Testprep Kart
Knowledge... Everywhere

Mathematics

Progression

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1. Introduction.

(1) **Sequence:** A sequence is a function whose domain is the set of natural numbers, N.

If f : N $\rightarrow$ C is a sequence, we usually denote it by $< f(n) > = < f(1), f(2), f(3), ... >$

It is not necessary that the terms of a sequence always follow a certain pattern or they are described by some explicit formula for the nth term. Terms of a sequence are connected by commas. Example: 1, 1, 2, 3, 5, 8, ... is a sequence.

(2) **Series:** By adding or subtracting the terms of a sequence, we get a series.

If $t_1, t_2, t_3, ... t_n, ...$ is a sequence, then the expression $t_1 + t_2 + t_3 + ... + t_n ....$ is a series.

A series is finite or infinite as the number of terms in the corresponding sequence is finite or infinite.

Example: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ...$ is a series.

(3) **Progression:** A progression is a sequence whose terms follow a certain pattern i.e. the terms are arranged under a definite rule.

Example: 1, 3, 5, 7, 9, ... is a progression whose terms are obtained by the rule : $T_n = 2n - 1$, where T_n denotes the nth term of the progression.

Progression is mainly of three types: Arithmetic progression, Geometric progression and Harmonic progression.

However, here we have classified the study of progression into five parts as:

- Arithmetic progression
- Geometric progression
- Arithmetico-geometric progression
- Harmonic progression
- Miscellaneous progressions

Arithmetic Progression (A.P)

2. Definition.

A sequence of numbers $< t_n >$ is said to be in arithmetic progression (A.P.) when the difference $t_n - t_{n-1}$ is a constant for all n $\in$ N. This constant is called the common difference of the A.P., and is usually denoted by the letter d.

If 'a' is the first term and 'd' the common difference, then an A.P. can be represented as $a, a+d, a+2d, a+3d, ...$

Example: 2, 7, 12, 17, 22, ... is an A.P. whose first term is 2 and common difference 5.

Algorithm to determine whether a sequence is an A.P. or not.

Step I: Obtain a_n (the nth term of the sequence).

Step II: Replace n by n – 1 in a_n to get a_{n-1}.

Step III: Calculate $a_n - a_{n-1}$.

If $a_n - a_{n-1}$ is independent of n, the given sequence is an A.P. otherwise it is not an A.P. An arithmetic progression is a linear function with domain as the set of natural numbers N.

$\therefore\ t_n = An + B$ represents the nth term of an A.P. with common difference A.

3. General Term of an A.P.

(1) Let 'a' be the first term and 'd' be the common difference of an A.P. Then its nth term is $a+(n-1)d$.

$$\boxed{T_n = a+(n-1)d}$$

(2) **pth term of an A.P. from the end :** Let 'a' be the first term and 'd' be the common difference of an A.P. having n terms. Then pth term from the end is $(n-p+1)^{th}$ term from the beginning.

$$\boxed{p^{th} \text{ term from the end} = T_{(n-p+1)} = a+(n-p)d}$$

Important Tips

- General term (T_n) is also denoted by l (last term).
- Common difference can be zero, +ve or –ve.
- n (number of terms) always belongs to set of natural numbers.
- If T_k and T_p of any A.P. are given, then formula for obtaining T_n is $\frac{T_n - T_k}{n-k} = \frac{T_p - T_k}{p-k}$.
- If $pT_p = qT_q$ of an A.P., then $T_{p+q} = 0$.
- If p^{th} term of an A.P. is q and the q^{th} term is p, then $T_{p+q} = 0$ and $T_n = p + q - n$.
- If the p^{th} term of an A.P. is $\frac{1}{q}$ and the q^{th} term is $\frac{1}{p}$, then its pq^{th} term is 1.
- If $T_n = pn + q$, then it will form an A.P. of common difference p and first term p + q.

4. Selection of Terms in an A.P.

When the sum is given, the following way is adopted in selecting certain number of terms:

Number of terms	Terms to be taken
3	a – d, a, a + d
4	a – 3d, a – d, a + d, a + 3d
5	a – 2d, a – d, a, a + d, a + 2d

In general, we take a – rd, a – (r – 1)d, ……., a – d, a, a + d, ……, a + (r – 1)d, a + rd, in case we have to take (2r + 1) terms (i.e. odd number of terms) in an A.P.

And, $a-(2r-1)d, a-(2r-3)d, ..., a-d, a+d, ..., a+(2r-1)d$, in case we have to take 2r terms in an A.P.

When the sum is not given, then the following way is adopted in selection of terms.

Number of terms	Terms to be taken
3	$a, a+d, a+2d$
4	$a, a+d, a+2d, a+3d$
5	$a, a+d, a+2d, a+3d, a+4d$

Sum of n terms of an A.P. : The sum of n terms of the series $a+(a+d)+(a+2d)+.......+\{a+(n-1)d\}$ is given by

$$\boxed{S_n = \frac{n}{2}[2a+(n-1)d]}$$

Also, $S_n = \frac{n}{2}(a+l)$, where l = last term = $a+(n-1)d$

Important Tips

- The common difference of an A.P is given by $d = S_2 - 2S_1$ where S_2 is the sum of first two terms and S_1 is the sum of first term or the first term.
- The sum of infinite terms $= \begin{cases} \infty, & \text{when } d>0 \\ -\infty, & \text{when } d<0 \end{cases}$.
- If sum of n terms S_n is given then general term $T_n = S_n - S_{n-1}$, where S_{n-1} is sum of (n – 1) terms of A.P.
- Sum of n terms of an A.P. is of the form $An^2 + Bn$ i.e. a quadratic expression in n, in such case, common difference is twice the coefficient of n^2 i.e. 2A.
- • If for the different A.P's $\frac{S_n}{S'_n} = \frac{f_n}{\phi_n}$, then $\frac{T_n}{T'_n} = \frac{f(2n-1)}{\phi(2n-1)}$
  - If for two A.P.'s $\frac{T_n}{T'_n} = \frac{An+B}{Cn+D}$ then $\frac{S_n}{S'_n} = \frac{A\left(\frac{n+1}{2}\right)+B}{C\left(\frac{n+1}{2}\right)+D}$
- Some standard results
  - Sum of first n natural numbers $= 1+2+3+........ +n = \sum_{r=1}^{n} r = \frac{n(n+1)}{2}$
  - Sum of first n odd natural numbers $= 1+3+5+..... +(2n-1) = \sum_{r=1}^{n}(2r-1) = n^2$
  - Sum of first n even natural numbers $= 2+4+6+...... +2n = \sum_{r=1}^{n} 2r = n(n+1)$
- • If for an A.P. sum of p terms is q and sum of q terms is p, then sum of (p + q) terms is {–(p + q)}.
- If for an A.P., sum of p terms is equal to sum of q terms, then sum of (p + q) terms is zero.
- If the p^th term of an A.P. is $\frac{1}{q}$ and q^th term is $\frac{1}{p}$, then sum of pq terms is given by $S_{pq} = \frac{1}{2}(pq+1)$

5. Arithmetic Mean.

(1) **Definitions**

(i) If three quantities are in A.P. then the middle quantity is called Arithmetic mean (A.M.) between the other two.

If a, A, b are in A.P., then A is called A.M. between a and b.

(ii) If $a, A_1, A_2, A_3, ..., A_n, b$ are in A.P., then $A_1, A_2, A_3, ..., A_n$ are called n A.M.'s between a and b.

(2) **Insertion of arithmetic means**

(i) **Single A.M. between a and b:** If a and b are two real numbers then single A.M. between a and b $= \frac{a+b}{2}$

(ii) **n A.M.'s between a and b :** If $A_1, A_2, A_3, ..., A_n$ are n A.M.'s between a and b, then

$$A_1 = a + d = a + \frac{b-a}{n+1},\ A_2 = a + 2d = a + 2\frac{b-a}{n+1},\ A_3 = a + 3d = a + 3\frac{b-a}{n+1},\ ....,\ A_n = a + nd = a + n\frac{b-a}{n+1}$$

Important Tips

- Sum of n A.M.'s between a and b is equal to n times the single A.M. between a and b.

i.e. $A_1 + A_2 + A_3 + ...\ + A_n = n\left(\frac{a+b}{2}\right)$

- If A_1 and A_2 are two A.M.'s between two numbers a and b, then $A_1 = \frac{1}{3}(2a+b), A_2 = \frac{1}{3}(a+2b)$.

- Between two numbers, $\frac{\text{Sum of } m \text{ A.M.'s}}{\text{Sum of } n \text{ A.M.'s}} = \frac{m}{n}$.

- If number of terms in any series is odd, then only one middle term exists which is $\left(\frac{n+1}{2}\right)^{th}$ term.

- If number of terms in any series is even then there are two middle terms, which are given by $\left(\frac{n}{2}\right)^{th}$ and $\left\{\left(\frac{n}{2}\right)+1\right\}^{th}$ term.

6. Properties of A.P.

(1) If $a_1, a_2, a_3.....$ are in A.P. whose common difference is d, then for fixed non-zero number K $\in$ R.

(i) $a_1 \pm K, a_2 \pm K, a_3 \pm K,.....$ will be in A.P., whose common difference will be d.

(ii) $Ka_1, Ka_2, Ka_3........$ will be in A.P. with common difference = Kd.

(iii) $\frac{a_1}{K}, \frac{a_2}{K}, \frac{a_3}{K}......$ will be in A.P. with common difference = d/K.

(2) The sum of terms of an A.P. equidistant from the beginning and the end is constant and is equal to sum of first and last term. i.e. $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = ...$

(3) Any term (except the first term) of an A.P. is equal to half of the sum of terms equidistant from the term i.e. $a_n = \frac{1}{2}(a_{n-k} + a_{n+k})$, k < n.

(4) If number of terms of any A.P. is odd, then sum of the terms is equal to product of middle term and number of terms.

(5) If number of terms of any A.P. is even then A.M. of middle two terms is A.M. of first and last term.

(6) If the number of terms of an A.P. is odd then its middle term is A.M. of first and last term.

(7) If $a_1, a_2, ...a_n$ and $b_1, b_2, ...b_n$ are the two A.P.'s. Then $a_1 \pm b_1, a_2 \pm b_2, ...a_n \pm b_n$ are also A.P.'s with common difference $d_1 \neq d_2$, where d_1 and d_2 are the common difference of the given A.P.'s.

(8) Three numbers a, b, c are in A.P. iff $2b = a + c$.

(9) If T_n, T_{n+1} and T_{n+2} are three consecutive terms of an A.P., then $2T_{n+1} = T_n + T_{n+2}$.

(10) If the terms of an A.P. are chosen at regular intervals, then they form an A.P.

7. Definition.

A progression is called a G.P. if the ratio of its each term to its previous term is always constant. This constant ratio is called its common ratio and it is generally denoted by r.

Example: The sequence 4, 12, 36, 108, ... is a G.P., because $\frac{12}{4}=\frac{36}{12}=\frac{108}{36}=.....=3$, which is constant.

Clearly, this sequence is a G.P. with first term 4 and common ratio 3.

The sequence $\frac{1}{3},-\frac{1}{2},\frac{3}{4},-\frac{9}{8},....$ is a G.P. with first term $\frac{1}{3}$ and common ratio $\left(-\frac{1}{2}\right)\Big/\left(\frac{1}{3}\right)=-\frac{3}{2}$

8. General Term of a G.P.

(1) We know that, $a, ar, ar^2, ar^3, ...ar^{n-1}$ is a sequence of G.P.

Here, the first term is 'a' and the common ratio is 'r'.

The general term or n^th term of a G.P. is $\boxed{T_n = ar^{n-1}}$

It should be noted that,

$$\boxed{r=\frac{T_2}{T_1}=\frac{T_3}{T_2}=......}$$

(2) **p^th term from the end of a finite G.P. :** If G.P. consists of 'n' terms, p^th term from the end $=(n-p+1)^{th}$ term from the beginning $=ar^{n-p}$.

Also, the p^th term from the end of a G.P. with last term l and common ratio r is $l\left(\frac{1}{r}\right)^{n-1}$

Important Tips

- If a, b, c are in G.P. $\Rightarrow \frac{b}{a} = \frac{c}{b}$ or $\boxed{b^2 = ac}$
- If T_k and T_p of any G.P. are given, then formula for obtaining T_n is

$$\boxed{\left(\frac{T_n}{T_k}\right)^{\frac{1}{n-k}} = \left(\frac{T_p}{T_k}\right)^{\frac{1}{p-k}}}$$

- If a, b, c are in G.P. then

$\Rightarrow \frac{b}{a} = \frac{c}{b} \Rightarrow \frac{a+b}{a-b} = \frac{b+c}{b-c}$ or $\frac{a-b}{b-c} = \frac{a}{b}$ or $\frac{a+b}{b+c} = \frac{a}{b}$

- Let the first term of a G.P be positive, then if r > 1, then it is an increasing G.P., but if r is positive and less than 1, i.e. 0< r < 1, then it is a decreasing G.P.
- Let the first term of a G.P. be negative, then if r > 1, then it is a decreasing G.P., but if 0< r < 1, then it is an increasing G.P.
- If a, b, c, d,... are in G.P., then they are also in continued proportion i.e. $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = ... = \frac{1}{r}$

9. Sum of First 'n' Terms of a G.P.

If a be the first term, r the common ratio, then sum S_n of first n terms of a G.P. is given by

$S_n = \frac{a(1-r^n)}{1-r}$, |r|< 1

$S_n = \frac{a(r^n - 1)}{r-1}$, |r|> 1

$S_n = na$, r = 1

10. Selection of Terms in a G.P.

(1) When the product is given, the following way is adopted in selecting certain number of terms:

Number of terms	Terms to be taken
3	$\frac{a}{r}, a, ar$
4	$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$
5	$\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

(2) When the product is not given, then the following way is adopted in selection of terms

Number of terms	Terms to be taken
3	a, ar, ar^2
4	a, ar, ar^2, ar^3
5	a, ar, ar^2, ar^3, ar^4

11. Sum of Infinite Terms of a G.P.

(1) When |r|< 1, (or $-1 < r < 1$)

$$S_\infty = \frac{a}{1-r}$$

(2) If r ≥ 1, then S_∞ doesn't exist

12. Geometric Mean.

(1) **Definition:** (i) If three quantities are in G.P., then the middle quantity is called geometric mean (G.M.) between the other two. If a, G, b are in G.P., then G is called G.M. between a and b.
(ii) If $a, G_1, G_2, G_3, ... G_n, b$ are in G.P. then $G_1, G_2, G_3, ... G_n$ are called n G.M.'s between a and b.

(2) **Insertion of geometric means:** (i) **Single G.M. between a and b:** If a and b are two real numbers then single G.M. between a and b $= \sqrt{ab}$
(ii) **n G.M.'s between a and b :** If $G_1, G_2, G_3, ..., G_n$ are n G.M.'s between a and b, then

$$G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}},\ G_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}},\ G_3 = ar^3 = a\left(\frac{b}{a}\right)^{\frac{3}{n+1}},\ ....,\ G_n = ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Important Tips

- Product of n G.M.'s between a and b is equal to nth power of single geometric mean between a and b.
  i.e. $G_1 G_2 G_3 ...... G_n = (\sqrt{ab})^n$
- G.M. of $a_1 a_2 a_3 ...... a_n$ is $(a_1 a_2 a_3 ..... a_n)^{1/n}$
- If G_1 and G_2 are two G.M.'s between two numbers a and b is $G_1 = (a^2 b)^{1/3}, G_2 = (ab^2)^{1/3}$.
- The product of n geometric means between a and $\frac{1}{a}$ is 1.
- If n G.M.'s inserted between a and b then $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

13. Properties of G.P.

(1) If all the terms of a G.P. be multiplied or divided by the same non-zero constant, then it remains a G.P., with the same common ratio.

(2) The reciprocal of the terms of a given G.P. form a G.P. with common ratio as reciprocal of the common ratio of the original G.P.

(3) If each term of a G.P. with common ratio r be raised to the same power k, the resulting sequence also forms a G.P. with common ratio r^k.

(4) In a finite G.P., the product of terms equidistant from the beginning and the end is always the same and is equal to the product of the first and last term.
i.e., if $a_1, a_2, a_3, ... a_n$ be in G.P. Then $a_1\, a_n = a_2\, a_{n-1} = a_3\, a_{n-2} = a_n\, a_{n-3} = ... = a_r \,. a_{n-r+1}$

(5) If the terms of a given G.P. are chosen at regular intervals, then the new sequence so formed also forms a G.P.

(6) If $a_1, a_2, a_3, ..., a_n ......$ is a G.P. of non-zero, non-negative terms, then $\log a_1, \log a_2, \log a_3, ... \log a_n, ...$ is an A.P. and vice-versa.

(7) Three non-zero numbers a, b, c are in G.P. iff $b^2 = ac$.

(8) Every term (except first term) of a G.P. is the square root of terms equidistant from it.
i.e. $T_r = \sqrt{T_{r-p} \cdot T_{r+p}}$; [r > p]

(9) If first term of a G.P. of n terms is a and last term is l, then the product of all terms of the G.P. is $(al)^{n/2}$.

(10) If there be n quantities in G.P. whose common ratio is r and S_m denotes the sum of the first m terms, then the sum of their product taken two by two is $\frac{r}{r+1} S_n\, S_{n-1}$.

14. Definition.

A progression is called a harmonic progression (H.P.) if the reciprocals of its terms are in A.P.

Standard form: $\frac{1}{a}+\frac{1}{a+d}+\frac{1}{a+2d}+....$

Example: The sequence $1,\frac{1}{3},\frac{1}{5},\frac{1}{7},\frac{1}{9},....$ is a H.P., because the sequence 1, 3, 5, 7, 9, ... is an A.P.

15. General Term of an H.P.

If the H.P. be as $\frac{1}{a},\frac{1}{a+d},\frac{1}{a+2d},....$ then corresponding A.P. is $a, a+d, a+2d,.....$

T_n of A.P. is $a+(n-1)d$

$\therefore T_n$ of H.P. is $\frac{1}{a+(n-1)d}$

In order to solve the question on H.P., we should form the corresponding A.P.

Thus, General term: $\boxed{T_n = \frac{1}{a+(n-1)d}}$ or $\boxed{T_n \text{ of H.P.} = \frac{1}{T_n \text{ of A.P.}}}$

16. Harmonic Mean.

(1) **Definition:** If three or more numbers are in H.P., then the numbers lying between the first and last are called harmonic means (H.M.'s) between them. For example 1, 1/3, 1/5, 1/7, 1/9 are in H.P. So 1/3, 1/5 and 1/7 are three H.M.'s between 1 and 1/9.
Also, if a, H, b are in H.P., then H is called harmonic mean between a and b.

(2) **Insertion of harmonic means:**

(i) Single H.M. between a and b $= \frac{2ab}{a+b}$

(ii) H, H.M. of n non-zero numbers $a_1, a_2, a_3, ..., a_n$ is given by $\frac{1}{H} = \frac{\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}}{n}$.

(iii) Let a, b be two given numbers. If n numbers $H_1, H_2, ... H_n$ are inserted between a and b such that the sequence $a, H_1, H_2, H_3 ...... H_n, b$ is an H.P., then $H_1, H_2, ... H_n$ are called n harmonic means between a and b.

Now, $a, H_1, H_2, ... H_n, b$ are in H.P. $\Rightarrow \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, ... \frac{1}{H_n}, \frac{1}{b}$ are in A.P.

Let D be the common difference of this A.P. Then,

$$\frac{1}{b} = (n+2)^{th} \text{ term} = T_{n+2}$$

$$\frac{1}{b} = \frac{1}{a} + (n+1)D \Rightarrow D = \frac{a-b}{(n+1)ab}$$

Thus, if n harmonic means are inserted between two given numbers a and b, then the common difference of the corresponding A.P. is given by $D = \frac{a-b}{(n+1)ab}$

Also, $\frac{1}{H_1} = \frac{1}{a} + D$, $\frac{1}{H_2} = \frac{1}{a} + 2D$,......., $\frac{1}{H_n} = \frac{1}{a} + nD$ where $D = \frac{a-b}{(n+1)ab}$

Important Tips

- If a, b, c are in H.P. then $b = \frac{2ac}{a+c}$.
- If H_1 and H_2 are two H.M.'s between a and b, then $H_1 = \frac{3ab}{a+2b}$ and $H_2 = \frac{3ab}{2a+b}$

17. Properties of H.P.

(1) No term of H.P. can be zero.

(2) If a, b, c are in H.P., then $\frac{a-b}{b-c}=\frac{a}{c}$.

(3) If H is the H.M. between a and b, then

(i) $\frac{1}{H-a}+\frac{1}{H-b}=\frac{1}{a}+\frac{1}{b}$ (ii) $(H-2a)(H-2b)=H^2$ (iii) $\frac{H+a}{H-a}+\frac{H+b}{H-b}=2$

Arithmetico-Geometric Progression (A.G.P.)

18. n^{th} Term of A.G.P.

If $a_1, a_2, a_3, ..., a_n, ...$ is an A.P. and $b_1, b_2, ..., b_n, ...$ is a G.P., then the sequence $a_1b_1, a_2b_2, a_3b_3,$ $......, a_nb_n, ...$ is said to be an arithmetico-geometric sequence.

Thus, the general form of an arithmetico geometric sequence is $a, (a+d)r, (a+2d)r^2, (a+3d)r^3, ...$

From the symmetry we obtain that the nth term of this sequence is $[a+(n-1)d]r^{n-1}$

Also, let $a, (a+d)r, (a+2d)r^2, (a+3d)r^3, ...$ be an arithmetico-geometric sequence. Then, $a+(a+d)r$ $+(a+2d)r^2+(a+3d)r^3+...$ is an arithmetico-geometric series.

19. Sum of A.G.P.

(1) **Sum of n terms:** The sum of n terms of an arithmetico-geometric sequence $a,(a+d)r,(a+2d)r^2,$ $(a+3d)r^3,.....$ is given by

$$S_n=\begin{cases}\dfrac{a}{1-r}+dr\dfrac{(1-r^{n-1})}{(1-r)^2}-\dfrac{\{a+(n-1)d\}r^n}{1-r}, \text{when } r\neq 1\\ \dfrac{n}{2}[2a+(n-1)d], \text{when } r=1\end{cases}$$

(2) **Sum of infinite sequence:** Let |r|< 1. Then $r^n, r^{n-1}\to 0$ as n $\to\infty$ and it can also be shown that $n.r^n\to 0$ as n $\to\infty$. So, we obtain that $S_n\to\dfrac{a}{1-r}+\dfrac{dr}{(1-r)^2}$, as n $\to\infty$.

In other words, when |r|< 1 the sum to infinity of an arithmetico-geometric series is $\boxed{S_\infty=\dfrac{a}{1-r}+\dfrac{dr}{(1-r)^2}}$

20. Method for Finding Sum.

This method is applicable for both sum of n terms and sum of infinite number of terms.

First suppose that sum of the series is S, then multiply it by common ratio of the G.P. and subtract. In this way, we shall get a G.P., whose sum can be easily obtained.

21. Method of Difference.

If the differences of the successive terms of a series are in A.P. or G.P., we can find nth term of the series by the following steps :

Step I: Denote the nth term by T_n and the sum of the series upto n terms by S_n.

Step II: Rewrite the given series with each term shifted by one place to the right.

Step III: By subtracting the later series from the former, find T_n.

Step IV: From T_n, S_n can be found by appropriate summation.

Miscellaneous series

22. Special Series.

There are some series in which n[th] term can be predicted easily just by looking at the series.

If $T_n = \alpha n^3 + \beta n^2 + \gamma n + \delta$

Then $S_n = \sum_{n=1}^{n} T_n = \sum_{n=1}^{n} (\alpha n^3 + \beta n^2 + \gamma n + \delta) = \alpha \sum_{n=1}^{n} n^3 + \beta \sum_{n=1}^{n} n^2 + \gamma \sum_{n=1}^{n} n + \delta \sum_{n=1}^{n} 1$

$$= \alpha \left(\frac{n(n+1)}{2} \right)^2 + \beta \left(\frac{n(n+1)(2n+1)}{6} \right) + \gamma \left(\frac{n(n+1)}{2} \right) + \delta n$$

Note: Sum of squares of first n natural numbers $= 1^2 + 2^2 + 3^2 + ... + n^2 = \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

- Sum of cubes of first n natural numbers $= 1^3 + 2^3 + 3^3 + 4^3 + ... + n^3 = \sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2} \right]^2$

23. V_n Method.

(1) To find the sum of the series $\frac{1}{a_1 a_2 a_3 .....a_r} + \frac{1}{a_2 a_3 .....a_{r+1}} + ... + \frac{1}{a_n a_{n+1} .....a_{n+r-1}}$

Let d be the common difference of A.P. Then $a_n = a_1 + (n-1)d$.

Let S_n and T_n denote the sum to n terms of the series and n[th] term respectively.

$$S_n = \frac{1}{a_1 a_2 .....a_r} + \frac{1}{a_2 a_3 .....a_{r+1}} + ... + \frac{1}{a_n a_{n+1} .....a_{n+r-1}}$$

$$\therefore \; T_n = \frac{1}{a_n a_{n+1} .....a_{n+r-1}}$$

Let $V_n = \frac{1}{a_{n+1} a_{n+2} .....a_{n+r-1}}$; $\quad V_{n-1} = \frac{1}{a_n a_{n+1} .....a_{n+r-2}}$

$$\Rightarrow V_n - V_{n-1} = \frac{1}{a_{n+1} a_{n+2} .....a_{n+r-1}} - \frac{1}{a_n a_{n+1} .....a_{n+r-2}} = \frac{a_n - a_{n+r-1}}{a_n a_{n+1} .....a_{n+r-1}}$$

$$= \frac{[a_1 + (n-1)d] - [a_1 + \{(n+r-1)-1\}d]}{a_n a_{n+1} .....a_{n+r-1}} = d(1-r)T_n$$

$\therefore\ T_n = \frac{1}{d(r-1)}\{V_{n-1} - V_n\}, \quad \therefore\ S_n = \sum_{n=1}^{n} T_n = \frac{1}{d(r-1)}(V_0 - V_n)$

$S_n = \frac{1}{(r-1)(a_2 - a_1)}\left\{\frac{1}{a_1 a_2 \ldots . a_{r-1}} - \frac{1}{a_{n+1} a_{n+2} \ldots \ldots a_{n+r-1}}\right\}$

Example: If $a_1, a_2, \ldots\ldots a_n$ are in A.P., then $\frac{1}{a_1 a_2 a_3} + \frac{1}{a_2 a_3 a_4} + \ldots + \frac{1}{a_n a_{n+1} a_{n+2}} = \frac{1}{2(a_2 - a_1)}\left\{\frac{1}{a_1 a_2} - \frac{1}{a_{n+1} a_{n+2}}\right\}$

(2) If $S_n = a_1 a_2 \ldots\ldots a_r + a_2 a_3 \ldots\ldots a_{r+1} \ldots . + a_n a_{n+1} \ldots a_{n+r-1}$

$T_n = a_n a_{n+1} \ldots\ldots a_{n+r-1}$

Let $V_n = a_n a_{n+1} \ldots . a_{n+r-1} a_{n+r}$, $\therefore\ V_{n-1} = a_{n-1} a_{n+1} \ldots\ldots a_{n+r-1}$

$\Rightarrow V_n - V_{n-1} = a_n a_{n+1} a_{n+2} \ldots\ldots a_{n+r-1}(a_{n+r} - a_{n-1}) = T_n\{[a_1 + (n+r-1)d] - [a_1 + (n-2)d]\} = T_n (r+1)d$

$\therefore\ T_n = \frac{V_n - V_{n-1}}{(r+1)d}$

$S_n = \sum_{n=1}^{n} T_n = \frac{1}{(r+1)d}\sum_{n=1}^{n}(V_n - V_{n-1}) = \frac{1}{(r+1)d}(V_n - V_0) = \frac{1}{(r+1)d}\{(a_n a_{n+1} \ldots . a_{n+r}) - (a_0 a_1 \ldots . a_r)\}$

$= \frac{1}{(r+1)(a_2 - a_1)}\{a_n a_{n+1} \ldots . a_{n+r} - a_0 a_1 \ldots\ldots a_r\}$

Example: $1.2.3.4 + 2.3.4.5 + \ldots\ldots + n(n+1)(n+2)(n+3) = \frac{1}{5.1}\{n(n+1)(n+2)(n+3) - 0.1.2.3\}$

$= \frac{1}{5}\{n(n+1)(n+2)(n+3)\}$

24. Properties of Arithmetic, Geometric and Harmonic means between Two given Numbers

Let A, G and H be arithmetic, geometric and harmonic means of two numbers a and b. Then,

$A = \frac{a+b}{2}, G = \sqrt{ab}$ and $H = \frac{2ab}{a+b}$

These three means possess the following properties:

(1) $A \ge G \ge H$

$A = \frac{a+b}{2}, G = \sqrt{ab}$ and $H = \frac{2ab}{a+b}$

$\therefore\ A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{(\sqrt{a} - \sqrt{b})^2}{2} \ge 0$

$\Rightarrow A \geq G$ .....(i)

$$G - H = \sqrt{ab} - \frac{2ab}{a+b} = \sqrt{ab}\left(\frac{a+b-2\sqrt{ab}}{a+b}\right) = \frac{\sqrt{ab}}{a+b}(\sqrt{a}-\sqrt{b})^2 \geq 0$$

$\Rightarrow G \geq H$ .....(ii)

From (i) and (ii), we get $A \geq G \geq H$

Note that the equality holds only when a = b

(2) A, G, H from a G.P., i.e. $G^2 = AH$

$$AH = \frac{a+b}{2} \times \frac{2ab}{a+b} = ab = (\sqrt{ab})^2 = G^2$$

Hence, $G^2 = AH$

(3) The equation having a and b as its roots is $x^2 - 2Ax + G^2 = 0$

The equation having a and b its roots is $x^2 - (a+b)x + ab = 0$

$\Rightarrow x^2 - 2Ax + G^2 = 0$ $\quad \left[\because A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}\right]$

The roots a, b are given by $A \pm \sqrt{A^2 - G^2}$

(4) If A, G, H are arithmetic, geometric and harmonic means between three given numbers a, b and c, then the equation having a, b, c as its roots is $x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0$

$A = \frac{a+b+c}{3}, G = (abc)^{1/3}$ and $\frac{1}{H} = \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}$

$\Rightarrow a+b+c = 3A, abc = G^3$ and $\frac{3G^3}{H} = ab+bc+ca$

The equation having a, b, c as its roots is $x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc = 0$

$$\Rightarrow x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0$$

25. Relation between A.P., G.P. and H.P.

(1) If A, G, H be A.M., G.M., H.M. between a and b, then $\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \begin{cases} A \text{ when } n=0 \\ G \text{ when } n=-1/2 \\ H \text{ when } n=-1 \end{cases}$

(2) If A_1, A_2 be two A.M.'s; G_1, G_2 be two G.M.'s and H_1, H_2 be two H.M.'s between two numbers a and b then $\frac{G_1G_2}{H_1H_2} = \frac{A_1+A_2}{H_1+H_2}$

(3) **Recognization of A.P., G.P., H.P.:** If a, b, c are three successive terms of a sequence.

Then if, $\frac{a-b}{b-c} = \frac{a}{a}$, then a, b, c are in A.P.

If, $\frac{a-b}{b-c} = \frac{a}{b}$, then a, b, c are in G.P.

If, $\frac{a-b}{b-c} = \frac{a}{c}$, then a, b, c are in H.P.

(4) If number of terms of any A.P./G.P./H.P. is odd, then A.M./G.M./H.M. of first and last terms is middle term of series.

(5) If number of terms of any A.P./G.P./H.P. is even, then A.M./G.M./H.M. of middle two terms is A.M./G.M./H.M. of first and last terms respectively.

(6) If p^{th}, q^{th} and r^{th} terms of a G.P. are in G.P. Then p, q, r are in A.P.

(7) If a, b, c are in A.P. as well as in G.P. then $a = b = c$.

(8) If a, b, c are in A.P., then x^a, x^b, x^c will be in G.P. $(x \neq \pm 1)$

26. Applications of Progressions.

There are many applications of progressions is applied in science and engineering. Properties of progressions are applied to solve problems of inequality and maximum or minimum values of some expression can be found by the relation among A.M., G.M. and H.M.