Mathematics

Regular Cartesian Co-ordinates

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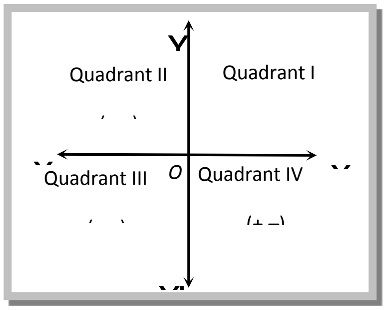
1. Introduction.



Co-ordinates of a point are the real variables associated in an order to a point to describe its location in some space. Here the space is the two dimensional plane. The work of describing the position of a point in a plane by an ordered pair of real numbers can be done in different ways.

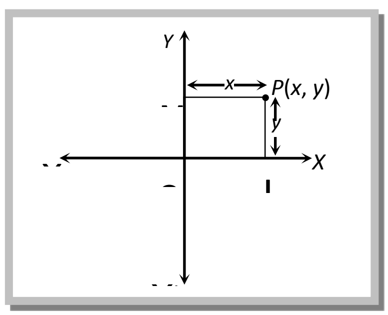
The two lines *XOX'* and *YOY'* divide the plane in four quadrants. *XOY, YOX', X'OY', Y'OX* are respectively called the first, the second, the third and the fourth quadrants. We assume the directions of *OX, OY* as positive while the directions of *OX', OY'* as negative.

Quadrant	***x*-coordinate**	***y*-coordinate**	**point**
First quadrant	+	+	(+,+)
Second quadrant	–	+	(–,+)
Third quadrant	–	–	(–,–)
Fourth quadrant	+	–	(+,–)

2. Cartesian Co-ordinates of a Point.

This is the most popular co-ordinate system.

Let us consider two intersecting lines *XOX'* and *YOY'*, which are perpendicular to each other. Let *P* be any point in the plane of lines. Draw the rectangle *OLPM* with its adjacent sides *OL,OM* along the lines *XOX', YOY'* respectively. The position of the point *P* can be fixed in the plane provided the locations as well as the magnitudes of *OL, OM* are known.



Axis of *x*: The line *XOX'* is called axis of *x*.

Axis of *y*: The line *YOY'* is called axis of *y*.

Co-ordinate axes: *x* axis and *y* axis together are called axis of co-ordinates or axis of reference.

Origin: The point '*O*' is called the origin of co-ordinates or the origin.

Oblique axes: If both the axes are not perpendicular then they are called as oblique axes.

Let $OL = x$ and $OM = y$ which are respectively called the abscissa (or x-coordinate) and the ordinate (or y-coordinate). The co-ordinate of P are (x, y).

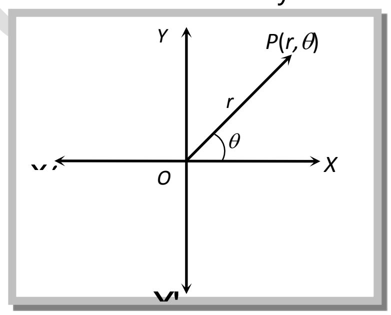
Note: Co-ordinates of the origin is (0, 0).

❑ The y co-ordinate of every point on x-axis is zero.

❑ The x co-ordinate of every point on y-axis is zero.

3. Polar Co-ordinates.

Let OX be any fixed line which is usually called the initial line and O be a fixed point on it. If distance of any point P from the O is 'r' and $\angle XOP = \theta$, then (r, θ) are called the polar co-ordinates of a point P.



If (x, y) are the Cartesian co-ordinates of a point P, then

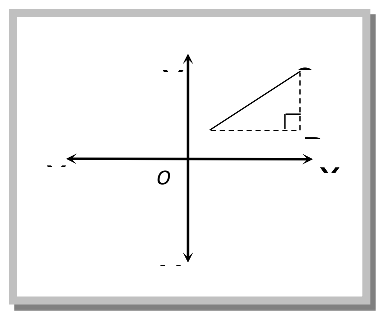
$x = r\cos\theta$; $y = r\sin\theta$ and $r = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

4. Distance Formula.

The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$PQ = \sqrt{(PR)^2 + (QR)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



Note: ❑ The distance of a point $M(x_0, y_0)$ from origin $O(0, 0)$

$OM = \sqrt{(x_0^2 + y_0^2)}$.

If distance between two points is given then use $\pm$ sign.

When the line PQ is parallel to the y-axis, the abscissa of point P and Q will be equal i.e, $x_1 = x_2$;

$\therefore\ PQ = |y_2 - y_1|$

When the segment PQ is parallel to the x-axis, the ordinate of the points P and Q will be equal i.e., $y_1 = y_2$.

Therefore $PQ = |x_2 - x_1|$

(1) **Distance between two points in polar co-ordinates**: Let O be the pole and OX be the initial line. Let P and Q be two given points whose polar co-ordinates are (r_1, θ_1) and (r_2, θ_2) respectively.

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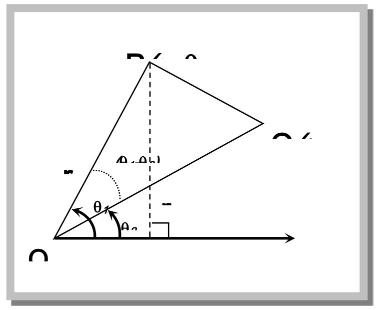
Then $OP = r_1, OQ = r_2$

$\angle POX = \theta_1$ and $\angle QOX = \theta_2$

Then $\angle POQ = (\theta_1 - \theta_2)$



In ΔPOQ, from cosine rule $\cos(\theta_1 - \theta_2) = \dfrac{(OP)^2 + (OQ)^2 - (PQ)^2}{2OP.OQ}$

$\therefore \quad (PQ)^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)$

$\therefore \quad PQ = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)}$

Note: Always taking θ_1 and θ_2 in radians.

5. Geometrical Conditions.

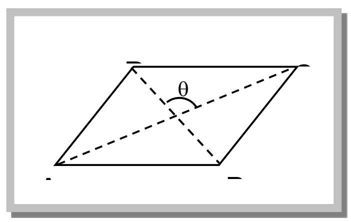
(1) **Properties of triangles**

(i) In any triangle $ABC, AB + BC > AC$ and $|AB - BC| < AC$.

(ii) The ΔABC is equilateral $\Leftrightarrow AB = BC = CA$.

(iii) The ΔABC is a right angled triangle $\Leftrightarrow AB^2 = AC^2 + BC^2$ or $AC^2 = AB^2 + BC^2$ or $BC^2 = AB^2 + AC^2$.

(iv) The ΔABC is isosceles $\Leftrightarrow AB = BC$ or $BC = CA$ or $AB = AC$.



(2) **Properties of quadrilaterals**

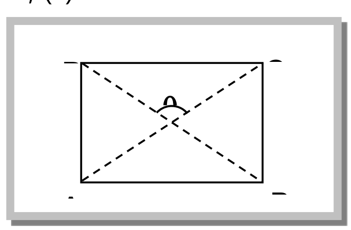
(i) The quadrilateral ABCD is a parallelogram if and only if

(a) $AB = DC, AD = BC$, or (b) the middle points of BD and AC are the same,

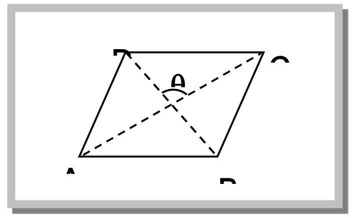
In a parallelogram diagonals AC and BD are not equal and $\theta \neq \dfrac{\pi}{2}$.

(ii) The quadrilateral $ABCD$ is a rectangle if and only if

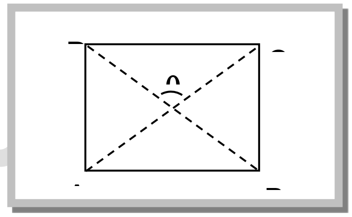
(a) $AB = CD, AD = BC$ and $AC^2 = AB^2 + BC^2$ or, (b) $AB = CD, AD = BC, AC = BD$ or, (c) the middle points of AC and BD are the same and AC=BD. $(\theta \neq \pi/2)$



(iii) The quadrilateral $ABCD$ is a rhombus (but not a square) if and only if (a) $AB = BC = CD = DA$ and $AC \neq BD$ or, (b) the middle points of AC and BD are the same and $AB = AD$ but $AC \neq BD$. $(\theta = \pi / 2)$



(iv) The quadrilateral $ABCD$ is a square if and only if
(a) $AB = BC = CD = DA$ and $AC = BD$ or (b) the middle points of AC and BD are the same and $AC = BD$, $(\theta = \pi / 2)$, $AB = AD$.



Note: Diagonals of square, rhombus, rectangle and parallelogram always bisect each other.

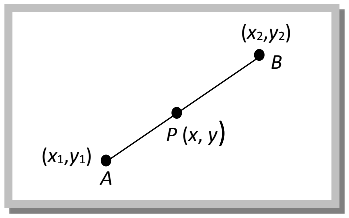
- ❑ Diagonals of rhombus and square bisect each other at right angle.
- ❑ Four given points are collinear, if area of quadrilateral is zero.

6. Section Formulae.

If $P(x,y)$ divides the join of $A(x_1,y_1)$ and $B(x_2,y_2)$ in the ratio $m_1 : m_2 (m_1, m_2 > 0)$

(1) **Internal division:** If $P(x,y)$ divides the segment AB internally in the ratio of $m_1 : m_2$

$$\Rightarrow \frac{PA}{PB} = \frac{m_1}{m_2}$$

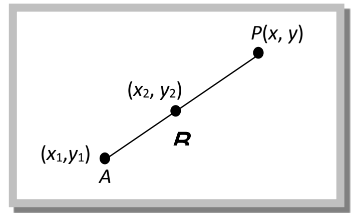


The co-ordinates of $P(x,y)$ are

$$x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2} \text{ and } y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}$$

(2) **External division:** If $P(x,y)$ divides the segment AB externally in the ratio of $m_1 : m_2$

$$\Rightarrow \quad \frac{PA}{PB} = \frac{m_1}{m_2}$$



The co-ordinates of $P(x,y)$ are $x = \dfrac{m_1 x_2 - m_2 x_1}{m_1 - m_2}$ and $y = \dfrac{m_1 y_2 - m_2 y_1}{m_1 - m_2}$

Note: If $P(x,y)$ divides the join of $A(x_1,y_1)$ and $B(x_2,y_2)$ in the ratio $\lambda : 1 (\lambda > 0)$, then $x = \dfrac{\lambda x_2 \pm x_1}{\lambda \pm 1}$; $y = \dfrac{\lambda y_2 \pm y_1}{\lambda \pm 1}$. Positive sign is taken for internal division and negative sign is taken for external division.

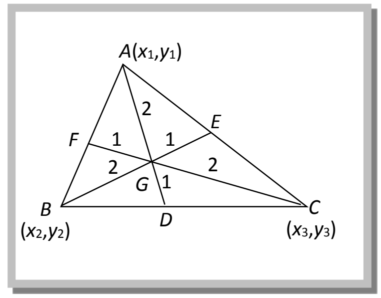
The midpoint of AB is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$ [Here $m_1 : m_2 :: 1 : 1$]

For finding ratio, use ratio $\lambda : 1$. If λ is positive, then divides internally and if λ is negative, then divides externally.

Straight line $ax + by + c = 0$ divides the join of points $A(x_1,y_1)$ and $B(x_2,y_2)$ in the ratio $\left(-\dfrac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right)$.

If ratio is –ve then divides externally and if ratio is +ve then divides internally.

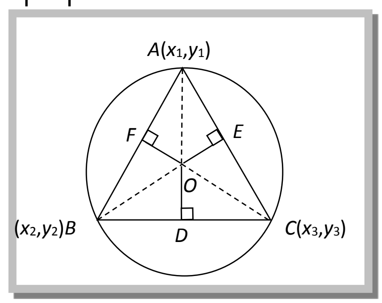
7. Some points of a Triangle.



(1) **Centroid of a triangle:** The centroid of a triangle is the point of intersection of its medians. The centroid divides the medians in the ratio $2:1$ (Vertex : base)

If $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are the vertices of a triangle. If G be the centroid upon one of the median (say) AD, then AG : GD = 2 : 1

$\Rightarrow$ Co-ordinate of G are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$



(2) **Circumcentre:** The circumcentre of a triangle is the point of intersection of the perpendicular bisectors of the sides of a triangle. It is the center of the circle which passes through the vertices of the triangle and so its distance from the vertices of the triangle is the same and this distance is known as the circum-radius of the triangle.

Let vertices A, B, C of the triangle ABC be $(x_1,y_1),(x_2,y_2)$ and (x_3,y_3) and let circumcentre be O(x, y) and then (x, y) can be found by solving

$(OA)^2=(OB)^2=(OC)^2$

i.e., $(x-x_1)^2+(y-y_1)^2=(x-x_2)^2+(y-y_2)^2=(x-x_3)^2+(y-y_3)^2$

Note: If a triangle is right angle, then its circumcentre is the midpoint of hypotenuse.

If angles of triangle i.e., A, B, C and vertices of triangle $A(x_1,y_1),B(x_2,y_2)$ and $C(x_3,y_3)$ are given, then circumcentre of the triangle ABC is

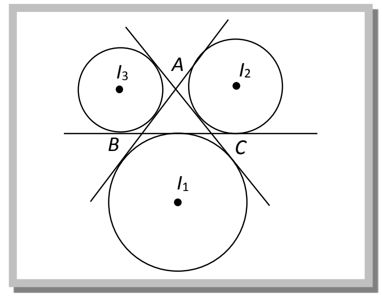
$$\left(\frac{x_1\sin 2A+x_2\sin 2B+x_3\sin 2C}{\sin 2A+\sin 2B+\sin 2C}, \frac{y_1\sin 2A+y_2\sin 2B+y_3\sin 2C}{\sin 2A+\sin 2B+\sin 2C}\right)$$

(3) **Incentre:** The incentre of a triangle is the point of intersection of internal bisector of the angles. Also it is a center of a circle touching all the sides of a triangle.

Co-ordinates of incentre $\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$

Where a, b, c are the sides of triangle ABC.

(4) **Excircle:** A circle touches one side outside the triangle and other two extended sides then circle is known as excircle. Let ABC be a triangle then there are three excircles with three excentres. Let I_1, I_2, I_3 opposite to vertices A,B and C respectively. If vertices of triangle are $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ then



$$I_1 \equiv \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c} \right)$$

$$I_2 \equiv \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c} \right), \ I_3 \equiv \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c} \right)$$

Note: Angle bisector divides the opposite sides in the ratio of remaining sides e.g. $\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}$

❑ Incentre divides the angle bisectors in the ratio $(b+c):a, (c+a):b$ and $(a+b):c$

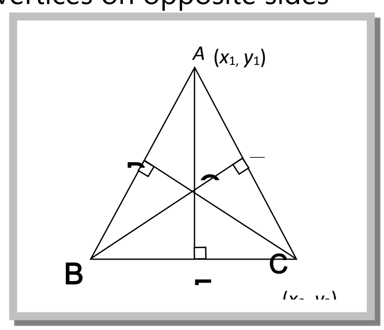
❑ **Excentre:** Point of intersection of one internal angle bisector and other two external angle bisector is called as excentre. There are three excentres in a triangle. Co-ordinate of each can be obtained by changing the sign of a,b,c respectively in the formula of in-center.

(5) **Orthocentre:** It is the point of intersection of perpendiculars drawn from vertices on opposite sides (called altitudes) of a triangle and can be obtained by solving the equation of any two altitudes.

Here O is the orthocentre since $AE \perp BC$, $BF \perp AC$ and $CD \perp AB$

then $OE \perp BC, \ OF \perp AC, OD \perp AB$

Solving any two we can get coordinate of O.



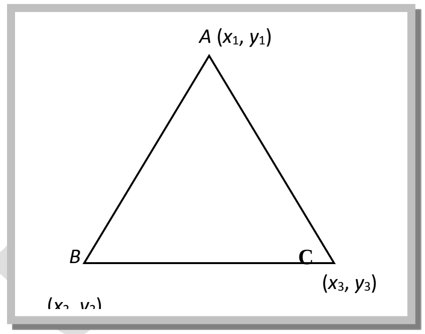
Note: If a triangle is right angled triangle, then orthocentre is the point where right angle is formed.
If the triangle is equilateral then centroid, incentre, orthocentre, circum-centre coincides.

Orthocentre, centroid and circum-centre are always collinear and centroid divides the line joining orthocentre and circum-centre in the ratio 2 : 1

In an isosceles triangle centroid, orthocentre, incentre, circum-centre lie on the same line.

8. Area of some Geometrical figures.

(1) **Area of a triangle**: The area of a triangle ABC with vertices $A(x_1,y_1);B(x_2,y_2)$ and $C(x_3,y_3)$. The area of triangle ABC is denoted by 'Δ'and is given as



$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}|(x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)|$$

In equilateral triangle

(i) Having sides a, area is $\frac{\sqrt{3}}{4}a^2$.

(ii) Having length of perpendicular as 'p' area is $\frac{(p^2)}{\sqrt{3}}$.

Note: If a triangle has polar co-ordinates $(r_1,\theta_1),(r_2,\theta_2)$ and (r_3,θ_3) then its area

$$\Delta = \frac{1}{2}[r_1r_2\sin(\theta_2-\theta_1)+r_2r_3\sin(\theta_3-\theta_2)+r_3r_1\sin(\theta_1-\theta_3)]$$

If area is a rational number. Then the triangle cannot be equilateral.

(2) **Collinear points:** Three points $A(x_1,y_1);B(x_2,y_2);C(x_3,y_3)$ are collinear. If area of triangle is zero,

i.e., (i) $\Delta = 0 \Rightarrow \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(ii) $AB+BC=AC$ or $AC+BC=AB$ or $AC+AB=BC$

(3) **Area of a quadrilateral**: If $(x_1,y_1);(x_2,y_2);(x_3,y_3)$ and (x_4,y_4) are vertices of a quadrilateral, then its

Area $=\frac{1}{2}[(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_4-x_4y_3)+(x_4y_1-x_1y_4)]$

Note: If two opposite vertex of rectangle are (x_1, y_1) and (x_2, y_2), then its area is $|(y_2 - y_1)(x_2 - x_1)|$.

It two opposite vertex of a square are $A(x_1, y_1)$ and $C(x_2, y_2)$, then its area is

$$= \frac{1}{2} AC^2 = \frac{1}{2}[(x_2 - x_1)^2 + (y_2 - y_1)^2]$$

(4) **Area of polygon**: The area of polygon whose vertices are $(x_1, y_1); (x_2, y_2); (x_3, y_3); .... (x_n, y_n)$ is

$$= \frac{1}{2} |\{(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + ... + (x_n y_1 - x_1 y_n)\}|$$

or **Stair method:** Repeat first co-ordinates one time in last for down arrow use positive sign and for up arrow use negative sign.

$$\therefore \quad \text{Area of polygon} = \frac{1}{2} \left| \begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \vdots & \vdots \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{array} \right| = \frac{1}{2} |\{(x_1 y_2 + x_2 y_3 + ... + x_n y_1) - (y_1 x_2 + y_2 x_3 + ... + y_n x_1)\}|$$

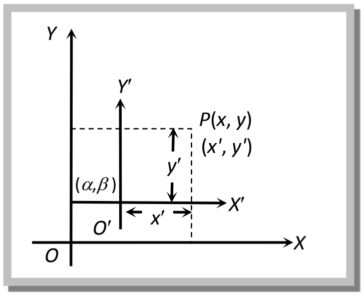
9. Transformation of Axes.

(1) **Shifting of origin without rotation of axes**: Let $P \equiv (x, y)$ with respect to axes OX and OY.

Let $O' \equiv (\alpha, \beta)$ with respect to axes OX and OY and let $P \equiv (x', y')$ with respect to axes O'X' and O'Y', where OX and O'X' are parallel and OY and O'Y' are parallel.

Then $x = x' + \alpha, y = y' + \beta$ or $x' = x - \alpha, y' = y - \beta$

Thus if origin is shifted to point (α, β) without rotation of axes, then new equation of curve can be obtained by putting $x + \alpha$ in place of x and $y + \beta$ in place of y.



(2) **Rotation of axes without changing the origin:** Let O be the origin. Let $P \equiv (x, y)$ with respect to axes OX and OY and let $P \equiv (x', y')$ with respect to axes OX′ and OY′ where

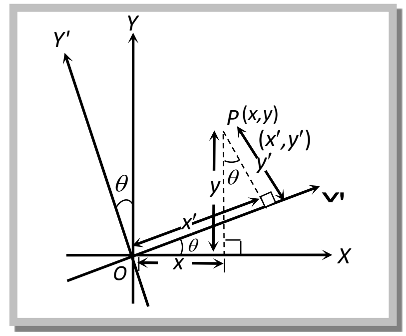
$\angle X'OX = \angle YOY' = \theta$

Then $x = x'\cos\theta - y'\sin\theta$

$y = x'\sin\theta + y'\cos\theta$

and $x' = x\cos\theta + y\sin\theta$

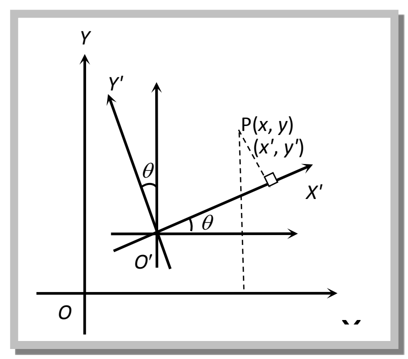
$y' = -x\sin\theta + y\cos\theta$



The above relation between (x, y) and (x', y') can be easily obtained with the help of following table

	$x\downarrow$	$y\downarrow$
$x'\rightarrow$	$\cos\theta$	$\sin\theta$
$y'\rightarrow$	$-\sin\theta$	$\cos\theta$

(3) **Change of origin and rotation of axes:** If origin is changed to $O'(\alpha, \beta)$ and axes are rotated about the new origin O' by an angle θ in the anticlock-wise sense such that the new co-ordinates of $P(x, y)$ become (x', y') then the equations of transformation will be $x = \alpha + x'\cos\theta - y'\sin\theta$ and $y = \beta + x'\sin\theta + y'\cos\theta$



(4) **Reflection (Image of a point):** Let (x, y) be any point, then its image with respect to

(i) x axis $\Rightarrow (x, -y)$ (ii) y-axis $\Rightarrow (-x, y)$ (iii) origin $\Rightarrow (-x, -y)$ (iv) line $y = x \Rightarrow (y, x)$

10. Locus.

Locus: The curve described by a point which moves under given condition or conditions is called its locus.

Equation to the locus of a point: The equation to the locus of a point is the relation, which is satisfied by the coordinates of every point on the locus of the point.

Algorithm to find the locus of a point

Step I: Assume the coordinates of the point say (h, k) whose locus is to be found.

Step II: Write the given condition in mathematical form involving h , k.

Step III: Eliminate the variable (s), if any.

Step IV: Replace h by x and k by y in the result obtained in step III. The equation so obtained is the locus of the point which moves under some stated condition (s)

Note: Locus of a point P which is equidistant from the two point A and B is a straight line and is a perpendicular bisector of line AB.

- In above case if PA = kPB where $k \neq 1$, then the locus of P is a circle.
- Locus of P if A and B is fixed.

(a) Circle, if $\angle APB$ = constant

(b) Circle with diameter AB, if $\angle APB = \frac{\pi}{2}$

(c) Ellipse, if PA +PB = constant

(d) Hyperbola, if PA – PB = constant