)\left(-\dfrac{3p}{7}\right) + \dfrac{p}{7} \times 1 + 2 \times -5 = 0$

$$\Rightarrow \quad \frac{9p}{7} + \frac{p}{7} - 10 = 0 \qquad \Rightarrow \quad \frac{10p}{7} = +10$$

$$\Rightarrow \quad p = +\frac{7 \times 10}{10} \qquad \Rightarrow \quad p = 7$$

The equation of line passing through (3, 2, –4) and parallel to l_1 is given by

$$\frac{x-3}{-3} = \frac{y-2}{\frac{p}{7}} = \frac{z+4}{2}$$

i.e., $$\frac{x-3}{-3} = \frac{y-2}{1} = \frac{z+4}{2} \qquad (\because p = 7)$$

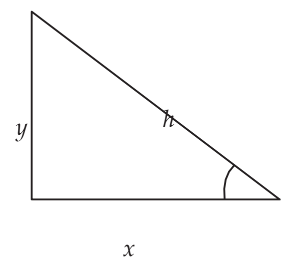
28. Let h and x be the length of hypotenuse and one side of a right triangle and y is length of the third side.

If A be the area of triangle, then

$$A = \frac{1}{2}xy = \frac{1}{2}x\sqrt{h^2 - x^2}$$

$$\left[\begin{array}{l}\text{also given} \\ h + x = k \text{ (constant)} \\ \therefore \quad h = k - x\end{array}\right]$$

$$A = \frac{1}{2}x\sqrt{(k-x)^2 - x^2} = \frac{1}{2}x\sqrt{k^2 - 2kx + x^2 - x^2}$$



$$\Rightarrow \quad A^2 = \frac{x^2}{4}(k^2 - 2kx) \quad \Rightarrow \quad A^2 = \frac{1}{4}(k^2x^2 - 2kx^3)$$

Differentiating w.r.t. x we get

$$\Rightarrow \quad \frac{d(A^2)}{dx} = \frac{1}{4}(2k^2x - 6kx^2) \qquad \ldots(i)$$

$\because$ For maxima or minima of A^2. $\frac{d(A^2)}{dx} = 0$.

$$\left[\begin{array}{l}\because V = lbh \\ 8 = lb2 \\ \therefore \quad b = \frac{8}{2l} = \frac{4}{l}\end{array}\right]$$

$$\Rightarrow \quad \frac{1}{4}(2k^2x - 6kx^2) = 0 \qquad \Rightarrow \qquad 2k^2x - 6kx^2 = 0$$

$$\Rightarrow \quad 2kx\,(k - 3x) = 0$$

$$\Rightarrow \quad k - 3x = 0\,; \qquad 2kx \neq 0$$

$$\Rightarrow \quad x = \frac{k}{3}$$

Differentiating (i) again w.r.t. x, we get

$$\frac{d^2(A^2)}{dx^2} = \frac{1}{4}(2k^2 - 12kx)$$

$$\left.\frac{d^2(A^2)}{dx^2}\right]_{x = k/3} = \frac{1}{4}\left(2k^2 - 12k\,.\frac{k}{3}\right) < 0$$

Hence, A^2 is maximum when $x = \frac{k}{3}$ and $h = k - \frac{k}{3} = \frac{2k}{3}$.

i.e., A is maximum when $x = \dfrac{k}{3}$, $h = \dfrac{2k}{3}$.

$\therefore \qquad \cos\theta = \dfrac{x}{h} = \dfrac{k}{3} \times \dfrac{3}{2k} = \dfrac{1}{2}$

$\Rightarrow \qquad \cos\theta = \dfrac{1}{2} \quad \Rightarrow \quad \theta = \dfrac{\pi}{3}$.

29. Let $\quad I = \displaystyle\int \frac{dx}{\sin^4 x + \sin^2 x.\cos^2 x + \cos^4 x} dx$

Dividing N^r and D^r by $\cos^4 x$, we get

$$I = \int \frac{\sec^4 x}{\tan^4 x + \tan^2 x + 1} dx$$

Put $\quad z = \tan x \quad \Rightarrow \quad dz = \sec^2 x\, dx$

$$\therefore \qquad I = \int \frac{(1+z^2)dz}{z^4 + z^2 + 1}$$

$$= \int \frac{z^2\left(1 + \frac{1}{z^2}\right)}{z^2\left\{z^2 + \frac{1}{z^2} + 1\right\}} dz = \int \frac{\left(1 + \frac{1}{z^2}\right)}{\left(z - \frac{1}{z}\right)^2 + 3} dz$$

$$= \int \frac{\left(1 + \frac{1}{z^2}\right)dz}{\left(z - \frac{1}{z}\right)^2 + (\sqrt{3})^2}$$

Again, let $z - \dfrac{1}{z} = t \qquad \Rightarrow \qquad \left(1 + \dfrac{1}{z^2}\right)dz = dt$

$$\Rightarrow \qquad I = \int \frac{dt}{t^2 + (\sqrt{3})^2} = \frac{1}{\sqrt{3}}\left(\tan^{-1}\frac{t}{\sqrt{3}}\right) + c$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{z - \frac{1}{z}}{\sqrt{3}}\right) + c \qquad \left[\because\ z - \frac{1}{z} = t\right]$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{z^2 - 1}{\sqrt{3}z}\right) + c$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{\tan^2 x - 1}{\sqrt{3}\tan x}\right) + c.$$

CBSE Examination Paper, Foreign-2014

Time allowed: 3 hours **Maximum marks: 100**

General Instructions: *As per given in CBSE Examination Paper Delhi-2014.*

SET–I

SECTION–A

Question numbers 1 to 10 carry 1 mark each.

1. Let $R = \{(a, a^3) : a$ is a prime number less than 5$\}$ be a relation. Find the range of R.
2. Write the value of $\cos^{-1}\left(-\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$.
3. Use elementary column operation $C_2 \to C_2 - 2C_1$ in the matrix equation $\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$.
4. If $\begin{bmatrix} a+4 & 3b \\ 8 & -6 \end{bmatrix} = \begin{bmatrix} 2a+2 & b+2 \\ 8 & a-8b \end{bmatrix}$ write the value of $a - 2b$.
5. If A is a 3×3 matrix, $|A| \neq 0$ and $|3A| = k\,|A|$, then write the value of k.
6. Evaluate:
$$\int \frac{dx}{\sin^2 x \cos^2 x}$$
7. Evaluate:
$$\int_0^{\pi/4} \tan x \; dx$$
8. Write the projection of vector $\hat{i} + \hat{j} + \hat{k}$ along the vector $\hat{j}$.
9. Find a vector in the direction of vector $2\hat{i} - 3\hat{j} + 6\hat{k}$ which has magnitude 21 units.
10. Find the angle between the lines $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda\,(3\hat{i} + 2\hat{j} + 6\hat{k})$ and $\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$.

SECTION–B

Question numbers 11 to 22 carry 4 marks each.

11. Let $f : W \to W$, be defined as $f(x) = x - 1$, if x is odd and $f(x) = x + 1$, if x is even. Show that f is invertible. Find the inverse of f, where W is the set of all whole numbers.

12. Solve for x :

$$\cos(\tan^{-1} x) = \sin\left(\cot^{-1} \frac{3}{4}\right)$$

OR

Prove that :

$$\cos^{-1} 7 + \cot^{-1} 8 + \cot^{-1} 18 = \cot^{-1} 3$$

13. Using properties of determinants, prove that

$$\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix} = a^2 (a + x + y + z)$$

14. If $x = a \cos\theta + b \sin\theta$ and $y = a \sin\theta - b \cos\theta$, show that $y^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 0$.

15. If $x^m y^n = (x + y)^{m+n}$, prove that $\dfrac{dy}{dx} = \dfrac{y}{x}$.

16. Find the approximate value of $f(3.02)$, upto 2 places of decimal, where $f(x) = 3x^2 + 5x + 3$.

OR

Find the intervals in which the function $f(x) = \dfrac{3}{2}x^4 - 4x^3 - 45x^2 + 51$ is

(a) strictly increasing

(b) strictly decreasing.

17. Evaluate: $\displaystyle\int \frac{x \cos^{-1} x}{\sqrt{1 - x^2}}\, dx$

OR

Evaluate: $\displaystyle\int (3x - 2)\sqrt{x^2 + x + 1}\, dx$

18. Solve the differential equation $(x^2 - yx^2)\, dy + (y^2 + x^2y^2)\, dx = 0$, given that $y = 1$, when $x = 1$.

19. Solve the differential equation $\dfrac{dy}{dx} + y \cot x = 2 \cos x$, given that $y = 0$, when $x = \dfrac{\pi}{2}$.

20. Show that the vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar if and only if $\vec{a} + \vec{b}$, and $\vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are coplanar.

OR

Find a unit vector perpendicular to both of the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ where $\vec{a} = \hat{i} + \hat{j} + \hat{k}$,

$\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}.$

21. Find the shortest distance between the lines whose vector equations are
$\vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$.

22. Three cards are drawn at random (without replacement) from a well shuffled pack of 52 playing cards. Find the probability distribution of number of red cards. Hence find the mean of the distribution.

SECTION–C

Question numbers 23 to 29 carry 6 marks each.

23. ***Two schools P and Q want to award their selected students on the values of Tolerance, Kindness and Leadership. The school P wants to award ₹ x each, ₹ y each and ₹ z each for the three respective values to 3, 2 and 1 students respectively with a total award money of ₹ 2,200. School Q wants to spend ₹ 3,100 to award its 4, 1 and 3 students on the respective values (by giving the same award money to the three values as school P). If the total amount of award for one prize on each value is ₹ 1,200, using matrices, find the award money for each value.***

Apart from these three values, suggest one more value that should be considered for award.

24. Show that a cylinder of a given volume which is open at the top has minimum total surface area, when its height is equal to the radius of its base.

25. Evaluate : $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\, dx$

26. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the line $\frac{x}{3} + \frac{y}{2} = 1$.

27. Find the equation of the plane that contains the point (1, –1, 2) and is perpendicular to both the planes $2x + 3y - 2z = 5$ and $x + 2y - 3z = 8$. Hence find the distance of point P(–2, 5, 5) from the plane obtained above.

OR

Find the distance of the point P(–1, –5, –10) from the point of intersection of the line joining the points A(2, –1, 2) and B(5, 3, 4) with the plane $x - y + z = 5$.

28. A cottage industry manufactures pedestal lamps and wooden shades, each requiring the use of a grinding/cutting machine and a sprayer. It takes 2 hours on the grinding/cutting machine and 3 hours on the sprayer to manufacture a pedestal lamp. It takes 1 hour on the grinding/cutting machine and 2 hours on the sprayer to manufacture a shade. On any day, the sprayer is available for at the most 20 hours and the grinding/cutting machine for at the most 12 hours. The profit from the sale of a lamp is ₹ 25 and that from a shade is ₹ 15. Assuming that the manufacturer can sell all the lamps and shades that he produces, how should he schedule his daily production in order to maximise his profit. Formulate an LPP and solve it graphically.

29. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probabilities of an accident for them are 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver or a car driver?

OR

Five cards are drawn one by one, with replacement, from a well shuffled deck of 52 cards, Find the probability that

(*i*) all the five cards are diamonds.

(*ii*) only 3 cards are diamonds.

(*iii*) none is a diamond.

SET–II

Only those questions, not included in Set I, are given

9. Evaluate :

$$\int_0^{\pi/4} \sin 2x \, dx$$

10. Write a unit vector in the direction of vector $\vec{PQ}$, where P and Q are the points (1, 3, 0) and (4, 5, 6) respectively.

19. Using properties of determinants, prove that:

$$\begin{vmatrix} x+\lambda & 2x & 2x \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix} = (5x+\lambda)(\lambda - x)^2$$

20. If $e^x + e^y = e^{x+y}$, prove that $\dfrac{dy}{dx} + e^{y-x} = 0$.

21. Find a particular solution of the differential equation $\dfrac{dy}{dx} + 2y \tan x = \sin x$, given that $y = 0$, when $x = \dfrac{\pi}{3}$.

22. Find the shortest distance between the following lines :

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}; \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$$

28. A window is of the form of a semi-circle with a rectangle on its diameter. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

29. Evaluate :

$$\int_0^{\pi} \frac{x\,dx}{a^2 \cos^2 x + b^2 \sin^2 x}$$

SET–III

Only those questions, not included in Set I and Set II are given.

9. Write the value of the following :

$$\hat{i} \times (\hat{j} + \hat{k}) + \hat{j} \times (\hat{k} + \hat{i}) + \hat{k} \times (\hat{i} + \hat{j})$$

10. Evaluate : $\int_0^1 x e^{x^2} dx$

19. Find the distance between the lines l_1 and l_2 given by

$$l_1 : \vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k});$$

$$l_2 : \vec{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + \mu(4\hat{i} + 6\hat{j} + 12\hat{k}).$$

20. Solve the differential equation $x \log x \dfrac{dy}{dx} + y = \dfrac{2}{x} \log x$.

21. If $\cos y = x \cos(a + y)$, where $\cos a \neq \pm 1$, prove that $\dfrac{dy}{dx} = \dfrac{\cos^2(a + y)}{\sin a}$.

22. Prove the following, using properties of determinants:

$$\begin{vmatrix} a^2 & bc & ac + c^2 \\ a^2 + ab & b^2 & ac \\ ab & b^2 + bc & c^2 \end{vmatrix} = 4a^2b^2c^2$$

28. The sum of the perimeters of a circle and a square is k, where k is some constant. Prove that the sum of their areas is least when the side of the square is equal to the diameter of the circle.

29. Evaluate:

$$\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$$

Solutions

SET–I

SECTION–A

1. Here $R = \{(a, a^3) : a \text{ is a prime number less than } 5\}$

$\Rightarrow \quad R = \{(2, 8), (3, 27)\}$

Hence Range of $R = \{8, 27\}$

2. We have, $\cos^{-1}\left(-\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$

$$\cos^{-1}\left(-\frac{1}{2}\right) = \cos^{-1}\left(\cos\frac{\pi}{3}\right) \qquad [\because \cos(-\theta) = \cos\theta]$$

$$\qquad \left[\because \frac{\pi}{3} \in [0, \pi]\right]$$

$$= \frac{\quad}{3}$$

Also $\quad \sin^{-1}\left(\frac{1}{2}\right) = \sin^{-1}\left(\sin\frac{\pi}{6}\right)$

$$= \frac{\pi}{6} \qquad \left[\because \frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right]$$

$\therefore \quad \cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} + 2\left(\frac{\pi}{6}\right)$

$$= \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$$

[**Note:** Principal value branches of $\sin x$ and $\cos x$ are $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $[0, \pi]$ respectively.]

3. Given $\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

Applying $C_2 \to C_2 - 2C_1$, we get

$$\begin{bmatrix} 4 & -6 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & -4 \\ 1 & -1 \end{bmatrix}$$

4. Given $\begin{bmatrix} a+4 & 3b \\ 8 & -6 \end{bmatrix} = \begin{bmatrix} 2a+2 & b+2 \\ 8 & a-8b \end{bmatrix}$

On equating, we get

$a + 4 = 2a + 2,\ 3b = b + 2,\ a - 8b = -6$

$\Rightarrow \quad a = 2, b = 1$

Now the value of $a - 2b = 2 - (2 \times 1) = 2 - 2 = 0$

5. Here, $|3A| = k|A|$

$\Rightarrow \quad 3^3 \; |A| = k|A| \qquad [\because |kA| = k^n |A|$ where n is order of $A]$

$\Rightarrow \quad 27 \; |A| = k|A|$

$\Rightarrow \quad k = 27$

6. Let $I = \int \dfrac{dx}{\sin^2 x.\cos^2 x}$

$$= \int \operatorname{cosec}^2 x.\sec^2 x\,dx = \int (1+\cot^2 x).\sec^2 x\,dx$$

$$= \int \sec^2 x\,dx + \int \cot^2 x.\sec^2 x\,dx$$

$$= \tan x + \int \frac{\sec^2 x\,dx}{\tan^2 x}$$

$$= \tan x + \int z^{-2}\,dz \qquad [\text{Let } \tan x = z \Rightarrow \sec^2 x\,dx = dz]$$

$$= \tan x + \frac{z^{-2+1}}{-2+1} + c = \tan x - \frac{1}{z} + c$$

$$= \tan x - \frac{1}{\tan x} + c \qquad [\text{Putting } z = \tan x]$$

7. Let $I = \int_0^{\pi/4} \tan x\,dx$

$$= \int_0^{\pi/4} \frac{\sin x}{\cos x}\,dx$$

Let $\cos x = z \Rightarrow -\sin x\,dx = dz \Rightarrow \sin x\,dx = -\,dz$

For limit, if $x = 0, \Rightarrow z = 1;\ x = \dfrac{\pi}{4} \Rightarrow z = \dfrac{1}{\sqrt{2}}$

$$\therefore \qquad I = -\int_1^{\frac{1}{\sqrt{2}}} \frac{dz}{z} = \int_{\frac{1}{\sqrt{2}}}^{1} \frac{dz}{z}$$

$$= [\log|z|]_{\frac{1}{\sqrt{2}}}^{1} = \log|1| - \log\left|\frac{1}{\sqrt{2}}\right|$$

$$= 0 - \log\frac{1}{\sqrt{2}} = \log\sqrt{2}$$

8. Required projection $= \dfrac{(\hat i + \hat j + \hat k).\hat j}{|\hat j|}$

$$= \frac{1}{\sqrt{0^2+1^2+0^2}} = \frac{1}{|1|} = 1$$

9. Required vector

$$= 21\left(\frac{2\hat i - 3\hat j + 6\hat k}{\sqrt{2^2+(-3)^2+6^2}}\right) = 21\left(\frac{2\hat i - 3\hat j + 6\hat k}{\sqrt{49}}\right)$$

$$= 21 \times \frac{2\hat i - 3\hat j + 6\hat k}{7} = 3(2\hat i - 3\hat j + 6\hat k)$$

$$= 6\hat i - 9\hat j + 18\hat k$$

10. Given two lines are $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$

$$\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k}).$$

Parallel vectors of both lines are

$$\vec{k_1} = 3\hat{i} + 2\hat{j} + 6\hat{k},\ \vec{k_2} = \hat{i} + 2\hat{j} + 2\hat{k}$$

$\therefore$ Required angle = angle between $\vec{k_1}$ and $\vec{k_2}$. If θ be required angle.

then $$\cos\theta = \frac{\vec{k_1}.\vec{k_2}}{|\vec{k_1}|.|\vec{k_2}|}$$

$\Rightarrow$ $$\cos\theta = \frac{3+4+12}{\sqrt{49}\sqrt{9}} \quad \Rightarrow \quad \cos\theta = \frac{19}{7\times 3}$$

$\Rightarrow$ $$\cos\theta = \frac{19}{21} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{19}{21}\right)$$

SECTION–B

11. In order to prove that f is invertible, we have to prove f is one-one onto function.

For one-one

Case I Let x_1, x_2 both be odd numbers

Now $f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 - 1 \quad \forall\ x_1, x_2 \in W$

$\Rightarrow x_1 = x_2$

i.e. f is one-one.

Case II Let x_1, x_2 both be even number

Now $f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 + 1$

$\Rightarrow x_1 = x_2$

i.e. f is one-one.

Case III Let x_1 be even and x_2 be odd.

If $f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 - 1 \Rightarrow x_1 - x_2 = -2 \Rightarrow x_2 - x_1 = 2$

Which is not possible as the difference of even and odd is always odd.

i.e. $f(x_1) \neq f(x_2)$ when x_1 is even and x_2 is odd.

i.e. $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

Hence f is one-one function. ...(A)

For Onto

Q $f(x) = x - 1$ if x is odd

$f(x) = x + 1$ if x is even

$\Rightarrow$ $\forall$ even number $y \in$ Wc $\exists$ odd number $(y + 1) \in$ Wd as f pre image and $\forall$ odd number $y \in$ Wc we have even number $(y - 1) \in$ Wd as f pre image.

Hence f is onto function ...(B)

A and B implies that f is one-one and onto function

i.e. f is invertible function.

For Inverse Function

Let $f^{-1}(x)$ be inverse of $f(x)$

$\therefore \quad fof^{-1} = \mathrm{I} \quad \Rightarrow \quad fof^{-1}(x) = \mathrm{I}x$

$\Rightarrow \quad f(f^{-1}(x)) = x \qquad [\because \mathrm{I}(x) = x]$

$\Rightarrow \quad f^{-1}(x) - 1 = x \qquad$ if $f^{-1}(x)$ is odd

and $\quad f^{-1}(x) + 1 = x \qquad$ if $f^{-1}(x)$ is even

$\Rightarrow \quad f^{-1}(x) = x + 1 \qquad$ if x is even

$\quad f^{-1}(x) = 1 - x \qquad$ if x is odd

i.e. $f^{-1} = f$.

12. Given $\cos(\tan^{-1} x) = \sin\left(\cot^{-1} \frac{3}{4}\right)$

$$\Rightarrow \quad \cos(\tan^{-1} x) = \cos\left(\frac{\pi}{2} - \cot^{-1} \frac{3}{4}\right)$$

$$\Rightarrow \quad \tan^{-1} x = \frac{\pi}{2} - \cot^{-1} \frac{3}{4}$$

$$\Rightarrow \quad \frac{\pi}{2} - \cot^{-1} x = \frac{\pi}{2} - \cot^{-1} \frac{3}{4} \qquad \Rightarrow \quad \cot^{-1} x = \cot^{-1} \frac{3}{4}$$

$$\Rightarrow \quad x = \frac{3}{4}$$

$\left[\textbf{Note:} \quad \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right), \quad \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\right]$

OR

We have,

$$\text{L.H.S.} = \cot^{-1} 7 + \cot^{-1} 8 + \cot^{-1} 18 = \left(\tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{8}\right) + \tan^{-1} \frac{1}{18}$$

$$= \tan^{-1}\left(\frac{\frac{1}{7} + \frac{1}{8}}{1 - \frac{1}{7} \times \frac{1}{8}}\right) + \tan^{-1} \frac{1}{18} \qquad \left[\because \frac{1}{7} \times \frac{1}{8} < 1\right]$$

$$= \tan^{-1} \frac{3}{11} + \tan^{-1} \frac{1}{18} = \tan^{-1}\left(\frac{\frac{3}{11} + \frac{1}{18}}{1 - \frac{3}{11} \times \frac{1}{18}}\right) \qquad \left[\because \frac{3}{11} \times \frac{1}{18} < 1\right]$$

$$= \tan^{-1}\left(\frac{\frac{65}{198}}{\frac{195}{198}}\right) = \tan^{-1}\left(\frac{65}{195}\right) = \tan^{-1} \frac{1}{3}$$

$$= \cot^{-1} 3 = \text{RHS}$$

13. L.H.S $= \begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$= \begin{vmatrix} a+x+y+z & y & z \\ a+x+y+z & a+y & z \\ a+x+y+z & y & a+z \end{vmatrix}$$

$$= (a+x+y+z) \begin{vmatrix} 1 & y & z \\ 1 & a+y & z \\ 1 & y & a+z \end{vmatrix}$$

Apply $R_1 \to R_1 - R_2$, we get

$$= (a+x+y+z) \begin{vmatrix} 0 & -a & 0 \\ 1 & a+y & z \\ 1 & y & a+z \end{vmatrix}$$

Expanding along R_1, we get

$$= (a+x+y+z)\ \{0 + a\ (a+z-z)\} = a^2(a+x+y+z)$$

14. Given

$$x = a\cos\theta + b\sin\theta$$

$$\Rightarrow \qquad \frac{dx}{d\theta} = -a\sin\theta + b\cos\theta \qquad \dots (i)$$

Also $\cancel{y =}\ a\sin\theta - b\cos$

$d\theta \frac{dy}{}$

$\theta \quad = a\cos\theta + b\sin$

$\cancel{\theta}\ \text{—} = -\ \text{————}$

$$\therefore \quad \frac{dy}{\sin\theta} = \frac{\frac{dy}{d\theta}}{dx} = \frac{a\cos\theta + b}{dx \quad -a\sin\theta + b}$$

$\cos\theta$

$dy \qquad d\theta x$

$$\Rightarrow \quad \text{—} = \frac{dy \quad a\cos\theta + b}{\sin\theta\ dx \quad b\cos\theta -}$$

$a\sin\theta$

$$\Rightarrow \quad \frac{}{dx} = -\frac{}{y}$$

Differentiating again w.r.t.x, we get

$$\frac{d^2y}{} \qquad \frac{y - x.\frac{dy}{dx}}{} \qquad \frac{y - x\left(\frac{-}{y}\right)}{}$$

$x \mid dx^2 = - \qquad y^2 \qquad = - \qquad y^2$

$$\frac{d^2y}{dx^2} \qquad \frac{y^2 + x^2}{y^3}$$

… (*ii*)

[From (*i*) and (*ii*)]

... (*iii*)

... (*iv*)

Now $y^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = y^2\left(-\dfrac{y^2 + x^2}{y^3}\right) - x\left(-\dfrac{x}{y}\right) + y$ [From (*iii*) and (*iv*)]

$$= \frac{-y^2 - x^2}{y} + \frac{x^2}{y} + y = -y - \frac{x^2}{y} + \frac{x^2}{y} + y = 0$$

15. Given $x^m y^n = (x+y)^{m+n}$

Taking logarithm of both sides, we get

$\log x^m y^n = \log (x+y)^{m+n}$

$\Rightarrow \log x^m + \log y^n = (m+n) \log (x+y)$ [By law of logarithm]

$\Rightarrow m \log x + n \log y = (m+n) \log (x+y)$

Differentiating both sides w.r.t. x we get

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y}\left(1 + \frac{dy}{dx}\right) \Rightarrow \frac{m}{x} - \frac{m+n}{x+y} = \left(\frac{m+n}{x+y} - \frac{n}{y}\right)\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{mx + my - mx - nx}{x(x+y)} = \left(\frac{my + ny - nx - ny}{(x+y).y}\right)\frac{dy}{dx}$$

$$\Rightarrow \frac{my - nx}{x(x+y)} = \frac{my - nx}{y(x+y)}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{y}{x}$$

16. Here $f(x) = 3x^2 + 5x + 3$

Let $x = 3$ and $dx = .02$ $\therefore x + dx = 3.02$

By definition, we have approximately

$$f'(x) = \frac{f(x+dx) - f(x)}{dx}$$

$\Rightarrow f'(3) = \dfrac{f(3 + .02) - f(3)}{.02}$ [Putting $x = 3$ and $dx = .02$]

$\Rightarrow f'(3) = \dfrac{f(3.02) - f(3)}{.02}$ ... (1)

Now $f(x) = 3x^2 + 5x + 3$

$\Rightarrow f'(x) = 6x + 5 \Rightarrow f'(3) = 23$

Also $f(3) = 3 \times 3^2 + 5 \times 3 + 3 = 27 + 15 + 3 = 45$

Putting in (1) we get

$$23 = \frac{f(3.02) - 45}{.02}$$

$\Rightarrow f(3.02) = 23 \times .02 + 45 = 45.46$

OR

Here, $f(x) = \dfrac{3}{2}x^4 - 4x^3 - 45x^2 + 51$

$\Rightarrow f'(x) = 6x^3 - 12x^2 - 90x$

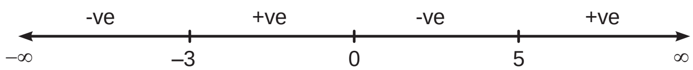
$$\Rightarrow \quad f'(x) = 6x(x^2 - 2x - 15)$$
$$= 6x(x + 3)(x - 5)$$

Now for critical point $f'(x) = 0$

$\Rightarrow \quad 6x(x+3)(x-5) = 0$

$\Rightarrow \quad x = 0, -3, 5$

i.e. $\quad -3, 0, 5$ are critical points which divides domain R of given function into four disjoint sub intervals $(-\infty, -3)$, $(-3, 0)$, $(0, 5)$, $(5, \infty)$



For (– ∞, –3)

$f'(x) = +\text{ve}\times-\text{ve}\times-\text{ve}\times-\text{ve} = -\text{ve}$

i.e. $f(x)$ is decreasing in $(-\infty, -3)$

For (– 3, 0)

$f'(x) = +\text{ve}\times-\text{ve}\times+\text{ve}\times-\text{ve} = +\text{ve}$

i.e. $f(x)$ is increasing in $(-3, 0)$

For (0, 5)

$f'(x) = +\text{ve}\times+\text{ve}\times+\text{ve}\times-\text{ve} = -\text{ve}$

i.e. $f(x)$ is decreasing in $(0, 5)$

For (5, ∞)

$f'(x) = +\text{ve}\times+\text{ve}\times+\text{ve}\times+\text{ve} = +\text{ve}$

i.e. $f(x)$ is increasing in $(5, \infty)$

Hence $f(x)$ is (a) strictly increasing in $(-3, 0)$ U $(5, \infty)$

(b) strictly decreasing in $(-\infty, -3)$ U $(0, 5)$

17. Let $\quad I = \int \frac{x\cos^{-1}x}{\sqrt{1-x^2}}\,dx$

Let $\cos^{-1}x = z \Rightarrow \quad -\frac{1}{\sqrt{\quad ^2}}\,dx = dz \quad \Rightarrow 1 - x \qquad\qquad \frac{1}{\sqrt{1-x^2}}\,dx = -dz$

$\therefore \quad I = -\int \cos z . z\,dz$

$= -\left(z . \sin z - \int \sin z\,dz + c\right)$

$= -(z . \sin z + \cos z - c)$

$= -z\sin z - \cos z + c$

$\Rightarrow \quad I = -\cos^{-1}x . \sqrt{1-x^2} - x + c \qquad [\because x = \cos z \Rightarrow \sin z = \sqrt{1-x^2}]$

$I = -\sqrt{1-x^2}\cos^{-1}x - x + c$

OR

Let $\quad I = \int (3x-2)\sqrt{x^2+x+1}\,dx$

Let $\quad 3x - 2 = \text{A}\frac{d}{dx}(x^2+x+1) + \text{B}$

$$\Rightarrow \qquad 3x - 2 = A\,(2x + 1) + B$$

$$\Rightarrow \qquad 3x - 2 = 2Ax + (A + B)$$

Equating we get

$$2A = 3 \quad \text{and } A + B = -2$$

$$\Rightarrow \quad A = \frac{3}{2} \quad \text{and } B = -2 - \frac{3}{2} = -\frac{7}{2}$$

Now, $I = \int \left\{\frac{3}{2}(2x+1) - \frac{7}{2}\right\}\sqrt{x^2+x+1}\, dx$

$$= \frac{3}{2}\int (2x+1)\sqrt{x^2+x+1}\, dx - \frac{7}{2}\int \sqrt{x^2+x+1}\, dx$$

$$\Rightarrow \quad I = \frac{3}{2} I_1 - \frac{7}{2} I_2 \qquad \ldots (i)$$

Where, $I_1 = \int (2x+1)\sqrt{x^2+x+1}\, dx$ and $I_2 = \int \sqrt{x^2+x+1}\, dx$

Now, $I_1 = \int (2x+1)\sqrt{x^2+x+1}\, dx$

Let $\quad x^2 + x + 1 = z \Rightarrow \; (2x+1)\, dx = dz$

$$\Rightarrow \quad I_1 = \int \sqrt{z}\, dz$$

$$= \frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c_1 = \frac{2}{3} z^{3/2} + c_1$$

$$I_1 = \frac{2}{3}(x^2+x+1)^{3/2} + c_1 \qquad \ldots (ii)$$

Again $I_2 = \int \sqrt{x^2+x+1}\, dx$

$$= \int \sqrt{x^2 + 2.x.\frac{1}{2} + \left(\frac{1}{2}\right)^2 - \frac{1}{4} + 1}\, dx$$

$$= \int \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\, dx$$

$$I_2 = \frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1} + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2 . \log\left|x + \frac{1}{2} + \sqrt{x^2+x+1}\right| + c_2 \qquad \ldots (iii)$$

Putting value of I_1 and I_2 from (ii), (iii) in (i), we get

$$I = (x^2+x+1)^{3/2} - \frac{7}{4}\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1} - \frac{21}{16}\log\left|\left(x+\frac{1}{2}\right) + \sqrt{x^2+x+1}\right| + c$$

[where $c = c_1 + c_2$]

18. The given differential equation is

$$(x^2 - yx^2)\, dy + (y^2 + x^2y^2)\, dx = 0$$

$$\Rightarrow \quad x^2(1-y)\, dy + y^2(1+x^2)\, dx = 0$$

$$\Rightarrow \quad x^2(1-y)\, dy = -y^2(1+x^2)\, dx$$

$$\Rightarrow \quad \frac{(1-y)}{y^2}\, dy = -\frac{(1+x^2)}{x^2}\, dx$$

$\Rightarrow \quad \left(\frac{1}{y^2}-\frac{1}{y}\right)dy = -\left(\frac{1}{x^2}+1\right)dx$

Integrating both sides, we get

$$\int\left(\frac{1}{y^2}-\frac{1}{y}\right)dy = -\int\left(\frac{1}{x^2}+1\right)dx$$

$\Rightarrow \quad \int y^{-2}dy - \int\frac{1}{y}\,dy = -\int x^{-2}dx - \int dx$

$\Rightarrow \quad -\frac{1}{y} - \log|y| = \frac{1}{x} - x + c$ It is general solution

Now putting $x = 1$ and $y = 1$ in general solution, we get

$-1-\log 1 = 1 - 1 + c \quad \Rightarrow \quad c = -1$

we have particular solution as

$$\log|y| + \frac{1}{y} = -\frac{1}{x} + x + 1$$

19. Given differential equation is

$\frac{dy}{dx} + y\cot x = 2\cos x \quad \Rightarrow \quad \frac{dy}{dx} + \cot x \,.\, y = 2\cos x$

It is in the form $\frac{dy}{dx} + Py = Q$. where $P = \cot x$, $Q = 2\cos x$

$\therefore \quad \text{I.F.} = e^{\int \cot x\, dx} = e^{\log|\sin x|} = \sin x.$

Therefore, general solution is

$y.\sin x = \int 2\cos x.\sin x\, dx + c$

$\Rightarrow \quad y\sin x = \int \sin 2x\, dx + c \quad \Rightarrow \quad y\sin x = -\frac{\cos 2x}{2} + c$

$\Rightarrow \quad y\sin x = -\frac{1}{2}\cos 2x + c$

Now put $y = 0$ and $x = \frac{\pi}{2}$ in the above equation, we get

$0 \times \sin\frac{\pi}{2} = \frac{-1}{2}\cos 2 \times \frac{\pi}{2} + c$

$0 = -\frac{1}{2}(-1) + c \qquad (\because \cos\pi = -1)$

$c = -\frac{1}{2}$

The particular solution is $y\sin x = -\frac{\cos 2x}{2} - \frac{1}{2}$ or $2y\sin x = -(1+\cos 2x)$

20. **If part:** Let $\vec{a}, \vec{b}, \vec{c}$ are coplanar

$\Rightarrow$ scalar triple product of $\vec{a}, \vec{b}$ and $\vec{c}$ is zero

$\Rightarrow \quad [a \;\; b \;\; c] = 0 \qquad\qquad \Rightarrow \quad \vec{a}.(\vec{b} \times \vec{c}) = \vec{b}.(\vec{c} \times \vec{a}) = \vec{c}.(\vec{a} \times \vec{b}) = 0$

Now, $[\vec{a}+\vec{b}\quad \vec{b}+\vec{c}\quad \vec{c}+\vec{a}] = (\vec{a}+\vec{b}).\{(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\}$

$= (\vec{a}+\vec{b}).\{\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}\}$

$= (\vec{a}+\vec{b}).\{\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}\}$ $\quad [\because \vec{c}\times\vec{c}=0]$

$= \vec{a}.(\vec{b}\times\vec{c}) + \vec{a}.(\vec{b}\times\vec{a}) + \vec{a}.(\vec{c}\times\vec{a}) + \vec{b}.(\vec{b}\times\vec{c}) + \vec{b}.(\vec{b}\times\vec{a}) + \vec{b}.(\vec{c}\times\vec{a})$

$= [\vec{a}\quad \vec{b}\quad \vec{c}] + 0 + 0 + 0 + 0 + [\vec{b}\quad \vec{c}\quad \vec{a}]$ [By property of scalar triple product]

$= [\vec{a}\quad \vec{b}\quad \vec{c}] + [\vec{a}\quad \vec{b}\quad \vec{c}] = 2[\vec{a}\quad \vec{b}\quad \vec{c}]$

$= 2\times 0 = 0$ $\quad [\because [\vec{a}\quad \vec{b}\quad \vec{c}] = 0]$

Hence, $\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$ are coplanar

Only if part: Let $\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$ are coplanar.

$\Rightarrow \quad [\vec{a}+\vec{b}\quad \vec{b}+\vec{c}\quad \vec{c}+\vec{a}] = 0$

$\Rightarrow \quad (\vec{a}+\vec{b}).\{(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\} = 0$

$\Rightarrow \quad (\vec{a}+\vec{b}).\{\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}\} = 0$

$\Rightarrow \quad (\vec{a}+\vec{b}).\{\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}\} = 0$

$\Rightarrow \quad \vec{a}.(\vec{b}\times\vec{c}) + \vec{a}.(\vec{b}\times\vec{a}) + \vec{a}.(\vec{c}\times\vec{a}) + \vec{b}.(\vec{b}\times\vec{c}) + \vec{b}.(\vec{b}\times\vec{a}) + \vec{b}.(\vec{c}\times\vec{a}) = 0$

$\Rightarrow \quad [\vec{a}\quad \vec{b}\quad \vec{c}] + 0 + 0 + 0 + 0 + [\vec{b}\quad \vec{c}\quad \vec{a}] = 0$

$\Rightarrow \quad 2[\vec{a}\quad \vec{b}\quad \vec{c}] = 0$ $\quad [\because [\vec{a}\quad \vec{b}\quad \vec{c}] = [\vec{b}\quad \vec{c}\quad \vec{a}]]$

$\Rightarrow \quad [\vec{a}\quad \vec{b}\quad \vec{c}] = 0$

Hence, $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

OR

$\vec{a}+\vec{b} = (\hat{i}+\hat{j}+\hat{k}) + (\hat{i}+2\hat{j}+3\hat{k}) = 2\hat{i}+3\hat{j}+4\hat{k}$

$\vec{a}-\vec{b} = (\hat{i}+\hat{j}+\hat{k}) - (\hat{i}+2\hat{j}+3\hat{k}) = -\hat{j}-2\hat{k}$

$\therefore \quad$ Perpendicular vector of $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b}) = (\vec{a}+\vec{b})\times(\vec{a}-\vec{b})$

$$= (2\hat{i}+3\hat{j}+4\hat{k})\times(-\hat{j}-2\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix}$$

$$= (-6+4)\hat{i} - (-4-0)\hat{j} + (-2-0)\hat{k} = -2\hat{i} + 4\hat{j} - 2\hat{k}$$

$\therefore$ Required unit vector perpendicular to $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$

$$= \frac{-2\hat{i}+4\hat{j}-2\hat{k}}{\sqrt{(-2)^2+4^2+(-2)^2}} = \frac{-2\hat{i}+4\hat{j}-2\hat{k}}{\sqrt{4+16+4}}$$

$$= \frac{-2\hat{i}+4\hat{j}-2\hat{k}}{\sqrt{24}} = \frac{-2}{2\sqrt{6}}\hat{i} + \frac{4\hat{j}}{2\sqrt{6}} - \frac{2}{2\sqrt{6}}\hat{k}$$

$$= -\frac{1}{\sqrt{6}}\hat{i} + \frac{2}{\sqrt{6}}\hat{j} - \frac{1}{\sqrt{6}}\hat{k}$$

21. Comparing the given equations with equations

$\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, we get $\vec{a}_1 = \hat{i}+\hat{j}, \vec{b}_1 = 2\hat{i}-\hat{j}+\hat{k}$

and $\vec{a}_2 = 2\hat{i}+\hat{j}-\hat{k}$, $\vec{b}_2 = 3\hat{i}-5\hat{j}+2\hat{k}$

Therefore, $\vec{a}_2 - \vec{a}_1 = (\hat{i}-\hat{k})$ and

$$\vec{b}_1 \times \vec{b}_2 = (2\hat{i}-\hat{j}+\hat{k}) \times (3\hat{i}-5\hat{j}+2\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix} = 3\hat{i}-\hat{j}-7\hat{k}$$

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{9+1+49} = \sqrt{59}$$

Hence, the shortest distance between the given lines is given by

$$d = \left|\frac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2-\vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right| = \left|\frac{3-0+7}{\sqrt{59}}\right| = \frac{10}{\sqrt{59}} \text{ units.}$$

22. Let the number of red card in a sample of 3 cards drawn be random variable X. Obviously X may have values 0,1,2,3.

Now $P(X=0) =$ Probability of getting no red card $= \frac{^{26}C_3}{^{52}C_3} = \frac{2600}{22100} = \frac{2}{17}$

$P(X=1) =$ Probability of getting one red card and two non-red cards

$$= \frac{^{26}C_1 \times {^{26}C_2}}{^{52}C_3} = \frac{8450}{22100} = \frac{13}{34}$$

$P(X=2) =$ Probability of getting two red card and one non-red card

$$= \frac{^{26}C_2 \times {^{26}C_1}}{^{52}C_3} = \frac{8450}{22100} = \frac{13}{34}$$

$P(X=3) =$ Probability of getting 3 red cards $= \frac{^{26}C_3}{^{52}C_3} = \frac{2600}{22100} = \frac{2}{17}$

Hence, the required probability distribution in table as

X	0	1	2	3
P(X)	$\frac{2}{17}$	$\frac{13}{34}$	$\frac{13}{34}$	$\frac{2}{17}$

$\therefore$ Required mean $= E(X) = \Sigma p_i x_i = 0 \times \frac{2}{17} + 1 \times \frac{13}{34} + 2 \times \frac{13}{34} + 3 \times \frac{2}{17}$

$$= \frac{13}{34} + \frac{26}{34} + \frac{6}{17} = \frac{13 + 26 + 12}{34} = \frac{51}{34} = \frac{3}{2}$$

23. According to question,

$$3x + 2y + z = 2200$$
$$4x + y + 3z = 3100$$
$$x + y + z = 1200$$

The above system of equation may be written in matrix form as

$$AX = B \Rightarrow X = A^{-1} B \text{ where}$$

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \; B = \begin{bmatrix} 2200 \\ 3100 \\ 1200 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 3(1 - 3) - 2(4 - 3) + 1(4 - 1) = -6 - 2 + 3 = -5 \neq 0$$

i.e., A^{-1} exist

Now, $A_{11} = (1 - 3) = -2$, $A_{12} = -(4 - 3) = -1$, $A_{13} = (4 - 1) = 3$,

$A_{21} = -(2 - 1) = -1$, $A_{22} = (3 - 1) = 2$, $A_{23} = -(3 - 2) = -1$

$A_{31} = (6 - 1) = 5$, $A_{32} = -(9 - 4) = -5$, $A_{33} = (3 - 8) = -5$

$$\therefore \quad \text{Adj}(A) = \begin{bmatrix} -2 & -1 & 3 \\ -1 & 2 & -1 \\ 5 & -5 & -5 \end{bmatrix}^T = \begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}(\text{Adj } A) = \frac{1}{-5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & 1 & -5 \\ 1 & -2 & 5 \\ -3 & 1 & 5 \end{bmatrix}$$

$$\therefore \quad X = A^{-1} B.$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & 1 & -5 \\ 1 & -2 & 5 \\ -3 & 1 & 5 \end{bmatrix}\begin{bmatrix} 2200 \\ 3100 \\ 1200 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4400 + 3100 - 6000 \\ 2200 - 6200 + 6000 \\ -6600 + 3100 + 6000 \end{bmatrix}$$

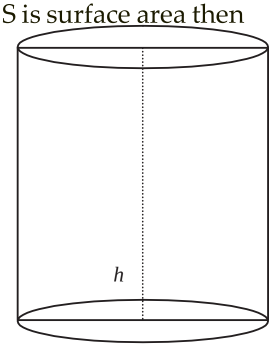
$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1500 \\ 2000 \\ 2500 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 300 \\ 400 \\ 500 \end{bmatrix}$$

$$\Rightarrow \quad x = 300, y = 400, z = 500$$

i.e., ₹ 300 for tolerance, ₹ 400 for kindness and ₹ 500 for leadership are awarded.

One more value like punctuality, honesty etc may be awarded.

24. Let r and h be radius and height of given cylinder having volume V. If S is surface area then



$$S = 2\pi rh + \pi r^2 \qquad \left[\because V = \pi r^2 h \;\Rightarrow\; h = \frac{V}{\pi r^2}\right]$$

$$\Rightarrow \quad S = 2\pi r . \frac{V}{\pi r^2} + \pi r^2 \qquad \Rightarrow \quad S = \frac{2V}{r} + \pi r^2$$

$$\Rightarrow \quad \frac{dS}{dr} = -\frac{2V}{r^2} + 2\pi r$$

For extremum value of S, $\frac{dS}{dr} = 0$

$$\Rightarrow \quad -\frac{2V}{r^2} + 2\pi r = 0 \qquad \Rightarrow \quad 2\pi r = \frac{2V}{r^2}$$

$$\Rightarrow \quad r^3 = \frac{2V}{2\pi} \qquad \Rightarrow \quad r^3 = \frac{V}{\pi}$$

Again $\frac{d^2S}{dr^2} = \frac{4V}{r^3} + 2\pi$ Now, $\left[\frac{d^2S}{dr^2}\right]_{r^3 = \frac{V}{\pi}} = +Ve$

Hence, for $r^3 = \frac{V}{\pi}$ S is minimum.

Therefore, for minimum value of surface area

$$r^3 = \frac{V}{\pi} \qquad \Rightarrow \qquad r^3 = \frac{\pi r^2 h}{\pi} \qquad [\because V = \pi r^2 h]$$

$$\Rightarrow \quad \frac{r^3}{r^2} = h \qquad \Rightarrow \qquad r = h$$

i.e. radius = height

25. Let $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\, dx$ ...(*i*)

As $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$

$$\therefore \quad I = \int_0^{\pi} \frac{(\pi - x) \tan (\pi - x)}{\sec (\pi - x) + \tan (\pi - x)}\, dx$$

$$= \int_0^{\pi} \frac{(\pi - x) \tan x}{\sec x + \tan x}\, dx \qquad \text{...(ii)}$$

By adding equations (*i*) and (*ii*), we get

$$2I = \pi \int_0^{\pi} \frac{\tan x}{\sec x + \tan x}\, dx$$

Multiplying and dividing by $(\sec x - \tan x)$, we get

$$2I = \pi \int_0^{\pi} \frac{\tan x\,(\sec x - \tan x)}{\sec^2 x - \tan^2 x}\,dx$$

$$= \pi \int_0^{\pi} (\sec x \tan x - \tan^2 x)\,dx$$

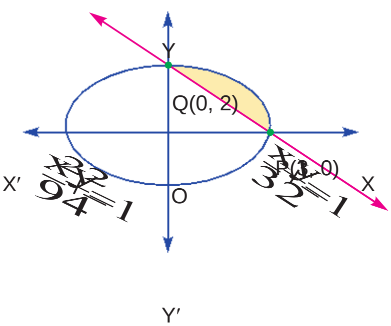
$$= \pi \int_0^{\pi} \sec x \tan x\,dx - \pi \int_0^{\pi} \sec^2 x\,dx + \int_0^{\pi} dx$$

$$= \pi\left[\sec x\right]_0^{\pi} - \pi\left[\tan x\right]_0^{\pi} + \pi\,[x]_0^{\pi} = \pi(-1-1) - 0 + \pi(\pi - 0) = \pi\,(\pi - 2)$$

$\Rightarrow \qquad 2I = \pi\,(\pi - 2) \qquad \Rightarrow \quad I = \dfrac{\pi}{2}(\pi - 2)$

26. Given curves are

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 \text{ and } \frac{x}{3} + \frac{y}{2} = 1$$



We have

$$y_1 = \frac{2}{3}\sqrt{9 - x^2} \text{ and } y_2 = \frac{2}{3}(3 - x)$$

So, area of required region

$$= \int_0^3 (y_1 - y_2)\,dx = \frac{2}{3}\int_0^3 [\sqrt{9 - x^2} - (3 - x)]\,dx$$

$$= \frac{2}{3}\int_0^3 [\sqrt{(3)^2 - x^2} - (3 - x)]\,dx$$

$$= \frac{2}{3}\left[\frac{x}{2}\sqrt{9 - x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3} - 3x + \frac{x^2}{2}\right]_0^3$$

$$= \frac{2}{3}\left[\frac{9}{2}\times\frac{\pi}{2} - 9 + \frac{9}{2} - 0\right] = \frac{2}{3}\left(\frac{9\pi}{4} - \frac{9}{2}\right) = \left(\frac{3\pi}{2} - 3\right) \text{ sq units.}$$

27. Equation of plane containing the point (1, –1, 2) is given by

$a(x - 1) + b\,(y + 1) + c\,(z - 2) = 0 \qquad \ldots (i)$

Q $\quad$ (*i*) is perpendicular to plane $2x + 3y - 2z = 5$

$\therefore \quad 2a + 3b - 2c = 0 \quad \ldots (ii)$

Also $\quad$ (*i*) is perpendicular to plane $x + 2y - 3z = 8$

$\therefore \quad a + 2b - 3c = 0 \quad \ldots (iii)$

From (*ii*) and (*iii*)

$$\frac{a}{-9 + 4} = \frac{b}{-2 + 6} = \frac{c}{4 - 3}$$

$\Rightarrow \quad \dfrac{a}{-5} = \dfrac{b}{4} = \dfrac{c}{1} = \lambda \text{ (say)} \qquad \Rightarrow \qquad a = -5\lambda,\ b = 4\lambda,\ c =$

λ
Putting these values in (*i*) we get

$-5\lambda\,(x - 1) + 4\lambda(y + 1) + \lambda(z - 2) = 0$

$\Rightarrow \quad -5\,(x - 1) + 4(y + 1) + (z - 2) = 0$

$\Rightarrow \quad -5x + 5 + 4y + 4 + z - 2 = 0$

$$\Rightarrow \quad -5x + 4y + z + 7 = 0$$

$\Rightarrow \quad 5x - 4y - z - 7 = 0$ ... (*iv*) It is required equation of plane.

Again, if d be the distance of point p (–2, 5, 5) to plane (*iv*)

Then

$$d = \left| \frac{5 \times -2 + (-4) \times 5 + (-1) \times 5 - 7}{\sqrt{5^2 + (-4)^2 + (-1)^2}} \right|$$

$$= \left| \frac{-10 - 20 - 5 - 7}{\sqrt{25 + 16 + 1}} \right| = \frac{42}{\sqrt{42}} = \sqrt{42} \text{ unit}$$

OR

The vector form of line and plane can be written as

$$\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda\,(3\hat{i} + 4\hat{j} + 2\hat{k}) \qquad \ldots(i)$$

$$\vec{r}\,.(\hat{i} - \hat{j} + \hat{k}) = 5 \qquad \ldots(ii)$$

For intersection point, we solve equations (*i*) and (*ii*) by putting the value of $\vec{r}$ from (*i*) in (*ii*).

$$[(2\hat{i} - \hat{j} + 2\hat{k}) + \lambda\,(3\hat{i} + 4\hat{j} + 2\hat{k})]\,.(\hat{i} - \hat{j} + \hat{k}) = 5$$

$\Rightarrow \quad (2 + 1 + 2) + \lambda\,(3 - 4 + 2) = 5 \quad \Rightarrow 5 + \lambda = 5 \quad \Rightarrow \lambda = 0$

Hence, position vector of intersecting point is $2\hat{i} - \hat{j} + 2\hat{k}$.

i.e., coordinates of intersection of line and plane is (2, –1, 2).

Hence, Required distance $= \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2}$

$$= \sqrt{9 + 16 + 144} = \sqrt{169} = 13 \text{ units}$$

28. Let the manufacturer produces x padestal lamps and y wooden shades; then time taken by x pedestal lamps and y wooden shades on grinding/cutting machines $= (2x + y)$ hours and time taken on the sprayer $= (3x + 2y)$ hours.

Since grinding/cutting machine is available for at the most 12 hours.

$$\therefore \qquad 2x + y \le 12$$

and sprayer is available for at most 20 hours.

Thus, we have

$$\therefore \qquad 3x + 2y \le 20$$

Now profit on the sale of x lamps and y shades is,

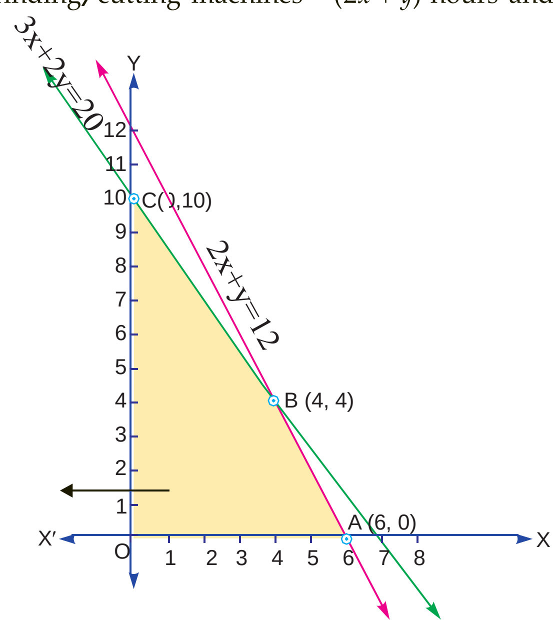
$$Z = 25x + 15y.$$

So, our problem is to find x and y so as to

Maximise $Z = 25x + 15y \qquad \ldots(i)$

Subject to the constraints:

$$3x + 2y \le 20 \qquad \ldots(ii)$$



$$2x + y \leq 12 \qquad \ldots(iii)$$

$$x \geq 0 \qquad \ldots(iv)$$

Y′

$y \geq 0 \qquad \ldots(v)$

The feasible region (shaded) $OABC$ determined by the linear inequalities (*ii*) to (*v*) is shown in the figure. The feasible region is bounded.

Let us evaluate the objective function at each corner point as shown below:

Corner Points	$Z = 25x + 15y$	
$O\,(0, 0)$	0	
$A\,(6, 0)$	150	
$B\,(4, 4)$	160	Maximum
$C\,(0, 10)$	150	

We find that maximum value of Z is ` 160 at $B\,(4, 4)$. Hence, manufacturer should produce 4 lamps and 4 shades to get maximum profit of `160.

29. Let E_1, E_2, E_3 and A be events such that

E_1 = Selection of scooter drivers

E_2 = Selection of car drivers.

E_3 = Selection of truck drivers.

A = meeting with an accident.

$$P(E_1) = \frac{2000}{12000} = \frac{1}{6},\ P(E_2) = \frac{4000}{12000} = \frac{1}{3},\ P(E_3) = \frac{6000}{12000} = \frac{1}{2}$$

$$P(A/E_1) = 0.01 = \frac{1}{100}$$

$$P(A/E_2) = 0.03 = \frac{3}{100}$$

$$P(A/E_3) = 0.15 = \frac{15}{100}$$

$$P(E_3/A) = \frac{P(E_3).P(A/E_3)}{P(E_1).P(A/E_1) + P(E_2).P(A/E_2) + P(E_3).P(A/E_3)}$$

$$= \frac{\frac{1}{2} \times \frac{15}{100}}{\frac{1}{6} \times \frac{1}{100} + \frac{1}{3} \times \frac{3}{100} + \frac{1}{2} \times \frac{15}{100}}$$

$$= \frac{\frac{15}{200}}{\frac{1}{600} + \frac{1}{100} + \frac{15}{200}} = \frac{15}{200} \times \frac{600}{52} = \frac{45}{52}$$

Therefore, required probability $= 1 - P\left(\frac{E_3}{A}\right) = 1 - \frac{45}{52} = \frac{7}{52}$

OR

Let number of diamond cards be taken as random variable X. X may have values 0, 1, 2, 3, 4, 5.

Here, p = probability of drawing diamond card in one draw

$$= \frac{13}{52} = \frac{1}{4}$$

$\therefore\ q$ = probability of drawing non diamond card in one draw

$$= 1 - \frac{1}{4} = \frac{3}{4}$$

Here, drawing a card is "Bernoullian trails" therefore we can apply $P(X = r) = {}^{n}C_{r}\ p^{r}\ q^{n-r}$ where $n = 5$.

(i) P (getting all the five cards diamond) $= P(X = 5)$

$$= {}^{5}C_{5}p^{5}.q^{0} = {}^{5}C_{5}\left(\frac{1}{4}\right)^{5}\left(\frac{3}{4}\right)^{0} = \left(\frac{1}{4}\right)^{5} = \frac{1}{1024}$$

(ii) P (getting only 3 cards diamond) $= P(X = 3)$

$$= {}^{5}C_{3}p^{3}.q^{2} = {}^{5}C_{3}\left(\frac{1}{4}\right)^{3}\left(\frac{3}{4}\right)^{2} = \frac{45}{512}$$

(iii) P (getting no card diamond) $= P(X = 0)$

$$= {}^{5}C_{0}.p^{0}q^{5} = {}^{5}C_{0} \times \left(\frac{1}{4}\right)^{0} \times \left(\frac{3}{4}\right)^{5} = \frac{243}{1024}$$

SET–II

9. $$I = \int_0^{\pi/4} \sin 2x\, dx$$

$$= -\left[\frac{\cos 2x}{2}\right]_0^{\pi/4} = -\frac{1}{2}\left[\cos\frac{\pi}{2} - \cos 0\right] = -\frac{1}{2}[0 - 1] = \frac{1}{2}$$

10. $$\overrightarrow{PQ} = (4-1)\,\hat{i} + (5-3)\hat{j} + (6-0)\hat{k} = 3\,\hat{i} + 2\hat{j} + 6\hat{k}$$

$\therefore$ Required unit vector $$= \frac{3\hat{i} + 2\hat{j} + 6\hat{k}}{\sqrt{3^2 + 2^2 + 6^2}} = \frac{3\hat{i} + 2\hat{j} + 6\hat{k}}{\sqrt{49}} = \frac{3}{7}\hat{i} + \frac{2}{7}\hat{j} + \frac{6}{7}\hat{k}$$

19. L.H.S. $$= \begin{vmatrix} x+\lambda & 2x & 2x \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 5x+\lambda & 2x & 2x \\ 5x+\lambda & x+\lambda & 2x \\ 5x+\lambda & 2x & x+\lambda \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 + C_2 + C_3]$$

$$= (5x+\lambda)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x+\lambda & 2x \\ 1 & 2x & x+\lambda \end{vmatrix} \quad [\text{Taking out } (5x+\lambda) \text{ common from } C_1]$$

$$= (5x+\lambda)\begin{vmatrix} 1 & 2x & 2x \\ 0 & \lambda - x & 0 \\ 0 & 0 & \lambda - x \end{vmatrix} \quad [\text{Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

Expanding along C_1, we get

$$= (5x+\lambda)(\lambda - x)^2 = \text{R.H.S.}$$

20. Given $e^x + e^y = e^{x+y}$

Differentiating both sides we get

$$e^x + e^y.\frac{dy}{dx} = e^{x+y}\left\{1 + \frac{dy}{dx}\right\}$$

$$\Rightarrow \quad e^x + e^y.\frac{dy}{dx} = e^{x+y} + e^{x+y}.\frac{dy}{dx} \qquad \Rightarrow \quad (e^{x+y} - e^y)\frac{dy}{dx} = e^x - e^{x+y}$$

$$\Rightarrow \quad (e^x + e^y - e^y)\frac{dy}{dx} = e^x - e^x - e^y \qquad [\because e^x + e^y = e^{x+y} \text{ (given)}]$$

$$\Rightarrow \quad e^x.\frac{dy}{dx} = -e^y \qquad \Rightarrow \quad \frac{dy}{dx} = -\frac{e^y}{e^x}$$

$$\Rightarrow \quad \frac{dy}{dx} = -e^{y-x} \qquad \Rightarrow \quad \frac{dy}{dx} + e^{y-x} = 0$$

21. Given differential equation is

$$\frac{dy}{dx} + 2\tan x.y = \sin x$$

Comparing it with $\frac{dy}{dx} + Py = Q$, we get

$$P = 2\tan x,\ Q = \sin x$$

$$\therefore \quad \text{I. F.} = e^{\int 2\tan x\,dx}$$

$$= e^{2\log \sec x} = e^{\log \sec^2 x} = \sec^2 x \qquad [\because e^{\log z} = z]$$

Hence general solution is

$$y.\sec^2 x = \int \sin x.\sec^2 x\,dx + C$$

$$y.\sec^2 x = \int \sec x.\tan x\,dx + C \Rightarrow y.\sec^2 x = \sec x + C$$

$$\Rightarrow \quad y = \cos x + C\cos^2 x$$

Putting $y = 0$ and $x = \frac{\pi}{3}$, we get

$$0 = \cos\frac{\pi}{3} + C.\cos^2\frac{\pi}{3}$$

$$0 = \frac{1}{2} + \frac{C}{4} \Rightarrow C = -2$$

$\therefore$ Required solution is $y = \cos x - 2\cos^2 x$

22. Let $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} = \lambda$ and $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} = k$

Now, let’s take a point on first line as

$A\,(\lambda + 3, -2\lambda + 5, \lambda + 7)$ and let

$B\,(7k - 1, -6k - 1, k - 1)$ be point on the second line

The direction ratio of the line AB

$$7k - \lambda - 4, -6k + 2\lambda - 6, k - \lambda - 8$$

Now as AB is the shortest distance between line 1 and line 2 so,

$$(7k - \lambda - 4) \times 1 + (-6k + 2\lambda - 6) \times (-2) + (k - \lambda - 8) \times 1 = 0 \quad ...(i)$$

and $$(7k - \lambda - 4) \times 7 + (-6k + 2\lambda - 6) \times (-6) + (k - \lambda - 8) \times 1 = 0 \quad ...(ii)$$

Solving equation (*i*) and (*ii*), we get

$$\lambda = 0 \text{ and } k = 0$$

$\therefore$ $A \equiv (3, 5, 7)$ and $B \equiv (-1, -1, -1)$

$\therefore$ $AB = \sqrt{(3+1)^2 + (5+1)^2 + (7+1)^2}$

$$= \sqrt{16 + 36 + 64} = \sqrt{116} \text{ units} = 2\sqrt{29} \text{ units}$$

28. Let $ABCED$ be required window having length $2x$ and width y. If A is the area of window.

Then $A = 2xy + \frac{1}{2}\pi x^2$ $\left[\begin{array}{l}\text{Given, Perimeter} = 10 \Rightarrow 2x + y + y + \frac{1}{2}\,2\pi x = 10 \\ \Rightarrow 2y = 10 - 2x - \pi x\end{array}\right]$

$$= x(10 - 2x - \pi x) + \frac{1}{2}\pi x^2$$

$$= 10x - 2x^2 - \pi x^2 + \frac{1}{2}\pi x^2$$

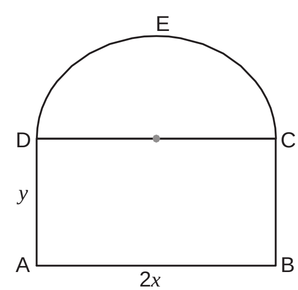
$$= 10x - 2x^2 - \frac{1}{2}\pi x^2 = 10x - \left(2 + \frac{1}{2}\pi\right)x^2$$

Obviously, window will admit maximum light and air if its area A is maximum.

Now, $\frac{dA}{dx} = 10 - 2x\left(2 + \frac{1}{2}\pi\right)$

For maxima or minima of A

$$\frac{dA}{dx} = 0$$



$\Rightarrow$ $10 - 2x\left(2 + \frac{1}{2}\pi\right) = 0 \Rightarrow 10 - x(4 + \pi) = 0$

$\Rightarrow$ $x = \frac{10}{4 + \pi}$ Also, $\frac{d^2A}{dx^2} = -(4 + \pi) < 0$

$\Rightarrow$ For maximum value of A, $x = \frac{10}{4 + \pi}$ and thus $y = \frac{10}{4 + \pi}$

Therefore, for maximum area, *i.e.*, for admitting maximum light and air,

Length of rectangular part of window $= 2x = \frac{20}{4 + \pi}$ and

10

$$\text{Width} = \frac{}{4 + \pi}$$

29. Let $I = \int_0^{\pi} \frac{x\,dx}{a^2 \cos^2 x + b^2 \sin^2 x}$ …(i)

$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)\,dx}{a^2 \cos^2(\pi - x) + b^2 \sin^2(\pi - x)}$ [using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$]

$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)\,dx}{a^2 \cos^2 x + b^2 \sin^2 x}$ …(ii)

Adding (i) and (ii)

$$2I = \int_0^{\pi} \frac{\pi}{a^2 \cos^2 x + b^2 \sin^2 x}\,dx \qquad \Rightarrow I = \frac{\pi}{2}\int_0^{\pi} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$$

Divide numerator and denominator by $\cos^2 x$

$$I = \frac{\pi}{2}\int_0^{\pi} \frac{\sec^2 x\,dx}{a^2 + b^2 \tan^2 x}$$

$\Rightarrow \quad I = \pi \int_0^{\frac{\pi}{2}} \frac{\sec^2 x\,dx}{a^2 + b^2 \tan^2 x}$ [using $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$]

Let $b \tan x = t \quad \Rightarrow \quad b \sec^2 x\,dx = dt$

When $\quad x = 0, t = 0 \quad$ and $\quad x = \frac{\pi}{2}, t \to \infty$

$$I = \frac{\pi}{b}\int_0^{\infty} \frac{dt}{t^2 + a^2} = \frac{\pi}{b}\cdot\frac{1}{a}\left[\tan^{-1}\frac{t}{a}\right]_0^{\infty}$$

$$I = \frac{\pi}{ab}(\tan^{-1}\infty - \tan^{-1}0) = \frac{\pi}{ab}\cdot\frac{\pi}{2} \qquad \Rightarrow I = \frac{\pi^2}{2ab}$$

SET–III

9. $\hat{i} \times (\hat{j} + \hat{k}) + \hat{j} \times (\hat{k} + \hat{i}) + \hat{k} \times (\hat{i} + \hat{j})$

$= \hat{i} \times \hat{j} + \hat{i} \times \hat{k} + \hat{j} \times \hat{k} + \hat{j} \times \hat{i} + \hat{k} \times \hat{i} + \hat{k} \times \hat{j}$

$= \hat{k} - \hat{j} + \hat{i} - \hat{k} + \hat{j} - \hat{i} = \vec{0}$

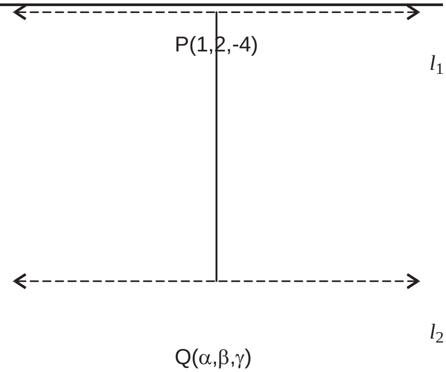
10. $I = \int_0^1 x\,e^{x^2}\,dx$

Let $\quad x^2 = z \Rightarrow 2x\,dx = dz \Rightarrow x\,dx = \frac{dz}{2}$

Also $\quad x = 0 \Rightarrow z = 0,\; x = 1 \Rightarrow z = 1$

$\therefore \quad I = \frac{1}{2}\int_0^1 e^z\,dz$

$= \frac{1}{2}[e^z]_0^1 = \frac{1}{2}(e^1 - e^0) = \frac{1}{2}(e - 1)$



19. Given lines are

$$l_1 : \vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k});$$

$$l_2 : \vec{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + \mu(4\hat{i} + 6\hat{j} + 12\hat{k})$$

After observation, we get $l_1 \parallel l_2$

Therefore, it is sufficient to find the perpendicular distance of a point of line l_1 to line l_2.

The co-ordinate of a point of l_1 is $P(1, 2, -4)$

Also the cartesian form of line l_2 is

$$\frac{x-3}{4} = \frac{y-3}{6} = \frac{z+5}{12} \qquad \ldots (i)$$

Let $Q(\alpha, \beta, \gamma)$ be foot of perpendicular drawn from P to line l_2

$\because$ $Q(\alpha, \beta, \gamma)$ lie on line l_2

$$\therefore \quad \frac{\alpha - 3}{4} = \frac{\beta - 3}{6} = \frac{\gamma + 5}{12} = \lambda \text{ (say)}$$

$$\Rightarrow \quad \alpha = 4\lambda + 3, \beta = 6\lambda + 3, \gamma = 12\lambda - 5$$

Again, $\because$ $\overrightarrow{PQ}$ is perpendicular to line l_2.

$\Rightarrow$ $\overrightarrow{PQ} \cdot \vec{b} = 0$, where $\vec{b}$ is parallel vector of l_2

$$\Rightarrow \quad (\alpha - 1).4 + (\beta - 2).6 + (\gamma + 4).12 = 0$$

$$\Rightarrow \quad 4\alpha - 4 + 6\beta - 12 + 12\gamma + 48 = 0$$

$$\Rightarrow \quad 4\alpha + 6\beta + 12\gamma + 32 = 0$$

$$\Rightarrow \quad 4(4\lambda + 3) + 6(6\lambda + 3) + 12(12\lambda - 5) + 32 = 0$$

$$\Rightarrow \quad 16\lambda + 12 + 36\lambda + 18 + 144\lambda - 60 + 32 = 0$$

$$\Rightarrow \quad 196\lambda + 2 = 0 \qquad \Rightarrow \qquad \lambda = \frac{-2}{196} = \frac{-1}{98}$$

$$\text{Co-ordinate of } Q \equiv \left(4 \times \left(-\frac{1}{98}\right) + 3, 6 \times \left(-\frac{1}{98}\right) + 3, 12 \times \left(-\frac{1}{98}\right) - 5\right)$$

$$\equiv \left(-\frac{2}{49} + 3, -\frac{3}{49} + 3, -\frac{6}{49} - 5\right) \equiv \left(\frac{145}{49}, \frac{144}{49}, -\frac{251}{49}\right)$$

Therefore required perpendicular distance is

$$\sqrt{\left(\frac{145}{49} - 1\right)^2 + \left(\frac{144}{49} - 2\right)^2 + \left(\frac{-251}{49} + 4\right)^2}$$

$$= \sqrt{\left(\frac{96}{49}\right)^2 + \left(\frac{46}{49}\right)^2 + \left(\frac{-55}{49}\right)^2} = \sqrt{\frac{96^2 + 46^2 + 55^2}{49^2}}$$

$$= \sqrt{\frac{9216 + 2116 + 3025}{49^2}} = \frac{\sqrt{14357}}{49} = \frac{7\sqrt{293}}{49} = \frac{\sqrt{293}}{7}$$

293 units

20. Given differential equation is

$$x \log x \frac{dy}{dx} + y = \frac{2}{x} \log x$$

$$\Rightarrow \quad \frac{dy}{dx} + \left(\frac{1}{x.\log x}\right).y = \frac{2}{x^2} \qquad \text{(Divide each term by } x \log x\text{)}$$

It is in the form $\frac{dy}{dx} + Py = Q$

where $P = \frac{1}{x.\log x}$, $Q = \frac{2}{x^2}$

$\therefore$ $I.F. = e^{\int P dx} = e^{\int \frac{dx}{x \log x}}$

put $\log x = z$ $= e^{\int \frac{1}{z} dz} = e^{\log z} = z = \log x$

$\therefore$ General solution is

$$y. \log x = \int \log x. \frac{2}{x^2} dx + c$$

$$\Rightarrow \quad y \log x = 2 \int \frac{\log x}{x^2} dx + c$$

Let $\log x = z \Rightarrow \frac{1}{x} dx = dz,$ Also $\log x = z \Rightarrow x = e^z$

$$\therefore \quad y \log x = 2 \int \frac{z}{e^z} dz + c$$

$$\Rightarrow \quad y \log x = 2 \int z.e^{-z} dz + c$$

$$\Rightarrow \quad y \log x = 2 \left[z.\frac{e^{-z}}{-1} - \int \frac{e^{-z}}{-1} dz \right] + c$$

$$\Rightarrow \quad y \log x = 2 [-ze^{-z} + \int e^{-z} dz] + c$$

$$\Rightarrow \quad y \log x = -2ze^{-z} - 2e^{-z} + c$$

$$y \log x = -2 \log x \, e^{-\log x} - 2e^{-\log x} + c$$

$$\Rightarrow \quad y \log x = -2 \log x. \frac{1}{x} - \frac{2}{x} + c \qquad \left[\because e^{-\log x} = e^{\log \frac{1}{x}} = \frac{1}{x}\right]$$

$$\Rightarrow \quad y \log x = -\frac{2}{x}(1 + \log x) + c$$

21. Given, $\cos y = x \cos (a + y)$

$$\therefore \quad x = \frac{\cos y}{\cos (a + y)}$$

Differentiating w.r.t. to y on both sides, we have

$$\frac{dx}{dy} = \frac{\cos (a + y) \times (-\sin y) - \cos y \times [-\sin (a + y)]}{\cos^2 (a + y)}$$

$$\Rightarrow \quad \cancel{dx = \cos y \sin (a + y) - \sin y \cos (a + y)}$$

$dy \cos^2(a + y)$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\sin(a+y-y)}{\cos^2(a+y)}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\sin a}{\cos^2(a+y)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

22. L.H.S. $\Delta = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$

Taking out a, b, c from C_1, C_2 and C_3

$$\Delta = abc \begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 - C_3$

$$\Delta = abc \begin{vmatrix} 0 & c & a+c \\ 2b & b & a \\ 2b & b+c & c \end{vmatrix}$$

Taking $2b$ from C_1

$$\Delta = 2ab^2c \begin{vmatrix} 0 & c & a+c \\ 1 & b & a \\ 1 & b+c & c \end{vmatrix}$$

Applying $R_2 \to R_2 - R_3$

$$\Delta = 2ab^2c \begin{vmatrix} 0 & c & a+c \\ 0 & -c & a-c \\ 1 & b+c & c \end{vmatrix}$$

Expanding by I column, we get

$$\Delta = 2ab^2c \cdot 1 \cdot \begin{vmatrix} c & a+c \\ -c & a-c \end{vmatrix}$$

$$= 2ab^2c(ac - c^2 + ac + c^2)$$

$$\Delta = 2ab^2c(2ac) = 4a^2b^2c^2 = \text{R.H.S.}$$

28. Let side of square be a units and radius of a circle be r units.

It is given,

$\therefore \quad 4a + 2\pi r = k$ where k is a constant $\Rightarrow \quad r = \dfrac{k-4a}{2\pi}$

Sum of areas, $A = a^2 + \pi r^2$

$$\Rightarrow \quad A = a^2 + \pi\left[\frac{k-4a}{2\pi}\right]^2 = a^2 + \frac{1}{4\pi}(k-4a)^2$$

Differentiating w.r.t. x, we get

$$\frac{dA}{da} = 2a + \frac{1}{4\pi}\cdot 2(k-4a).(-4) = 2a - \frac{2(k-4a)}{\pi} \qquad \ldots(i)$$

For minimum area, $\frac{dA}{da} = 0$

$$\Rightarrow \quad 2a - \frac{2(k-4a)}{\pi} = 0$$

$$\Rightarrow \quad 2a = \frac{2(k-4a)}{\pi}$$

$$\Rightarrow \quad 2a = \frac{2(2\pi r)}{\pi} \qquad [\text{As } k = 4a + 2\pi r \text{ given}]$$

$$\Rightarrow \quad a = 2r$$

Now, again differentiating equation (i) w.r.t. x

$$\frac{d^2A}{da^2} = 2 - \frac{2}{\pi}(-4) = 2 + \frac{8}{\pi}$$

at $a = 2\pi$, $\frac{d^2A}{da^2} = 2 + \frac{8}{\pi} > 0$

$\therefore$ For $a = 2r$, sum of areas is least.

Hence, sum of areas is least when side of the square is double the radius of the circle.

29. Let $I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\sin 2x}\,dx$

Here, we express denominator in terms of $\sin x - \cos x$ which is integral of the numerator.

We have, $(\sin x - \cos x)^2 = \sin^2 x + \cos^2 x - 2\sin x\cos x = 1 - \sin 2x$

$$\Rightarrow \quad \sin 2x = 1 - (\sin x - \cos x)^2$$

$$\therefore \quad I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\{1 - (\sin x - \cos x)^2\}}\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4} \frac{\sin x + \cos x}{25 - 16(\sin x - \cos x)^2}\,dx$$

Let $\sin x - \cos x = t$. Then,

$$d(\sin x - \cos x) = dt$$

$$\Rightarrow \quad (\cos x + \sin x)\,dx = dt$$

Also, $x = 0 \Rightarrow t = \sin 0 - \cos 0 = -1$ and $x = \frac{\pi}{4}$

$$\Rightarrow \quad t = \sin\frac{\pi}{4} - \cos\frac{\pi}{4} = 0$$

$$\therefore \quad I = \int_{-1}^{0} \frac{dt}{25 - 16t^2}$$

$$\Rightarrow \quad I = \frac{1}{16}\int_{-1}^{0} \frac{dt}{\frac{25}{16} - t^2} = \frac{1}{16}\int_{-1}^{0} \frac{dt}{\left(\frac{5}{4}\right)^2 - t^2}$$

$$\Rightarrow \quad I = \frac{1}{16} \cdot \frac{1}{2\left(\frac{5}{4}\right)} \left[\log \left|\frac{\frac{5}{4} + t}{\frac{5}{4} - t}\right|\right]_{-1}^{0}$$

$$\Rightarrow \quad I = \frac{1}{40}\left[\log 1 - \log\left(\frac{1/4}{9/4}\right)\right]$$

$$\Rightarrow \quad I = \frac{1}{40}\left[0 - \log\left(\frac{1}{9}\right)\right] = \frac{1}{40}\log 9$$

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