Mathematics

Sets Theory

Table of Content

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1. Definitions.

A set is a well-defined class or collection of objects. By a well-defined collection we mean that there exists a rule with the help of which it is possible to tell whether a given object belongs or does not belong to the given collection. The objects in sets may be anything, numbers, people, mountains, rivers etc. The objects constituting the set are called elements or members of the set.

A set is often described in the following two ways.

(1) **Roster method or Listing method:** In this method a set is described by listing elements, separated by commas, within braces {}. The set of vowels of English alphabet may be described as {a, e, i, o, u}.

The set of even natural numbers can be described as {2, 4, 6.........}. Here the dots stand for 'and so on'.

Note: ❑ The order in which the elements are written in a set makes no difference. Thus {a, e, i, o, u} and {e, a, i, o, u} denote the same set. Also the repetition of an element has no effect. For example, {1, 2, 3, and 2} is the same set as {1, 2, and 3}

(2) **Set-builder method or Rule method:** In this method, a set is described by a characterizing property P(x) of its elements x. In such a case the set is described by {x: P(x) holds} or {x | P(x) holds}, which is read as 'the set of all x such that P(x) holds'. The symbol '|' or ':' is read as 'such that'.

The set E of all even natural numbers can be written as

E = {x | x is natural number and x = 2n for n $\in$ N}

or E = {x | x $\in$ N, x = 2n, n $\in$ N}

or E = {x $\in$ N | x = 2n, n $\in$ N}

The set $A = \{0, 1, 4, 9, 16, ...\}$ can be written as $A = \{x^2 \mid x \in Z\}$

Note: Symbols

Symbol	Meaning
$\Rightarrow$	Implies
$\in$	Belongs to
A $\subset$ B	A is a subset of B
$\Leftrightarrow$	Implies and is implied by
$\notin$	Does not belong to
s.t. $\forall$	Such that For every
$\exists$	There exists

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Symbol	Meaning
iff	If and only if
&	And
a | b	a is a divisor of b
N	Set of natural numbers
I or Z	Set of integers
R	Set of real numbers
C	Set of complex numbers
Q	Set of rational numbers

2. Types of Sets.

(1) **Null set or Empty set:** The set which contains no element at all is called the null set. This set is sometimes also called the 'empty set' or the 'void set'. It is denoted by the symbol ϕ or {}.

A set which has at least one element is called a non-empty set.

Let $A = \{x : x^2 + 1 = 0 \text{ and } x$ is real)

Since there is no real number which satisfies the equation $x^2 + 1 = 0$, therefore the set A is empty set.

Note: If A and B are any two empty sets, then $x \in A$ iff $x \in B$ is satisfied because there is no element x in either A or B to which the condition may be applied. Thus A = B. Hence, there is only one empty set and we denote it by ϕ. Therefore, article 'the' is used before empty set.

(2) **Singleton set:** A set consisting of a single element is called a singleton set. The set {5} is a singleton set.

(3) **Finite set:** A set is called a finite set if it is either void set or its elements can be listed (counted, labelled) by natural number 1, 2, 3, ... and the process of listing terminates at a certain natural number n (say).

Cardinal number of a finite set: The number n in the above definition is called the cardinal number or order of a finite set A and is denoted by n(A) or O(A).

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(4) **Infinite set:** A set whose elements cannot be listed by the natural numbers 1, 2, 3, ..., n, for any natural number n is called an infinite set.

(5) **Equivalent set:** Two finite sets A and B are equivalent if their cardinal numbers are same i.e. n (A) = n (B).

Example: $A = \{1, 3, 5, 7\}$; $B = \{10,\ 12, 14, 16\}$ are equivalent sets $[\because\ O(A) = O(B) = 4]$

(6) **Equal set:** Two sets A and B are said to be equal iff every element of A is an element of B and also every element of B is an element of A. We write "A = B" if the sets A and B are equal and "A ≠ B" if the sets A and B are not equal. Symbolically, A = B if x ∈ A ⇔ x ∈ B.

The statement given in the definition of the equality of two sets is also known as the axiom of extension.

Example: If $A = \{2, 3, 5, 6\}$ and $B = \{6, 5, 3, 2\}$. Then $A = B,$ because each element of A is an element of B and vice-versa.

Note: Equal sets are always equivalent but equivalent sets may need not be equal set.

(7) **Universal set:** A set that contains all sets in a given context is called the universal set.

or

A set containing of all possible elements which occur in the discussion is called a universal set and is denoted by U.

Thus in any particular discussion, no element can exist out of universal set. It should be noted that universal set is not unique. It may differ in problem to problem.

(8) **Power set**: If S is any set, then the family of all the subsets of S is called the power set of S.

The power set of S is denoted by P(S). Symbolically, P(S) = {T: T ⊆ S}. Obviously ϕ and S are both elements of P(S).

Example**:** Let S = {a, b, c}, then P(S) = {ϕ , {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}}.

Note: ❑ If $A = \phi,$ then $P(A)$ has one element ϕ , $\therefore n[P(A)] = 1$

❑ Power set of a given set is always non-empty.

❑ If A has n elements, then P (A) has 2^n elements.

❑ $P(\phi) = \{\phi\}$

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$P(P(\phi)) = \{\phi, \{\phi\}\} \Rightarrow P[P(P(\phi))] = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$

Hence $n\{P[P(P(\phi))]\} = 4$.

(9) **Subsets (Set inclusion):** Let A and B be two sets. If every element of A is an element of B, then A is called a subset of B.

If A is subset of B, we write A $\subseteq$ B, which is read as "A is a subset of B" or "A is contained in B".

Thus, A $\subseteq$ B $\Rightarrow$ a $\in$ A $\Rightarrow$ a $\in$ B.

Note: ❑ Every set is a subset of itself.

❑ The empty set is a subset of every set.

❑ The total number of subset of a finite set containing n elements is 2^n.

Proper and improper subsets: If A is a subset of B and $A \neq B$, then A is a proper subset of B. We write this as $A \subset B$.

The null set ϕ is subset of every set and every set is subset of itself, i.e., $\phi \subset A$ and $A \subseteq A$ for every set A. They are called improper subsets of A. Thus every non-empty set has two improper subsets. It should be noted that ϕ has only one subset ϕ which is improper. Thus A has two improper subsets iff it is non-empty.

All other subsets of A are called its proper subsets. Thus, if $A \subset B$, $A \neq B$, $A \neq \phi$, then A is said to be proper subset of B.

Example: Let $A = \{1, 2\}$. Then A has $\phi; \{1\}, \{2\}, \{1, 2\}$ as its subsets out of which ϕ and {1, 2} are improper and {1} and {2} are proper subsets.

3. Venn-Euler Diagrams.

The combination of rectangles and circles are called Venn-Euler diagrams or simply **Venn-diagrams**.

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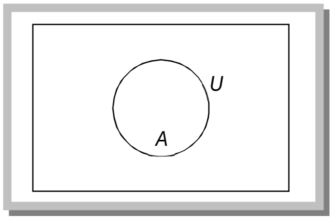
In venn-diagrams the universal set U is represented by points within a rectangle and its subsets are represented by points in closed curves (usually circles) within the rectangle. If a set A is a subset of a set B, then the circle representing A is drawn inside the circle representing B. If A and B are not equal but they have some common elements, then to represent A and B we draw two intersecting circles. Two disjoints sets are represented by two non-intersecting circles.



4. Operations on Sets.

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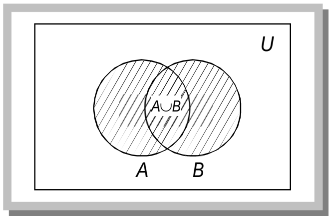
(1) **Union of sets:** Let A and B be two sets. The union of A and B is the set of all elements which are in set A or in B. We denote the union of A and B by $A \cup B$

Which is usually read as "A union B".

Symbolically, $A \cup B = \{x : x \in A \text{ or } x \in B\}$.



It should be noted here that we take standard mathematical usage of "or". When we say that x $\in$ A or x $\in$B we do not exclude the possibility that x is a member of both A and B.

Note: If $A_1, A_2, \ldots\ldots, A_n$ is a finite family of sets, then their union is denoted by $\bigcup_{i=1}^{n} A_i$ or $A_1 \cup A_2 \cup A_3 \ldots\ldots \cup A_n$.

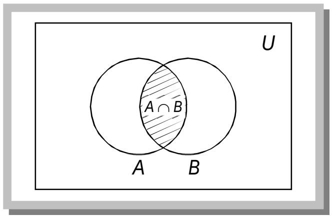
(2) **Intersection of sets:** Let A and B be two sets. The intersection of A and B is the set of all those elements that belong to both A and B.

The intersection of A and B is denoted by A $\cap$ B (read as "A intersection B")

Thus, A $\cap$ B = {x : x $\in$ A and x $\in$ B}.

Clearly, x $\in$ A $\cap$ B $\Leftrightarrow$ x $\in$ A and x $\in$ B.

In fig. the shaded region represents A $\cap$ B. Evidently A $\cap$ B $\subseteq$ A, A $\cap$ B $\subseteq$ B.



Note: If $A_1, A_2, A_3 \ldots\ldots, A_n$ is a finite family of sets, then their intersection is denoted by $\bigcap_{i=1}^{n} A_i$ or

$$A_1 \cap A_2 \cap A_3 \cap \ldots\ldots \cap A_n.$$

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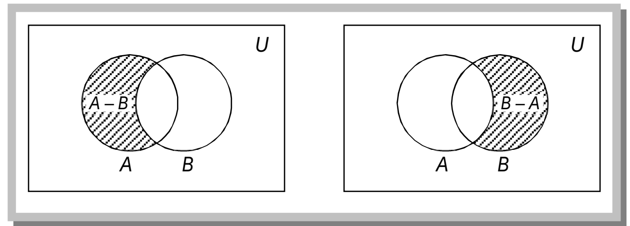
(3) **Disjoint sets:** Two sets A and B are said to be disjoint, if $A \cap B = \phi$. If $A \cap B \neq \phi$, then A and B are said to be non-intersecting or non-overlapping sets.

In other words, if A and B have no element in common, then A and B are called disjoint sets.

Example: Sets {1, 2}; {3, 4} are disjoint sets.

(4) **Difference of sets:** Let A and B be two sets. The difference of A and B written as A – B, is the set of all those elements of A which do not belong to B.



Thus, $A - B = \{x: x \in A \text{ and } x \notin B\}$

or $\quad A - B = \{x \in A: x \notin B\}$

Clearly, $x \in A - B \Leftrightarrow x \in A$ and $x \notin B$. In fig. the shaded part represents A – B.

Similarly, the difference $B - A$ is the set of all those elements of B that do not belong to A i.e.

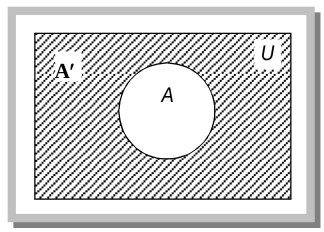
$B - A = \{x \in B : x \notin A\}$

Example: Consider the sets $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A - B = \{1, 2\}; B - A = \{4, 5\}$

As another example, $R - Q$ is the set of all irrational numbers.

(5) **Symmetric difference of two sets:** Let A and B be two sets. The symmetric difference of sets A and B is the set $(A - B) \cup (B - A)$ and is denoted by $A \Delta B$. Thus, $A \Delta B = (A - B) \cup (B - A) = \{x : x \notin A \cap B\}$

(6) **Complement of a set:** Let U be the universal set and let A be a set such that $A \subset U$. Then, the complement of A with respect to U is denoted by A′ or A^c or $C(A)$ or U – A and is defined the set of all those elements of U which are not in A.



Thus, $\quad A' = \{x \in U: x \notin A\}$.

Clearly, $\qquad x \in A' \Leftrightarrow x \notin A$

Example: Consider $U = \{1, 2, ..., 10\}$ and $A = \{1, 3, 5, 7, 9\}$.

Then $A' = \{2, 4, 6, 8, 10\}$

5. Some Important Results on Number of Elements in Sets.

If A, B and C are finite sets and U be the finite universal set, then

(1) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

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(2) $n(A \cup B) = n(A) + n(B) \Leftrightarrow A, B$ are disjoint non-void sets.

(3) $n(A - B) = n(A) - n(A \cap B)$ i.e. $n(A - B) + n(A \cap B) = n(A)$

(4) $n(A \Delta B)$ = Number of elements which belong to exactly one of A or B

$= n((A - B) \cup (B - A))$

$= n(A - B) + n(B - A)$ [$\because$ (A – B) and (B – A) are disjoint]

$= n(A) - n(A \cap B) + n(B) - n(A \cap B) = n(A) + n(B) - 2n(A \cap B)$

(5) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

(6) n (Number of elements in exactly two of the sets A, B, C) $= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$

(7) n (Number of elements in exactly one of the sets A, B, C) $= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$

(8) $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$

(9) $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$

6. Laws of Algebra of Sets.

(1) **Idempotent laws:** For any set A, we have

(i) $A \cup A = A$ (ii) $A \cap A = A$

(2) **Identity laws:** For any set A, we have

(i) $A \cup \phi = A$ (ii) $A \cap U = A$

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i.e. ϕ and U are identity elements for union and intersection respectively.

(3) **Commutative laws:** For any two sets A and B, we have

(i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$ (iii) $A \Delta B = B \Delta A$

i.e. union, intersection and symmetric difference of two sets are commutative.

(iv) $A - B \neq B - A$ (iv) $A \times B \neq B \times A$

i.e., difference and Cartesian product of two sets are not commutative

(4) **Associative laws:** If A, B and C are any three sets, then

(i) $(A \cup B) \cup C = A \cup (B \cup C)$ (ii) $A \cap (B \cap C) = (A \cap B) \cap C$ (iii) $(A \Delta B) \Delta C = A \Delta (B \Delta C)$

i.e., union, intersection and symmetric difference of two sets are associative.

(iv) $(A - B) - C \neq A - (B - C)$ (v) $(A \times B) \times C \neq A \times (B \times C)$

i.e., difference and Cartesian product of two sets are not associative.

(5) **Distributive law:** If A, B and C are any three sets, then

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

i.e. union and intersection are distributive over intersection and union respectively.

(iii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$ (iv) $A \times (B \cup C) = (A \times B) \cup (A \times C)$ (v) $A \times (B - C) = (A \times B) - (A \times C)$

(6) **De-Morgan's law:** If A and B are any two sets, then

(i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

(iii) $A - (B \cup C) = (A - B) \cap (A - C)$ (iv) $A - (B \cap C) = (A - B) \cup (A - C)$

Note: **Theorem 1:** If A and B are any two sets, then

(i) $A - B = A \cap B'$ (ii) $B - A = B \cap A'$

(iii) $A - B = A \Leftrightarrow A \cap B = \phi$ (iv) $(A - B) \cup B = A \cup B$

(v) $(A - B) \cap B = \phi$ (vi) $A \subseteq B \Leftrightarrow B' \subseteq A'$

(viii) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

Theorem 2: If A, B and C are any three sets, then

(i) $A - (B \cap C) = (A - B) \cup (A - C)$ (ii) $A - (B \cup C) = (A - B) \cap (A - C)$

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(iii) $A \cap (B - C) = (A \cap B) - (A \cap C)$ (iv) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$

7. Cartesian Product of Sets.

Cartesian product of sets: Let A and B be any two non-empty sets. The set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called the Cartesian product of the sets A and B and is denoted by $A \times B$.

Thus, $A \times B = [(a, b): a \in A \text{ and } b \in B]$

If $A = \phi$ or $B = \phi$, then we define $A \times B = \phi$.

Example: Let A = {a, b, c} and B = {p, q}.

Then A × B = {(a, p), (a, q), (b, p), (b, q), (c, p), (c, q)}

Also B × A = {(p, a), (p, b), (p, c), (q, a), (q, b), (q, c)}

Important theorems on Cartesian product of sets:

Theorem 1: For any three sets A, B, C

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$ (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Theorem 2: For any three sets A, B, C

$$A \times (B - C) = (A \times B) - (A \times C)$$

Theorem 3: If A and B are any two non-empty sets, then

$$A \times B = B \times A \Leftrightarrow A = B$$

Theorem 4: If $A \subseteq B$, then $A \times A \subseteq (A \times B) \cap (B \times A)$

Theorem 5: If $A \subseteq B$, then $A \times C \subseteq B \times C$ for any set C.

Theorem 6: If $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$

Theorem 7: For any sets A, B, C, D

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$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$

Theorem 8: For any three sets A, B, C

$A \times (B' \cup C')' = (A \times B) \cap (A \times C)$

(ii) $A \times (B' \cap C')' = (A \times B) \cup (A \times C)$

Theorem 9: Let A and B two non-empty sets having n elements in common, then $A \times B$ and $B \times A$ have n^2 elements in common.

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