Testprep Kart

Knowledge... Everywhere

Mathematics

Matrices

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1. Definition.

A rectangular arrangement of numbers (which may be real or complex numbers) in rows and columns, is called a matrix. This arrangement is enclosed by small ( ) or big [ ] brackets. The numbers are called the elements of the matrix or entries in the matrix. A matrix is represented by capital letters *A, B, C etc.* and its elements by small letters *a, b, c, x, y etc.* The following are some examples of matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix},\ B = \begin{bmatrix} 2+i & -3 & 2 \\ 1 & -3+i & -5 \end{bmatrix},\ C = [1,\ 4,\ 9],\ \ D = \begin{bmatrix} a \\ g \\ h \end{bmatrix},\ \ E = [l]$$

2. Order of a Matrix.

A matrix having m rows and n columns is called a matrix of order $m \times n$ or simply $m \times n$ matrix (read as 'an m by n matrix). A matrix A of order $m \times n$ is usually written in the following manner

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & ...a_{1j} & ...a_{1n} \\ a_{21} & a_{22} & a_{23} & ...a_{2j} & ...a_{2n} \\ ..... & ..... & ..... & ..... & ..... \\ a_{i1} & a_{i2} & a_{i3} & ...a_{ij} & ...a_{in} \\ ..... & ..... & ..... & ..... & ..... \\ a_{m1} & a_{m2} & a_{m3} & ...a_{mj} & ...a_{mn} \end{bmatrix} \text{or } A = [a_{ij}]_{m \times n}, \text{ where } \begin{matrix} i = 1,\ 2, ...m \\ j = 1,\ 2, ...n \end{matrix}$$

Here a_{ij} denotes the element of i^{th} row and j^{th} column. *Example:* order of matrix $\begin{bmatrix} 3 & -1 & 5 \\ 6 & 2 & -7 \end{bmatrix}$ is 2×3

Note: A matrix of order $m \times n$ contains mn elements. Every row of such a matrix contains n elements and every column contains m elements.

3. Equality of Matrices.

Two matrix *A* and *B* are said to be equal matrix if they are of same order and their corresponding elements are equal *Example*: If $A=\begin{bmatrix}1 & 6 & 3\\ 5 & 2 & 1\end{bmatrix}$ and $B=\begin{bmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{bmatrix}$ are equal matrices.

Then $a_1=1, a_2=6, a_3=3, b_1=5, b_2=2, b_3=1$

4. Types of Matrices.

(1) **Row matrix**: A matrix is said to be a row matrix or row vector if it has only one row and any number of columns. *Example*: [5 0 3] is a row matrix of order 1× 3 and [2] is a row matrix of order 1×1.

(2) **Column matrix:** A matrix is said to be a column matrix or column vector if it has only one column and any number of rows. *Example*: $\begin{bmatrix}2\\ 3\\ -6\end{bmatrix}$ is a column matrix of order 3×1 and [2] is a column matrix of order 1×1. Observe that [2] is both a row matrix as well as a column matrix.

(3) **Singleton matrix**: If in a matrix there is only one element then it is called singleton matrix. Thus, $A=[aij]_{m\times n}$ is a singleton matrix if $m=n=1$ *Example:* [2], [3], [a], [–3] are singleton matrices.

(4) **Null or zero matrix:** If in a matrix all the elements are zero then it is called a zero matrix and it is generally denoted by *O*. Thus $A=[a_{ij}]_{m\times n}$ is a zero matrix if $a_{ij}=0$ for all *i* and *j*.

Example: $[0], \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, [0\ \ 0]$ are all zero matrices, but of different orders.

(5) **Square matrix**: If number of rows and number of columns in a matrix are equal, then it is called a square matrix. Thus $A=[a_{ij}]_{m\times n}$ is a square matrix if $m=n$. *Example*: $\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$ is a square matrix of order 3×3

(i) If $m \neq n$ then matrix is called a rectangular matrix.

(ii) The elements of a square matrix A for which $i = j, i.e.\ a_{11}, a_{22}, a_{33}, ... a_{nn}$ are called diagonal elements and the line joining these elements is called the principal diagonal or leading diagonal of matrix *A.*

(iii) **Trace of a matrix:** The sum of diagonal elements of a square matrix. *A* is called the trace of matrix A , which is denoted by tr A. $tr\ A = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + ... a_{nn}$

Properties of trace of a matrix: Let $A = [a_{ii}]_{n \times n}$ and $B = [b_{ij}]_{n \times n}$ and λ be a scalar

(i) $tr(\lambda A) = \lambda\, tr(A)$

(ii) $tr(A - B) = tr(A) - tr(B)$

(iii) $tr(AB) = tr(BA)$

(iv) $tr(A) = tr(A')$ or $tr\ (A^T)$

(v) $tr(I_n) = n$

(vi) $tr(0) = 0$

(vii) $tr(AB) \neq tr\,A\,.\,tr\,B$

(6) **Diagonal matrix:** If all elements except the principal diagonal in a square matrix are zero, it is called a diagonal matrix. Thus a square matrix $A = [a_{ij}]$ is a diagonal matrix if $a_{ij} = 0$, when $i \neq j$.

Example: $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a diagonal matrix of order 3×3, which can be denoted by diag [2, 3, and 4]

Note: No element of principal diagonal in a diagonal matrix is zero.

- Number of zeros in a diagonal matrix is given by $n^2 - n$ where *n* is the order of the matrix.
- A diagonal matrix of order $n \times n$ having $d_1, d_2, ..., d_n$ as diagonal elements is denoted by $\text{diag}\ [d_1, d_2, ..., d_n]$.

(7) **Identity matrix**: A square matrix in which elements in the main diagonal are all '1' and rest are all zero is called an identity matrix or unit matrix. Thus, the square matrix $A=[a_{ij}]$ is an identity matrix, if

$$a_{ij}=\begin{cases}1, \text{ if } i=j\\ 0, \text{ if } i\neq j\end{cases}$$

We denote the identity matrix of order *n* by I_n.

Example: [1], $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ are identity matrices of order 1, 2 and 3 respectively.

(8) **Scalar matrix** : A square matrix whose all non-diagonal elements are zero and diagonal elements are equal is called a scalar matrix. Thus, if $A=[a_{ij}]$ is a square matrix and $a_{ij}=\begin{cases}\alpha, \text{if } i=j\\ 0, \text{ if } i\neq j\end{cases}$, then *A* is a scalar matrix.

Example: [2], $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $\begin{bmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 5\end{bmatrix}$ are scalar matrices of order 1, 2 and 3 respectively.

Note: Unit matrix and null square matrices are also scalar matrices.

(9) **Triangular Matrix**: A square matrix $[a_{ij}]$ is said to be triangular matrix if each element above or below the principal diagonal is zero. It is of two types

(i) **Upper Triangular matrix:** A square matrix $[a_{ij}]$ is called the upper triangular matrix, if $a_{ij}=0$ when $i>j$.

Example: $\begin{bmatrix}3 & 1 & 2\\ 0 & 4 & 3\\ 0 & 0 & 6\end{bmatrix}$ is an upper triangular matrix of order 3×3.

(ii) **Lower Triangular matrix:** A square matrix $[a_{ij}]$ is called the lower triangular matrix, if $a_{ij}=0$ when $i<j$.

Example: $\begin{bmatrix}1 & 0 & 0\\ 2 & 3 & 0\\ 4 & 5 & 2\end{bmatrix}$ is a lower triangular matrix of order 3×3.

Note: Minimum number of zeros in a triangular matrix is given by $\frac{n(n-1)}{2}$ where n is order of matrix.

- Diagonal matrix is both upper and lower triangular.
- A triangular matrix $a = [a_{ij}]_{n\times n}$ is called strictly triangular if $a_{ij} = 0$ for $1 \le i \le n$

5. Addition and Subtraction of Matrices.

If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are two matrices of the same order then their sum $A+B$ is a matrix whose each element is the sum of corresponding elements. *i.e.* $A + B = [a_{ij} + b_{ij}]_{m\times n}$

Example: If $A = \begin{bmatrix} 5 & 2 \\ 1 & 3 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 \\ 2 & 2 \\ 3 & 3 \end{bmatrix}$, then $A + B = \begin{bmatrix} 5+1 & 2+5 \\ 1+2 & 3+2 \\ 4+3 & 1+3 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 3 & 5 \\ 7 & 4 \end{bmatrix}$

Similarly, their subtraction $A - B$ is defined as $A - B = [a_{ij} - b_{ij}]_{m\times n}$

i.e. in above example $A - B = \begin{bmatrix} 5-1 & 2-5 \\ 1-2 & 3-2 \\ 4-3 & 1-3 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \\ 1 & -2 \end{bmatrix}$

Note: Matrix addition and subtraction can be possible only when matrices are of the same order.

Properties of matrix addition: If *A*, *B* and *C* are matrices of same order, then

(i) $A + B = B + A$ (Commutative law)

(ii) $(A + B) + C = A + (B + C)$ (Associative law)

(iii) $A + O = O + A = A$, Where O is zero matrix which is additive identity of the matrix.

(iv) $A + (-A) = 0 = (-A) + A$, where $(-A)$ is obtained by changing the sign of every element of *A*, which is additive inverse of the matrix.

(v) $\left.\begin{matrix} A + B = A + C \\ B + A = C + A \end{matrix}\right\} \Rightarrow B = C$ (Cancellation law)

6. Scalar Multiplication of Matrices.

Let $A=[a_{ij}]_{m\times n}$ be a matrix and *k* be a number, then the matrix which is obtained by multiplying every element of *A* by *k* is called scalar multiplication of *A* by *k* and it is denoted by *kA*.

Thus, if $A=[a_{ij}]_{m\times n}$, then $kA = Ak = [ka_{ij}]_{m\times n}$. *Example*: If $A=\begin{bmatrix}2 & 4\\ 3 & 1\\ 4 & 6\end{bmatrix}$, then $5A=\begin{bmatrix}10 & 20\\ 15 & 5\\ 20 & 30\end{bmatrix}$

Properties of scalar multiplication:

If A, B are matrices of the same order and λ, μ are any two scalars then

(i) $\lambda(A+B)=\lambda A+\lambda B$ (ii) $(\lambda+\mu)A=\lambda A+\mu A$

(iii) $\lambda(\mu A)=(\lambda\mu A)=\mu(\lambda A)$ (iv) $(-\lambda A)=-(\lambda A)=\lambda(-A)$

Note: All the laws of ordinary algebra hold for the addition or subtraction of matrices and their multiplication by scalars.

7. Multiplication of Matrices.

Two matrices A and B are conformable for the product AB if the number of columns in A (pre-multiplier) is same as the number of rows in *B* (post multiplier).Thus, if $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{n\times p}$ are two matrices of order $m\times n$ and $n\times p$ respectively, then their product *AB* is of order $m\times p$ and is defined as

$$(AB)_{ij}=\sum_{r=1}^{n} a_{ir}b_{rj}$$

$=[a_{i1}a_{i2}...a_{in}]\begin{bmatrix}b_{1j}\\ b_{2j}\\ \vdots\\ b_{nj}\end{bmatrix}$ = (i^{th} row of A)(j^{th} column of *B*) .....(i), where *i*=1, 2, ..., *m* and *j*=1, 2, ...*p*

Now we define the product of a row matrix and a column matrix.

Let $A=[a_1 a_2 .... a_n]$ be a row matrix and $B=\begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_n\end{bmatrix}$ be a column matrix.

Then $AB=[a_1b_1+a_2b_2+....+a_nb_n]$ ...(ii). Thus, from (i),

$(AB)_{ij}$ = Sum of the product of elements of i^{th} row of *A* with the corresponding elements of j^{th} column of *B*.

Properties of matrix multiplication

If A, B and C are three matrices such that their product is defined, then

(i) $AB \neq BA$ (Generally not commutative)

(ii) $(AB)C = A(BC)$ (Associative Law)

(iii) $IA = A = AI$ Where I is identity matrix for matrix multiplication

(iv) $A(B+C) = AB + AC$ (Distributive law)

(v) If $AB = AC \nRightarrow B = C$ (Cancellation law is not applicable)

(vi) If $AB = 0$ It does not mean that $A = 0$ or $B = 0$, again product of two non zero matrix may be a zero matrix.

Note: If A and B are two matrices such that AB exists, then BA may or may not exist.

❑ The multiplication of two triangular matrices is a triangular matrix.

❑ The multiplication of two diagonal matrices is also a diagonal matrix and diag $(a_1, a_2, ...a_n) \times \text{diag } (b_1, b_2, ...b_n) = \text{diag } (a_1 b_1, a_2 b_2, ...a_n b_n)$

❑ The multiplication of two scalar matrices is also a scalar matrix.

❑ If A and B are two matrices of the same order, then

(i) $(A+B)^2 = A^2 + B^2 + AB + BA$

(ii) $(A - B^2) = A^2 + B^2 - AB - BA$

(iii) $(A-B)(A+B) = A^2 - B^2 + AB - BA$

(iv) $(A+B)(A-B) = A^2 - B^2 - AB + BA$

(v) $A(-B) = (-A)B = -(AB)$

8. Positive Integral Powers of a Matrix.

The positive integral powers of a matrix A are defined only when A is a square matrix. Also then $A^2 = A.A$, $A^3 = A.A.A = A^2 A$. Also for any positive integers m, n.

(i) $A^m A^n = A^{m+n}$

(ii) $(A^m)^n = A^{mn} = (A^n)^m$

(iii) $I^n = I, I^m = I$

(iv) $A^0 = I_n$ Where A is a square matrix of order n.

9. Matrix Polynomial.

Let $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$ be a polynomial and let *A* be a square matrix of order *n*. Then $f(A) = a_0 A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots + a_{n-1} A + a_n I_n$ is called a matrix polynomial.

Example: If $f(x) = x^2 - 3x + 2$ is a polynomial and A is a square matrix, then $A^2 - 3A + 2I$ is a matrix polynomial.

10. Transpose of a Matrix.

The matrix obtained from a given matrix A by changing its rows into columns or columns into rows is called transpose of Matrix *A* and is denoted by A^T or A'.
From the definition it is obvious that if order of *A* is $m \times n$, then order of A^T is $n \times m$

Example: Transpose of matrix $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}_{2\times 3}$ is $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}_{3\times 2}$

Properties of transpose: Let *A* and *B* be two matrices then

(i) $(A^T)^T = A$
(ii) $(A+B)^T = A^T + B^T$, A and *B* being of the same order
(iii) $(kA)^T = kA^T, k$ be any scalar (real or complex)
(iv) $(AB)^T = B^T A^T$, A and B being conformable for the product *AB*
(v) $(A_1 A_2 A_3 \ldots.. A_{n-1} A_n)^T = A_n{}^T A_{n-1}{}^T \ldots\ldots. A_3{}^T A_2{}^T A_1{}^T$
(vi) $I^T = I$

11. Determinant of a Matrix.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix, then its determinant, denoted by $|A|$ or Det (A) is defined as

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Properties of determinant of a matrix

(i) $|A|$ Exists $\Leftrightarrow A$ is square matrix

(ii) $|AB| = |A||B|$

(iii) $|A^T| = |A|$

(iv) $|kA| = k^n |A|$, If A is a square matrix of order n

(v) If A and B are square matrices of same order then $|AB| = |BA|$

(vi) If A is a skew symmetric matrix of odd order then $|A| = 0$

(vii) If $A = \text{diag}(a_1, a_2, ...a_n)$ then $|A| = a_1 a_2 ... a_n$

(viii) $|A|^n = |A^n|$, $n \in N$.

12. Special Types of Matrices.

(1) **Symmetric and skew-symmetric matrix**

(i) **Symmetric matrix**: A square matrix $A = [a_{ij}]$ is called symmetric matrix if $a_{ij} = a_{ji}$ for all i, j or $A^T = A$

Example: $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

Note: Every unit matrix and square zero matrix are symmetric matrices.

❑ Maximum number of different elements in a symmetric matrix is $\frac{n(n+1)}{2}$

(ii) **Skew-symmetric matrix**: A square matrix $A=[a_{ij}]$ is called skew- symmetric matrix if $a_{ij}=-a_{ji}$ for all *i, j*

or $A^T=-A$. *Example*: $\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$

Note: All principal diagonal elements of a skew- symmetric matrix are always zero because for any diagonal element. $a_{ij}=-a_{ij} \Rightarrow a_{ij}=0$

❑ Trace of a skew symmetric matrix is always 0.

Properties of symmetric and skew-symmetric matrices:

(i) If *A* is a square matrix, then $A+A^T, AA^T, A^TA$ are symmetric matrices, while $A-A^T$ is skew- symmetric matrix.

(ii) If *A* is a symmetric matrix, then $-A, KA, A^T, A^n, A^{-1}, B^TAB$ are also symmetric matrices, where $n \in N$, $K \in R$ and *B* is a square matrix of order that of *A*

(iii) If A is a skew-symmetric matrix, then
(a) A^{2n} is a symmetric matrix for $n \in N$,
(b) A^{2n+1} is a skew-symmetric matrix for $n \in N$,
(c) *kA* is also skew-symmetric matrix, where $k \in R$,
(d) B^TAB is also skew- symmetric matrix where *B* is a square matrix of order that of *A.*

(iv) If *A*, *B* are two symmetric matrices, then
(a) $A \pm B,\ AB+BA$ are also symmetric matrices,
(b) $AB-BA$ is a skew- symmetric matrix,
(c) *AB* is a symmetric matrix, when $AB=BA$.

(v) If *A,B* are two skew-symmetric matrices, then
(a) $A \pm B,\ AB-BA$ are skew-symmetric matrices,
(b) $AB+BA$ is a symmetric matrix.

(vi) If A a skew-symmetric matrix and C is a column matrix, then $C^T AC$ is a zero matrix.

(vii) Every square matrix A can uniquely be expressed as sum of a symmetric and skew-symmetric matrix *i.e.*

$$A = \left[\frac{1}{2}(A + A^T)\right] + \left[\frac{1}{2}(A - A^T)\right].$$

(2) **Singular and Non-singular matrix:** Any square matrix A is said to be non-singular if $|A| \neq 0$, and a square matrix A is said to be singular if $|A| = 0$. Here $|A|$ (or det(A) or simply det $|A|$ means corresponding determinant of square matrix A.

Example: $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ then $|A| = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 10 - 12 = -2 \Rightarrow A$ is a non-singular matrix.

(3) **Hermitian and skew-Hermitian matrix:** A square matrix $A = [a_{ij}]$ is said to be hermitian matrix if $a_{ij} = \bar{a}_{ji}$ $\forall i,j$ *i.e.* $A = A^\theta$. *Example*: $\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$, $\begin{bmatrix} 3 & 3-4i & 5+2i \\ 3+4i & 5 & -2+i \\ 5-2i & -2-i & 2 \end{bmatrix}$ are Hermitian matrices.

Note: If A is a Hermitian matrix then $a_{ii} = \bar{a}_{ii} \Rightarrow a_{ii}$ is real $\forall i$, thus every diagonal element of a Hermitian matrix must be real.

❑ A Hermitian matrix over the set of real numbers is actually a real symmetric matrix and a square matrix, $A=|a_{ij}|$ is said to be a skew-Hermitian if $a_{ij} = -\bar{a}_{ji}. \forall i, j$ *i.e.* $A^\theta = -A$.

Example: $\begin{bmatrix} 0 & -2+i \\ 2-i & 0 \end{bmatrix}$, $\begin{bmatrix} 3i & -3+2i & -1-i \\ 3+2i & -2i & -2-4i \\ 1-i & 2-4i & 0 \end{bmatrix}$ are skew-Hermitian matrices.

❑ If A is a skew-Hermitian matrix, then $a_{ii} = -\bar{a}_{ii} \Rightarrow a_{ii} + \bar{a}_{ii} = 0$ i.e. a_{ii} must be purely imaginary or zero.

❑ A skew-Hermitian matrix over the set of real numbers is actually a real skew-symmetric matrix.

(4) **Orthogonal matrix:** A square matrix A is called orthogonal if $AA^T = I = A^T A$ *i.e.* if $A^{-1} = A^T$

Example: $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ is orthogonal because $A^{-1} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = A^T$

In fact every unit matrix is orthogonal.

(5) **Idempotent matrix**: A square matrix A is called an idempotent matrix if $A^2 = A$.

Example: $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ is an idempotent matrix, because $A^2 = \begin{bmatrix} 1/4+1/4 & 1/4+1/4 \\ 1/4+1/4 & 1/4+1/4 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = A$.

Also, $A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$ are idempotent matrices because $A^2=A$ and $B^2=B$.
In fact every unit matrix is indempotent.

(6) **Involutory matrix:** A square matrix A is called an involutory matrix if $A^2=I$ or $A^{-1}=A$

Example: $A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ is an involutory matrix because $A^2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$

In fact every unit matrix is involutory.

(7) **Nilpotent matrix:** A square matrix A is called a nilpotent matrix if there exists a $p \in N$ such that $A^p=0$

Example: $A=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$ is a nilpotent matrix because $A^2=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=0$ (Here *P = 2*)

(8) **Unitary matrix:** A square matrix is said to be unitary, if $\bar{A}'A=I$ since $|\bar{A}'|=|A|$ and $|\bar{A}'A|=|\bar{A}'||A|$ therefore if $\bar{A}'A$=I, we have $|\bar{A}'||A|=1$
Thus the determinant of unitary matrix is of unit modulus. For a matrix to be unitary it must be non-singular.
Hence $\bar{A}'A=I \Rightarrow A\bar{A}'=I$

(9) **Periodic matrix**: A matrix *A* will be called a periodic matrix if $A^{k+1}=A$ where *k* is a positive integer. If, however *k* is the least positive integer for which $A^{k+1}=A$, then *k* is said to be the period of *A*.

(10) **Differentiation of a matrix:** If $A=\begin{bmatrix}f(x) & g(x)\\ h(x) & l(x)\end{bmatrix}$ then $\frac{dA}{dx}=\begin{bmatrix}f'(x) & g'(x)\\ h'(x) & l'(x)\end{bmatrix}$ is a differentiation of matrix *A*.

Example: If $A=\begin{bmatrix}x^2 & \sin x\\ 2x & 2\end{bmatrix}$ then $\frac{dA}{dx}=\begin{bmatrix}2x & \cos x\\ 2 & 0\end{bmatrix}$

(11) **Submatrix :** Let *A* be $m \times n$ matrix, then a matrix obtained by leaving some rows or columns or both, of *A* is called a sub matrix of *A*. *Example* : If $A'=\begin{bmatrix}2 & 1 & 0\\ 3 & 2 & 2\\ 2 & 5 & 3\end{bmatrix}$ and $\begin{bmatrix}2 & 2\\ 5 & 3\end{bmatrix}$ are sub matrices of matrix

$$A=\begin{bmatrix}2 & 1 & 0 & -1\\ 3 & 2 & 2 & 4\\ 2 & 5 & 3 & 1\end{bmatrix}$$

(12) **Conjugate of a matrix:** The matrix obtained from any given matrix A containing complex number as its elements, on replacing its elements by the corresponding conjugate complex numbers is called

conjugate of A and is denoted by $\overline{A}$. *Example*: $A = \begin{bmatrix} 1+2i & 2-3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix}$ then

$$\overline{A} = \begin{bmatrix} 1-2i & 2+3i & 3-4i \\ 4+5i & 5-6i & 6+7i \\ 8 & 7-8i & 7 \end{bmatrix}$$

Properties of conjugates:

(i) $\overline{(\overline{A})} = A$

(ii) $\overline{(A+B)} = \overline{A} + \overline{B}$

(iii) $\overline{(\alpha A)} = \overline{\alpha}\overline{A}, \alpha$ being any number

(iv) $\overline{(AB)} = \overline{A}\,\overline{B}, A$ and B being conformable for multiplication.

(13) **Transpose conjugate of a matrix:** The transpose of the conjugate of a matrix *A* is called transposed conjugate of A and is denoted by A^θ. The conjugate of the transpose of A is the same as the transpose of the conjugate of *A i.e.* $\overline{(A')} = (\overline{A})' = A^\theta$.

If $A = [a_{ij}]_{m\times n}$ then $A^\theta = [b_{ji}]_{n\times m}$ where $b_{ji} = \overline{a}_{ij}$ *i.e.* the $(j, i)^{th}$ element of A^θ = the conjugate of $(i, j)^{th}$ element of *A*.

Example: If $A = \begin{bmatrix} 1+2i & 2-3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix}$, then $A^\theta = \begin{bmatrix} 1-2i & 4+5i & 8 \\ 2+3i & 5-6i & 7-8i \\ 3-4i & 6+7i & 7 \end{bmatrix}$

Properties of transpose conjugate

(i) $(A^\theta)^\theta = A$

(ii) $(A+B)^\theta = A^\theta + B^\theta$

(iii) $(kA)^\theta = \overline{K}A^\theta$, *K* being any number

(iv) $(AB)^\theta = B^\theta A^\theta$

13. Adjoint of a Square Matrix.

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be cofactor of a_{ij} in A. Then the transpose of the matrix of cofactors of elements of *A* is called the adjoint of *A* and is denoted by *adj* A

Thus, $adj\ A = [C_{ij}]^T \Rightarrow (adj\,A)_{ij} = C_{ji} =$ cofactor of a_{ji} in *A*.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $adj\,A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$;

Where C_{ij} denotes the cofactor of a_{ij} in *A*.

Example: $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}, C_{11} = s, C_{12} = -r, C_{21} = -q, C_{22} = p$

$$\therefore adj\,A = \begin{bmatrix} s & -r \\ -q & p \end{bmatrix}^T = \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$$

Note: The adjoint of a square matrix of order 2 can be easily obtained by interchanging the diagonal elements and changing signs of off diagonal elements.