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Knowledge... Everywhere

Mathematics

Binomial Theorem

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1. Binomial Expression.

An algebraic expression consisting of two terms with $+ve$ or $-ve$ sign between them is called a binomial expression.

For example: $(a+b),(2x-3y),\left(\frac{p}{x^2}-\frac{q}{x^4}\right),\left(\frac{1}{x}+\frac{4}{y^3}\right)$ etc.

2. Binomial Theorem for Positive Integral Index.

The rule by which any power of binomial can be expanded is called the binomial theorem.
If n is a positive integer and $x, y \in C$ then

$$(x+y)^n = {}^nC_0 x^{n-0} y^0 + {}^nC_1 x^{n-1} y^1 + {}^nC_2\, x^{n-2} y^2 + \dots\dots + {}^nC_r x^{n-r} y^r + \dots\dots + {}^nC_{n-1} x y^{n-1} + {}^nC_n x^0 y^n$$

i.e., $(x+y)^n = \sum_{r=0}^{n} {}^nC_r . x^{n-r} . y^r$ .....(i)

Here ${}^nC_0, {}^nC_1, {}^nC_2, \dots\dots {}^nC_n$ are called binomial coefficients and ${}^nC_r = \frac{n!}{r!(n-r)!}$ for $0 \le r \le n$.

Important Tips

- *The number of terms in the expansion of* $(x+y)^n$ *are (n + 1).*
- *The expansion contains decreasing power of x and increasing power of y. The sum of the powers of x and y in each term is equal to n.*
- *The binomial coefficients* ${}^nC_0, {}^nC_1, {}^nC_2 \dots\dots$ *equidistant from beginning and end are equal i.e.,* ${}^nC_r = {}^nC_{n-r}$.
- $(x+y)^n =$ *Sum of odd terms + sum of even terms.*

3. Some Important Expansions.

(1) Replacing y by $-y$ in (i), we get,

$$(x-y)^n = {}^nC_0\, x^{n-0}.y^0 - {}^nC_1 x^{n-1}.y^1 + {}^nC_2 x^{n-2}.y^2 .... + (-1)^r\ {}^nC_r x^{n-r}.y^r + ... + (-1)^n\ {}^nC_n x^0.y^n$$

i.e., $(x-y)^n = \sum_{r=0}^{n} (-1)^r\ {}^nC_r x^{n-r}.y^r$ .....(ii)

The terms in the expansion of $(x-y)^n$ are alternatively positive and negative, the last term is positive or negative according as n is even or odd.

(2) Replacing x by 1 and y by x in equation (i) we get,

$(1+x)^n = {}^nC_0 x^0 + {}^nC_1 x^1 + {}^nC_2 x^2 + ... + {}^nC_r x^r + ... + {}^nC_n x^n$ *i.e.,* $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$

This is expansion of $(1+x)^n$ in ascending power of x.

(3) Replacing x by 1 and y by $-x$ in (i) we get,

$(1-x)^n = {}^nC_0 x^0 - {}^nC_1 x^1 + {}^nC_2 x^2 - ...... + (-1)^r\ {}^nC_r x^r + ... + (-1)^n\ {}^nC_n x^n$ *i.e.,*

$(1-x)^n = \sum_{r=0}^{n} (-1)^r\ {}^nC_r x^r$

(4) $(x+y)^n + (x-y)^n = 2[{}^nC_0 x^n y^0 + {}^nC_2 x^{n-2} y^2 + {}^nC_4 x^{n-4} y^4 + ...]$ and

$(x+y)^n - (x-y)^n = 2[{}^nC_1 x^{n-1} y^1 + {}^nC_3 x^{n-3} y^3 + {}^nC_5 x^{n-5} y^5 + ...]$

(5) The coefficient of $(r+1)^{th}$ term in the expansion of $(1+x)^n$ is ${}^nC_r$.

(6) The coefficient of x^r in the expansion of $(1+x)^n$ is ${}^nC_r$.

Note: If n is odd, then $(x+y)^n+(x-y)^n$ and $(x+y)^n-(x-y)^n$, both have the same number of terms equal to $\left(\frac{n+1}{2}\right)$.

If n is even, then $(x+y)^n+(x-y)^n$ has $\left(\frac{n}{2}+1\right)$ terms and $(x+y)^n-(x-y)^n$ has $\frac{n}{2}$ terms.

4. General Term.

$(x+y)^n = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + ... + {}^nC_r x^{n-r} y^r + ... + {}^nC_n x^0 y^n$

The first term = ${}^nC_0 x^n y^0$

The second term = ${}^nC_1 x^{n-1} y^1$. The third term = ${}^nC_2 x^{n-2} y^2$ and so on

The term ${}^nC_r x^{n-r} y^r$ is the $(r+1)^{th}$ term from beginning in the expansion of $(x+y)^n$.

Let T_{r+1} denote the (r + 1)th term $\therefore$ $T_{r+1} = {}^nC_r x^{n-r} y^r$

This is called general term, because by giving different values to r, we can determine all terms of the expansion.

In the binomial expansion of $(x-y)^n, T_{r+1} = (-1)^r\ {}^nC_r x^{n-r} y^r$

In the binomial expansion of $(1+x)^n, T_{r+1} = {}^nC_r x^r$

In the binomial expansion of $(1-x)^n, T_{r+1} = (-1)^r\ {}^nC_r x^r$

Note: In the binomial expansion of $(x+y)^n$, the pth term from the end is $(n-p+2)^{th}$ term from beginning.

Important Tips

- In the expansion of $(x+y)^n, n \in N$

$\frac{T_{r+1}}{T_r} = \left(\frac{n-r+1}{r}\right)\frac{y}{x}$

- The coefficient of x^{n-1} in the expansion of $(x-1)(x-2).......(x-n) = -\frac{n(n+1)}{2}$

- The coefficient of x^{n-1} in the expansion of $(x+1)(x+2)......(x+n) = \frac{n(n+1)}{2}$

5. Independent Term or Constant Term.

Independent term or constant term of a binomial expansion is the term in which exponent of the variable is zero.

Condition: $(n-r)$ [Power of x] + r. [Power of y] = 0, in the expansion of $[x+y]^n$.

6. Number of Terms in the Expansion of (a + b + c) ^{n} and (a + b + c + d) ^{n}.

$(a+b+c)^n$ can be expanded as : $(a+b+c)^n = \{(a+b)+c\}^n$

$$= (a+b)^n + {}^nC_1(a+b)^{n-1}(c)^1 + {}^nC_2(a+b)^{n-2}(c)^2 + ... + {}^nC_n c^n$$

$$= (n+1)\text{term} + n\,\text{term} + (n-1)\text{term} + ... + 1\text{term}$$

$\therefore$ Total number of terms $= (n+1)+(n)+(n-1)+ ...... +1 = \frac{(n+1)(n+2)}{2}$.

Similarly, Number of terms in the expansion of $(a+b+c+d)^n = \frac{(n+1)(n+2)(n+3)}{6}$.

7. Middle Term.

The middle term depends upon the value of n.

(1) **When n is even,** then total number of terms in the expansion of $(x+y)^n$ is $n+1$ (odd). So there is only one middle term i.e., $\left(\frac{n}{2}+1\right)^{\text{th}}$ term is the middle term. $T_{\left[\frac{n}{2}+1\right]} = {}^nC_{n/2}x^{n/2}y^{n/2}$

(2) **When n is odd**, then total number of terms in the expansion of $(x+y)^n$ is $n+1$ (even). So, there are two middle terms i.e., $\left(\frac{n+1}{2}\right)^{\text{th}}$ and $\left(\frac{n+3}{2}\right)^{\text{th}}$ are two middle terms. $T_{\left(\frac{n+1}{2}\right)} = {}^nC_{\frac{n-1}{2}}x^{\frac{n+1}{2}}y^{\frac{n-1}{2}}$ and

$$T_{\left(\frac{n+3}{2}\right)} = {}^nC_{\frac{n+1}{2}}x^{\frac{n-1}{2}}y^{\frac{n+1}{2}}$$

Note: When there are two middle terms in the expansion then their binomial coefficients are equal.

Binomial coefficient of middle term is the greatest binomial coefficient.

8. To Determine a Particular Term in the Expansion.

In the expansion of $\left(x^{\alpha} \pm \frac{1}{x^{\beta}}\right)^{n}$, if x^m occurs in T_{r+1}, then r is given by $n\alpha - r(\alpha+\beta) = m \Rightarrow r = \frac{n\alpha - m}{\alpha+\beta}$

Thus in above expansion if constant term which is independent of x, occurs in T_{r+1} then r is determined by

$$n\alpha - r(\alpha+\beta) = 0 \Rightarrow r = \frac{n\alpha}{\alpha+\beta}$$

9. Greatest Term and Greatest Coefficient.

(1) **Greatest term:** If T_r and T_{r+1} be the r^th and $(r+1)^{th}$ terms in the expansion of $(1+x)^n$, then

$$\frac{T_{r+1}}{T_r} = \frac{{}^nC_r x^r}{{}^nC_{r-1} x^{r-1}} = \frac{n-r+1}{r} x$$

Let numerically, T_{r+1} be the greatest term in the above expansion. Then $T_{r+1} \geq T_r$ or $\frac{T_{r+1}}{T_r} \geq 1$

$$\therefore \quad \frac{n-r+1}{r}|x| \geq 1 \quad \text{or} \quad r \leq \frac{(n+1)}{(1+|x|)}|x| \qquad ......(i)$$

Now substituting values of n and x in (i), we get $r \leq m+f$ or $r \leq m$

Where m is a positive integer and f is a fraction such that $0 < f < 1$.

When n is even T_{m+1} is the greatest term, when n is odd T_m and T_{m+1} are the greatest terms and both are equal.

Short cut method: To find the greatest term (numerically) in the expansion of $(1+x)^n$.

(i) Calculate m = $\left|\frac{x(n+1)}{x+1}\right|$

(ii) If m is integer, then T_m and T_{m+1} are equal and both are greatest term.

(iii) If m is not integer, there $T_{[m]+1}$ is the greatest term, where [.] denotes the greatest integral part.

(2) **Greatest coefficient**

(i) If n is even, then greatest coefficient is ${}^nC_{n/2}$

(ii) If n is odd, then greatest coefficient are ${}^nC_{\frac{n+1}{2}}$ and ${}^nC_{\frac{n+3}{2}}$

Important Tips

☞ For finding the greatest term in the expansion of $(x+y)^n$. we rewrite the expansion in this form $(x+y)^n = x^n\left[1+\frac{y}{x}\right]^n$.

Greatest term in (x + y)^n $= x^n$. Greatest term in $\left(1+\frac{y}{x}\right)^n$

10. Properties of Binomial Coefficients.

In the binomial expansion of $(1+x)^n$, $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + ... + {}^nC_r x^r + ... + {}^nC_n x^n$.

Where ${}^nC_0, {}^nC_1, {}^nC_2, ..., {}^nC_n$ are the coefficients of various powers of x and called binomial coefficients, and they are written as $C_0, C_1, C_2, ...C_n$.

Hence, $(1+x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_r x^r + ... + C_n x^n$ .....(i)

(1) The sum of binomial coefficients in the expansion of $(1+x)^n$ is 2^n.

Putting $x=1$ in (i), we get $2^n = C_0 + C_1 + C_2 + ... + C_n$ .....(ii)

(2) Sum of binomial coefficients with alternate signs : Putting $x=-1$ in (i)

We get, $0 = C_0 - C_1 + C_2 - C_3 + ...$ .....(iii)

(3) Sum of the coefficients of the odd terms in the expansion of $(1+x)^n$ is equal to sum of the coefficients of even terms and each is equal to 2^{n-1}.

From (iii), we have $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ...$ ......(iv)
i.e., sum of coefficients of even and odd terms are equal.

From (ii) and (iv), $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$ ......(v)

(4) ${}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1} = \frac{n}{r}.\frac{n-1}{r-1}\,{}^{n-2}C_{r-2}$ and so on.

(5) Sum of product of coefficients: Replacing x by $\frac{1}{x}$ in (i) we get $\left(1+\frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + ...\frac{C_n}{x^n} + ...$

(vi)

Multiplying (i) by (vi), we get $\frac{(1+x)^{2n}}{x^n} = (C_0 + C_1x + C_2x^2 + ...)\left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + ...\right)$

Now comparing coefficient of x^r on both sides.

We get, ${}^{2n}C_{n+r} = C_0C_r + C_1C_{r+1} + ...C_{n-r}.C_n$ ......(vii)

(6) Sum of squares of coefficients: Putting $r = 0$ in (vii), we get ${}^{2n}C_n = C_0^2 + C_1^2 + ...C_n^2$

(7) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

11. An Important Theorem.

If $(\sqrt{A}+B)^n = I+f$ where I and n are positive integers, n being odd and $0 \le f < 1$ then $(I+f).f = K^n$ where $A-B^2 = K > 0$ and $\sqrt{A}-B<1$.

Note: If n is even integer then $(\sqrt{A}+B)^n + (\sqrt{A}-B)^n = I+f+f'$

Hence L.H.S. and I are integers.

$\therefore$ $f+f'$ is also integer; $\Rightarrow f+f'=1$; $\therefore$ $f'=(1-f)$

Hence $(I+f)(1-f)=(I+f)f' = (\sqrt{A}+B)^n(\sqrt{A}-B)^n = (A-B^2)^n = K^n$.

12. Multinomial Theorem (For positive integral index).

If n is positive integer and $a_1, a_2, a_3, ...a_n \in C$ then

$$(a_1+a_2+a_3+...+a_m)^n = \sum \frac{n!}{n_1!n_2!n_3!...n_m!} a_1^{n_1} a_2^{n_2} ... a_m^{n_m}$$

Where $n_1, n_2, n_3, ...n_m$ are all non-negative integers subject to the condition, $n_1+n_2+n_3+.....n_m = n$.

(1) The coefficient of $a_1^{n_1}.a_2^{n_2}.....a_m^{n_m}$ in the expansion of $(a_1+a_2+a_3+....a_m)^n$ is $\frac{n!}{n_1!n_2!n_3!....n_m!}$

(2) The greatest coefficient in the expansion of $(a_1+a_2+a_3+....a_m)^n$ is $\frac{n!}{(q!)^{m-r}[(q+1)!]^r}$

Where q is the quotient and r is the remainder when n is divided by m.

(3) If n is +ve integer and $a_1, a_2, ...a_m \in C$, $a_1^{n_1}.a_2^{n_2} ....a_m^{n_m}$ then coefficient of x^r in the expansion of $(a_1+a_2x+.....a_mx^{m-1})^n$ is $\sum \frac{n!}{n_1!n_2!n_3!.....n_m!}$

Where $n_1, n_2.....n_m$ are all non-negative integers subject to the condition: $n_1+n_2+.....n_m = n$ and $n_2+2n_3+3n_4+....+(m-1)n_m = r$.

(4) The number of distinct or dissimilar terms in the multinomial expansion $(a_1+a_2+a_3+....a_m)^n$ is ${}^{n+m-1}C_{m-1}$.

13. Binomial Theorem for any Index.

Statement: $(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)}{3!}x^3 + ... + \frac{n(n-1)......(n-r+1)}{r!}x^r + ...\text{terms up to} \infty$

When n is a negative integer or a fraction, where $-1 < x < 1$, otherwise expansion will not be possible.

If $x < 1$, the terms of the above expansion go on decreasing and if x be very small a stage may be reached when we may neglect the terms containing higher power of x in the expansion, then $(1+x)^n = 1 + nx$.

Important Tips

- Expansion is valid only when $-1 < x < 1$.
- ${}^nC_r$ cannot be used because it is defined only for natural number, so ${}^nC_r$ will be written as $\frac{(n)(n-1)......(n-r+1)}{r!}$
- The number of terms in the series is infinite.
- If first term is not 1, then make first term unity in the following way, $(x+y)^n = x^n\left[1+\frac{y}{x}\right]^n$, if $\left|\frac{y}{x}\right| < 1$.

General term: $T_{r+1} = \frac{n(n-1)(n-2)......(n-r+1)}{r!}x^r$

Some important expansions:

(i) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ... + \frac{n(n-1)(n-2)......(n-r+1)}{r!}x^r + ...$

(ii) $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - .... + \frac{n(n-1)(n-2).....(n-r+1)}{r!}(-x)^r + ...$

(iii) $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + ... + \frac{n(n+1)......(n+r-1)}{r!}x^r + ...$

(iv) $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + ... + \frac{n(n+1)......(n+r-1)}{r!}(-x)^r + ...$

(a) **Replace n by 1 in (iii):** $(1-x)^{-1} = 1 + x + x^2 + ... + x^r + ...\infty$, General term, $T_{r+1} = x^r$

(b) **Replace n by 1 in (iv):** $(1+x)^{-1} = 1 - x + x^2 - x^3 + ... + (-x)^r + ...\infty$, General term, $T_{r+1} = (-x)^r$.

(c) **Replace n by 2 in (iii):** $(1-x)^{-2} = 1 + 2x + 3x^2 + ... + (r+1)x^r + ...\infty$, General term, $T_{r+1} = (r+1)x^r$.

(d) **Replace n by 2 in (iv):** $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ... + (r+1)(-x)^r + ...\infty$

General term, $T_{r+1} = (r+1)(-x)^r$.

(e) **Replace n by 3 in (iii):** $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + ... + \frac{(r+1)(r+2)}{2!} x^r + ...\infty$

General term, $T_{r+1} = (r+1)(r+2)/2!.\, x^r$

(f) **Replace n by 3 in (iv):** $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + ... + \frac{(r+1)(r+2)}{2!}(-x)^r + ...\infty$

General term, $T_{r+1} = \frac{(r+1)(r+2)}{2!}(-x)^r$

14. Three / Four Consecutive terms or Coefficients.

(1) **If consecutive coefficients are given:** In this case divide consecutive coefficients pair wise. We get equations and then solve them.

(2) **If consecutive terms are given :** In this case divide consecutive terms pair wise i.e. if four consecutive terms be $T_r, T_{r+1}, T_{r+2}, T_{r+3}$ then find $\frac{T_r}{T_{r+1}}, \frac{T_{r+1}}{T_{r+2}}, \frac{T_{r+2}}{T_{r+3}} \Rightarrow \lambda_1, \lambda_2, \lambda_3$ (say) then divide λ_1 by λ_2 and λ_2 by λ_3 and solve.

15. Some Important Points.

(1) **Pascal's Triangle:**

1	$(x+y)^0$
1 1	$(x+y)^1$
1 2 1	$(x+y)^2$
1 3 3 1	$(x+y)^3$
1 4 6 4 1	$(x+y)^4$
1 5 10 10 5 1	$(x+y)^5$

Pascal's triangle gives the direct binomial coefficients.

Example: $(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

(2) **Method for finding terms free from radical or rational terms in the expansion of $(a^{1/p} + b^{1/q})^N \forall a, b \in$ prime numbers:** Find the general term $T_{r+1} = {}^NC_r (a^{1/p})^{N-r} (b^{1/q})^r = {}^NC_r\, a^{\frac{N-r}{p}} . b^{\frac{r}{q}}$

Putting the values of $0 \le r \le N$, when indices of a and b are integers.

Note: Number of irrational terms = Total terms – Number of rational terms.

16. First Principle of Mathematical Induction.

The proof of proposition by mathematical induction consists of the following three steps:

Step I: (Verification step): Actual verification of the proposition for the starting value "i"

Step II: (Induction step)**:** Assuming the proposition to be true for "k", $k \ge i$ and proving that it is true for the value $(k+1)$ which is next higher integer.

Step III: (Generalization step)**:** To combine the above two steps

Let $p(n)$ be a statement involving the natural number n such that

(i) $p(1)$ is true *i.e.* $p(n)$ is true for $n = 1$.

(ii) $p(m+1)$ is true, whenever $p(m)$ is true *i.e.* $p(m)$ is true $\Rightarrow p(m+1)$ is true.

Then $p(n)$ is true for all natural numbers n.

17. Second Principle of Mathematical Induction.

The proof of proposition by mathematical induction consists of following steps:

Step I: (Verification step): Actual verification of the proposition for the starting value *i* and (*i* + 1).

Step II: (Induction step) : Assuming the proposition to be true for *k* – 1 and *k* and then proving that it is true for the value *k* + 1; $k \geq i + 1$.

Step III: (Generalization step): Combining the above two steps.

Let *p*(*n*) be a statement involving the natural number *n* such that

(i) *p*(1) is true *i.e.* *p*(*n*) is true for *n* = 1 and

(ii) *p*(*m* + 1) is true, whenever *p*(*n*) is true for all *n*, where $i \leq n \leq m$

Then *p*(*n*) is true for all natural numbers.

For $a \neq b$, The expression $a^n - b^n$ is divisible by

(a) *a* + *b* if *n* is even.

(b) *a* – *b* is *n* if odd or even.

18. Some Formulae based on Principle of Induction.

For any natural number *n*

(i) $\sum n = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$ (ii)

$\sum n^2 = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$

(iii) $\sum n^3 = 1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2(n+1)^2}{4} = \left(\sum n\right)^2$

19. Divisibility Problems.

To show that an expression is divisible by an integer

(i) If *a*, *p*, *n*, *r* are positive integers, then first of all we write $a^{pn+r} = a^{pn}.a^{r} = (a^{p})^{n}.a^{r}$.

(ii) If we have to show that the given expression is divisible by *c*.S

Then express, $a^{p} = [1+(a^{p}-1]$, if some power of $(a^{p}-1)$ has *c* as a factor.

$a^{p} = [2+(a^{p}-2)]$, if some power of $(a^{p}-2)$ has *c* as a factor.

$a^{p} = [K+(a^{p}-K)]$, if some power of $(a^{p}-K)$ has *c* as a factor.