Mathematics

Determinants

Table of Content

1. Definition.

(1) Consider two equations, $a_1x + b_1y = 0$ **.....**(i) and $a_2x + b_2y = 0$ **.....**(ii)

Multiplying (i) by b_2 and (ii) by b_1 and subtracting, dividing by *x*, we get, $a_1b_2 - a_2b_1 = 0$

The result $a_1b_2 - a_2b_1$ is represented by $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

Which is known as determinant of order two and $a_1b_2 - a_2b_1$ is the expansion of this determinant. The horizontal lines are called rows and vertical lines are called columns.

Now let us consider three homogeneous linear equations

$a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$ and $a_3x + b_3y + c_3z = 0$

Eliminated *x, y, z* from above three equations we obtain

$a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) = 0$ .....(iii)

The L.H.S. of (iii) is represented by $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

It contains three rows and three columns, it is called a determinant of third order.

Note: The number of elements in a second order is $2^2 = 4$ and the number of elements in a third order determinant is $3^2 = 9$.

(2) **Rows and columns of a determinant:** In a determinant horizontal lines counting from top 1st, 2nd, 3rd,..... respectively known as rows and denoted by R_1, R_2, R_3, ... and vertical lines counting left to right, 1st, 2nd, 3rd,..... respectively known as columns and denoted by C_1, C_2, C_3,.....

(3) **Shape and constituents of a determinant:** Shape of every determinant is square. If a determinant of n order then it contains n rows and n columns.

i.e., Number of constituents in determinants = n^2

(4) **Sign system for expansion of determinant:** Sign system for order 2, order 3, order 4, are given by

$\begin{vmatrix} + & - \\ - & + \end{vmatrix}$, $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$, $\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$,.....

2. Expansion of Determinants.

Unlike a matrix, determinant is not just a table of numerical data but (quite differently) a short hand way of writing algebraic expression, whose value can be computed when the values of terms or elements are known.

(1) The 4 numbers a_1, b_1, a_2, b_2 arranged as $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is a determinant of second order. These numbers are called elements of the determinant. The value of the determinant is defined as $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ $= a_1b_2 - a_2b_1$.

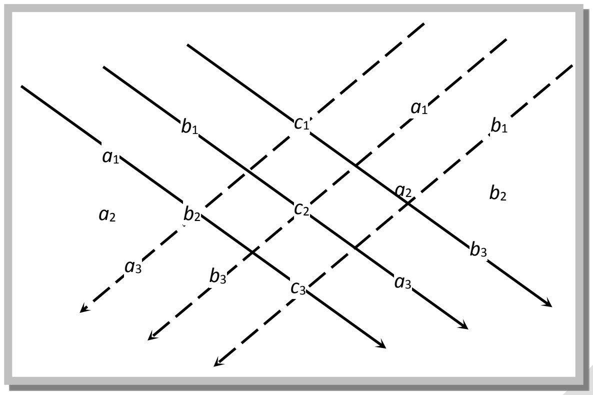
The expanded form of determinant has 2! terms.

(2) The 9 numbers $a_r, b_r, c_r\ (r = 1, 2, 3)$ arranged as $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is a determinant of third order. Take any row (or column); the value of the determinant is the sum of products of the elements of the row (or column) and the corresponding determinant obtained by omitting the row and the column of the element with a proper sign, given by the rule $(-1)^{i+j}$, where i and j are the number of rows and the number of columns respectively of the element of the row (or the column) chosen.

Thus $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

The diagonal through the left-hand top corner which contains the element a_1, b_2, c_3 is called the leading diagonal or principal diagonal and the terms are called the leading terms. The expanded form of determinant has 3! terms.

Short cut method or Sarrus diagram method: To find the value of third order determinant, following method is also useful



Taking product of R.H.S. diagonal elements positive and L.H.S. diagonal elements negative and adding them. We get the value of determinant as $= a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - c_1b_2a_3 - a_1c_2b_3 - b_1a_2c_3$

Note: This method does not work for determinants of order greater than three.

3. Evaluation of Determinants.

If A is a square matrix of order 2, then its determinant can be easily found. But to evaluate determinants of square matrices of higher orders, we should always try to introduce zeros at maximum number of places in a particular row (column) by using the properties and then we should expand the determinant along that row (column).

We shall be using the following notations to evaluate a determinant:

(1) R_i to denote i^{th} row.

(2) $R_i \leftrightarrow R_j$ to denote the interchange of i^{th} and j^{th} rows.

(3) $R_i \to R_i + \lambda R_j$ to denote the addition of λ times the elements of j^{th} row to the corresponding elements of i^{th} row.

(4) $R_i(\lambda)$ to denote the multiplication of all element of i^{th} row by λ.

Similar notations are used to denote column operations if R is replaced by C.

4. Properties of Determinants.

P-1: The value of determinant remains unchanged, if the rows and the columns are interchanged.

If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$. Then $D' = D$, D and D' are transpose of each other.

Note: Since the determinant remains unchanged when rows and columns are interchanged, it is obvious that any theorem which is true for 'rows' must also be true for 'columns'.

P-2: If any two rows (or columns) of a determinant be interchanged, the determinant is unaltered in numerical value but is changed in sign only.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$. Then $D' = -D$

P-3: If a determinant has two rows (or columns) identical, then its value is zero.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$. Then, D = 0

P-4: If all the elements of any row (or column) be multiplied by the same number, then the value of determinant is multiplied by that number.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. Then $D' = kD$

P-5: If each element of any row (or column) can be expressed as a sum of two terms, then the determinant can be expressed as the sum of the determinants.

e.g., $\begin{vmatrix} a_1 + x & b_1 + y & c_1 + z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

P-6: The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column)

e.g., $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} a_1 + ma_2 & b_1 + mb_2 & c_1 + mc_2 \\ a_2 & b_2 & c_2 \\ a_3 - na_1 & b_3 - nb_1 & c_3 - nc_1 \end{vmatrix}$. Then

$D' = D$

Note: It should be noted that while applying **P-6** at least one row (or column) must remain unchanged.

P-7 : If all elements below leading diagonal or above leading diagonal or except leading diagonal elements are zero then the value of the determinant equal to multiplied of all leading diagonal elements.

e.g., $\begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{vmatrix} = a_1 b_2 c_3$

P-8: If a determinant D becomes zero on putting $x = \alpha$, then we say that $(x - \alpha)$ is factor of determinant.

e.g., if $D = \begin{vmatrix} x & 5 & 2 \\ x^2 & 9 & 4 \\ x^3 & 16 & 8 \end{vmatrix}$. At $x = 2,\ D = 0$ (because C_1 and C_2 are identical at $x = 2$)

Hence $(x - 2)$ is a factor of D.

Note: It should be noted that while applying operations on determinants then at least one row (or column) must remain unchanged or Maximum number of operations = order or determinant –1

It should be noted that if the row (or column) which is changed by multiplied a non-zero number, then the determinant will be divided by that number.

5. Minors and Cofactors.

(1) **Minor of an element:** If we take the element of the determinant and delete (remove) the row and column containing that element, the determinant left is called the minor of that element. It is denoted by M_{ij}

Consider the determinant $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then determinant of minors $M = \begin{vmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{vmatrix}$,

Where M_{11} = minor of $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, M_{12} = minor of $a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

M_{13} = Minor of $a_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Similarly, we can find the minors of other elements. Using this concept the value of determinant can be
$\Delta = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$
or, $\Delta = -a_{21}M_{21} + a_{22}M_{22} - a_{23}M_{23}$ or, $\Delta = a_{31}M_{31} - a_{32}M_{32} + a_{33}M_{33}$.

(2) **Cofactor of an element:** The cofactor of an element a_{ij} (i.e. the element in the i^{th} row and j^{th} column) is defined as $(-1)^{i+j}$ times the minor of that element. It is denoted by C_{ij} or A_{ij} or F_{ij}.

$C_{ij} = (-1)^{i+j} M_{ij}$

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then determinant of cofactors is $C = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix}$, where

$C_{11} = (-1)^{1+1} M_{11} = +M_{11}$, $C_{12} = (-1)^{1+2} M_{12} = -M_{12}$ and $C_{13} = (-1)^{1+3} M_{13} = +M_{13}$
Similarly, we can find the cofactors of other elements.

Note: The sum of products of the element of any row with their corresponding cofactor is equal to the value of determinant i.e. $\Delta = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$

Where the capital letters C_{11}, C_{12}, C_{13} etc. denote the cofactors of a_{11}, a_{12}, a_{13} etc.

❑ In general, it should be noted

$a_{i1}C_{j1} + a_{i2}C_{j2} + a_{i3}C_{j3} = 0,$ if $i \neq j$ or $a_{1i}C_{1j} + a_{2i}C_{2j} + a_{3i}C_{3j} = 0,$ if $i \neq j$

❑ If Δ' is the determinant formed by replacing the elements of a determinant Δ by their corresponding cofactors, then if $\Delta = 0$, then $\Delta^C = 0$, $\Delta' = \Delta^{n-1}$, where n is the order of the determinant.

6. Product of two Determinants.

Let the two determinants of third order be,

$D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$. Let D be their product.

(1) **Method of multiplying (Row by row):** Take the first row of D_1 and the first row of D_2 i.e. a_1, b_1, c_1 and $\alpha_1, \beta_1, \gamma_1$ multiplying the corresponding elements and add. The result is $a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1$ is the first element of first row of D.

Now similar product first row of D_1 and second row of D_2 gives $a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2$ is the second element of first row of D, and the product of first row D_1 and third row of D_2 gives $a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3$ is the third element of first row of D. The second row and third row of D is obtained by multiplying second row and third row of D_1 with 1st, 2nd, 3rd row of D_2, in the above manner.

Hence, $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$

Note: We can also multiply rows by columns or columns by rows or columns by columns.

7. Summation of Determinants.

Let $\Delta_r = \begin{vmatrix} f(r) & a & l \\ g(r) & b & m \\ h(r) & c & n \end{vmatrix}$, where a, b, c, l, m and n are constants, independent of r.

Then, $\sum_{r=1}^{n} \Delta_r = \begin{vmatrix} \sum_{r=1}^{n} f(r) & a & l \\ \sum_{r=1}^{n} g(r) & b & m \\ \sum_{r=1}^{n} h(r) & c & n \end{vmatrix}$. Here function of r can be the elements of only one row or one column.

8. Differentiation and Integration of Determinants.

(1) **Differentiation of a determinant:**

(i) Let $\Delta(x)$ be a determinant of order two. If we write $\Delta(x) = |C_1 \quad C_2|$, where C_1 and C_2 denote the 1st and 2nd columns, then

$\Delta'(x) = |C'_1 \quad C_2| + |C_1 \quad C'_2|$

Where C'_i denotes the column which contains the derivative of all the functions in the i^{th} column C_i.

In a similar fashion, if we write $\Delta(x) = \begin{vmatrix} R_1 \\ R_2 \end{vmatrix}$, then $\Delta'(x) = \begin{vmatrix} R'_1 \\ R_2 \end{vmatrix} + \begin{vmatrix} R_1 \\ R'_2 \end{vmatrix}$

(ii) Let $\Delta(x)$ be a determinant of order three. If we write $\Delta(x) = |C_1 \quad C_2 \quad C_3|$, then

$\Delta'(x) = |C'_1 \quad C_2 \quad C_3| + |C_1 \quad C'_2 \quad C_3| + |C_1 \quad C_2 \quad C'_3|$

and similarly if we consider $\Delta(x) = \begin{vmatrix} R_1 \\ R_2 \\ R_3 \end{vmatrix}$, then $\Delta'(x) = \begin{vmatrix} R'_1 \\ R_2 \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R'_2 \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R_2 \\ R'_3 \end{vmatrix}$

(iii) If only one row (or column) consists functions of x and other rows (or columns) are constant, viz.

Let $\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$,

Then $\Delta'(x) = \begin{vmatrix} f'_1(x) & f'_2(x) & f'_3(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ and in general $\Delta^n(x) = \begin{vmatrix} f_1^n(x) & f_2^n(x) & f_3^n(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Where n is any positive integer and $f^n(x)$ denotes the n^{th} derivative of $f(x)$.

9. Application of Determinants in solving a system of Linear Equations.

Consider a system of simultaneous linear equations is given by
$$\left.\begin{aligned} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{aligned}\right\} \quad .....(i)$$

A set of values of the variables x, y, z which simultaneously satisfy these three equations is called a solution. A system of linear equations may have a unique solution or many solutions, or no solution at all, if it has a solution (whether unique or not) the system is said to be consistent. If it has no solution, it is called an inconsistent system.

If $d_1=d_2=d_3=0$ in (i) then the system of equations is said to be a homogeneous system. Otherwise it is called a non-homogeneous system of equations.

Theorem 1: (Cramer's rule) the solution of the system of simultaneous linear equations

$a_1x+b_1y=c_1$ .....(i) and $a_2x+b_2y=c_2$ .....(ii)

is given by $x=\frac{D_1}{D}, y=\frac{D_2}{D}$, where $D=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, D_1=\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ and $D_2=\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$, provided that

$D \neq 0$

Note: Here $D=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is the determinant of the coefficient matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$.

The determinant D_1 is obtained by replacing first column in D by the column of the right hand side of the given equations. The determinant D_2 is obtained by replacing the second column in D by the right most column in the given system of equations.

(1) **Solution of system of linear equations in three variables by Cramer's rule:**

Theorem 2: (Cramer's Rule) the solution of the system of linear equations

$a_1x + b_1y + c_1z = d_1$ .....(i)

$a_2x + b_2y + c_2z = d_2$ .....(ii)

$a_3x + b_3y + c_3z = d_3$ .....(iii)

is given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$, where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$,

$D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, and $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$, Provided that $D \neq 0$

Note: Here D is the determinant of the coefficient matrix. The determinant D_1 is obtained by replacing the elements in first column of D by d_1, d_2, d_3. D_2 is obtained by replacing the element in the second column of D by d_1, d_2, d_3 and to obtain D_3, replace elements in the third column of D by d_1, d_2, d_3.

Theorem 3: (Cramer's Rule) let there be a system of n simultaneous linear equation n unknown given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n &= b_n \end{aligned}$$

Let $D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{vmatrix}$ and let D_j, be the determinant obtained from D after replacing the j^{th} column by $\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$. Then, $x_1 = \frac{D_1}{D}$, $x_2 = \frac{D_2}{D}$,....., $x_n = \frac{D_n}{D}$, Provided that $D \neq 0$

(2) **Conditions for consistency**

Case 1: For a system of 2 simultaneous linear equations with 2 unknowns

(i) If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by $x = \frac{D_1}{D}, y = \frac{D_2}{D}$.

(ii) If $D = 0$ and $D_1 = D_2 = 0$, then the system is consistent and has infinitely many solutions.

(iii) If $D = 0$ and one of D_1 and D_2 is non-zero, then the system is inconsistent.

Case 2: For a system of 3 simultaneous linear equations in three unknowns

(i) If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$

(ii) If $D = 0$ and $D_1 = D_2 = D_3 = 0$, then the given system of equations is consistent with infinitely many solutions.

(iii) If $D = 0$ and at least one of the determinants D_1, D_2, D_3 is non-zero, then given of equations is inconsistent.

(3) **Algorithm for solving a system of simultaneous linear equations by Cramer's rule (Determinant method)**

Step 1: Obtain D, D_1, D_2 and D_3

Step 2: Find the value of D. If $D \neq 0$, then the system of the equations is consistent has a unique solution. To find the solution, obtain the values of D_1, D_2 and D_3. The solutions is given by $x = \frac{D_1}{D}, y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$. If $D = 0$ go to step 3.

Step 3: Find the values of D_1, D_2, D_3. If at least one of these determinants is non-zero, then the system is inconsistent. If $D_1 = D_2 = D_3 = 0$, then go to step 4

Step 4: Take any two equations out of three given equations and shift one of the variables, say z on the right hand side to obtain two equations in x, y. Solve these two equations by Cramer's rule to obtain x, y, in terms of z.

Note: The system of following homogeneous equations $a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$, $a_3x + b_3y + c_3z = 0$ is always consistent.

If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$, then this system h

As the unique solution $x = y = z = 0$ known as **trivial**

10. Application of Determinants in Co-ordinate Geometry.

(1) Area of triangle whose vertices are $(x_r, y_r);\ r = 1, 2, 3$ is

$$\Delta = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

(2) If $a_rx + b_ry + c_r = 0,\ (r = 1, 2, 3)$ are the sides of a triangle, then the area of the triangle is given by

$\Delta = \dfrac{1}{2C_1C_2C_3}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$, where $C_1 = a_2b_3 - a_3b_2,\ C_2 = a_3b_1 - a_1b_3,\ C_3 = a_1b_2 - a_2b_1$ are the cofactors of the elements c_1, c_2, c_3 respectively in the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

(3) The equation of a straight line passing through two points (x_1, y_1) and (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

(4) If three lines $a_r x + b_r y + c_r = 0;\ (r = 1,\ 2,\ 3)$ are concurrent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

(5) If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines then

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

(6) The equation of circle through three non-collinear points $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ is

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

11. Some Special Determinants.

(1) **Symmetric determinant:** A determinant is called symmetric determinant if for its every element $a_{ij} = a_{ji} \forall\ i, j$ e.g., $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

(2) **Skew-symmetric determinant:** A determinant is called skew symmetric determinant if for its every element $a_{ij} = -a_{ji}\ \forall i, j$ e.g., $\begin{vmatrix} 0 & 3 & -1 \\ -3 & 0 & 5 \\ 1 & -5 & 0 \end{vmatrix}$

Note: Every diagonal element of a skew symmetric determinant is always zero.

The value of a skew symmetric determinant of even order is always a perfect square and that of odd order is always zero.

(ii) $\begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix} = 0 + a^2 = a^2$ (Perfect square)

(iii) $\begin{vmatrix} 0 & a-b & e-f \\ b-a & 0 & l-m \\ f-e & m-l & 0 \end{vmatrix} = 0$

(3) **Cyclic order:** If elements of the rows (or columns) are in cyclic order.

i.e.

(i) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(ii) $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

(iii) $\begin{vmatrix} a & bc & abc \\ b & ca & abc \\ c & ab & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a)$

(iv) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

(v) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$

Note: These results direct applicable in lengthy questions (As behavior of standard results)