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Knowledge... Everywhere

Mathematics

Conic Section - Parabola

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1. Definition.

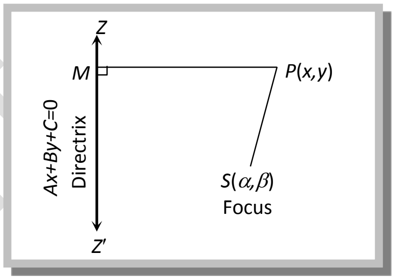
A parabola is the locus of a point which moves in a plane such that its distance from a fixed point (*i.e.,* focus) in the plane is always equal to its distance from a fixed straight line (*i.e.*, directrix) in the same plane.

General equation of a parabola: Let *S* be the focus, ZZ' be the directrix and let *P* be any point on the parabola. Then by definition,

$SP = PM \qquad (\because e = 1)$

$\sqrt{(x-\alpha)^2 + (y-\beta)^2} = \dfrac{Ax + By + C}{\sqrt{A^2 + B^2}}$

Or $\quad (A^2 + B^2)\{(x-\alpha)^2 + (y-\beta)^2\} = (Ax + By + C)^2$



2. Standard equation of the Parabola.

Let *S* be the focus ZZ' be the directrix of the parabola and (x, y) be any point on parabola.

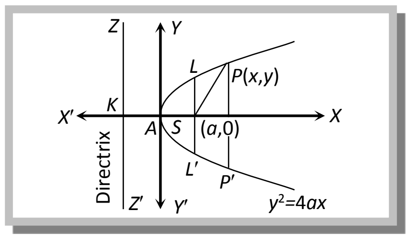
Let $AS = AK = a(>0)$ then coordinate of *S* is (*a*, 0) and the equation of *KZ* is $x = -a$ or $x + a = 0$

Now $SP = PM \Rightarrow (SP)^2 = (PM)^2$

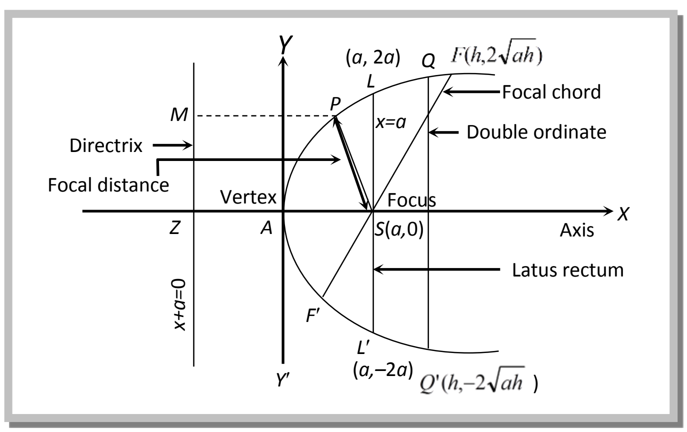
$\Rightarrow (x-a)^2 + (y-0)^2 = (a+x)^2$

$\therefore \boxed{y^2 = 4ax}$

Which is the equation of the parabola in its standard form.



Some terms related to parabola



For the parabola $y^2 = 4ax$,

(1) **Axis:** A straight line passes through the focus and perpendicular to the directrix is called the axis of parabola.
For the parabola $y^2 = 4ax$, *x*-axis is the axis. Here all powers of y are even in $y^2 = 4ax$. Hence parabola $y^2 = 4ax$ is symmetrical about *x*-axis.

(2) **Vertex:** The point of intersection of a parabola and its axis is called the vertex of the parabola. The vertex is the middle point of the focus and the point of intersection of axis and the directrix.
For the parabola $y^2 = 4ax$, $A(0,0)$ i.e., the origin is the vertex.

(3) **Double-ordinate:** The chord which is perpendicular to the axis of parabola or parallel to directrix is called double ordinate of the parabola.
Let QQ' be the double-ordinate. If abscissa of Q is h then ordinate of Q, $y^2 = 4ah$ or $y = 2\sqrt{ah}$ (for I^{st} Quadrant) and ordinate of Q' is $y = -2\sqrt{ah}$ (for IV[th] Quadrant). Hence coordinates of Q and Q' are $(h, 2\sqrt{ah})$ and $(h, -2\sqrt{ah})$ respectively.

(4) **Latus-rectum:** If the double-ordinate passes through the focus of the parabola, then it is called latus-rectum of the parabola.
Coordinates of the extremeties of the latus rectum are $L(a, 2a)$ and $L'(a, -2a)$ respectively.
Since $LS = L'S = 2a$ $\therefore$ Length of latus rectum $LL' = 2(LS) = 2(L'S) = 4a$.

(5) **Focal Chord:** A chord of a parabola which is passing through the focus is called a focal chord of the parabola. Here *PP'* and *LL'* are the focal chords.

(6) **Focal distance (Focal length):** The focal distance of any point *P* on the parabola is its distance from the focus *S* *i.e.*, *SP*.
Here, Focal distance *SP* = *PM* = $x + a$

Note : ❑ If length of any double ordinate of parabola $y^2 = 4ax$ is 2l, then coordinates of end points of this double ordinate are $\left(\frac{l^2}{4a}, l\right)$ and $\left(\frac{l^2}{4a}, -l\right)$.

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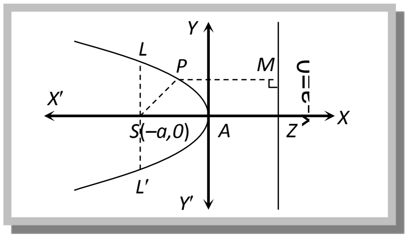
Important Tips

- The area of the triangle inscribed in the parabola $y^2 = 4ax$ is $\frac{1}{8a}(y_1 \sim y_2)(y_2 \sim y_3)(y_3 \sim y_1)$, where $y_1, y_2 y_3$ are the ordinate of the vertices
- The length of the side of an equilateral triangle inscribed in the parabola $y^2 = 4ax$ is $8a\sqrt{3}$ (one angular point is at the vertex).

3. Some other standard forms of Parabola.

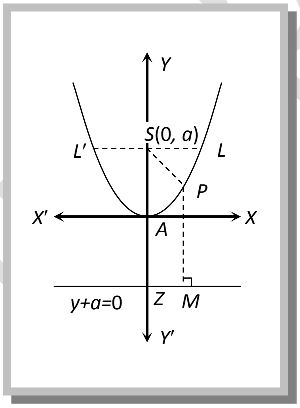
(1) Parabola opening to left wards

(i.e. $y^2 = -4ax$); (a > 0)



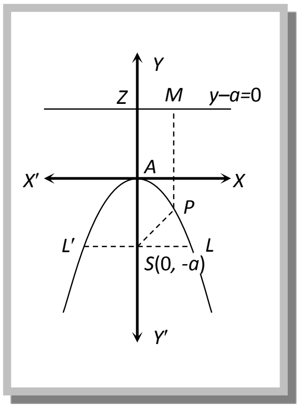
(2) Parabola opening upwards

(i.e. $x^2 = 4ay$); (a>0)



(3) Parabola opening down

(i.e. $x^2 = -4ay$); (a > 0)



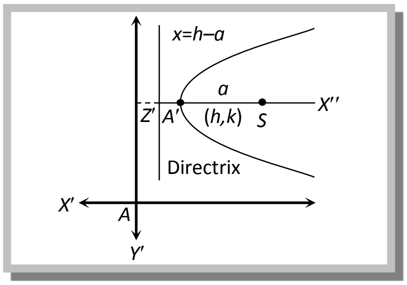
Important terms	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Coordinates of vertex	(0, 0)	(0, 0)	(0, 0)	(0, 0)
Coordinates of focus	(a, 0)	(–a, 0)	(0, a)	(0, –a)
Equation of the directrix	$x = -a$	$x = a$	$y = -a$	y = a
Equation of the axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Length of the latusrectum	$4a$	$4a$	$4a$	$4a$
Focal distance of a point $P(x, y)$	$x + a$	$a - x$	$y + a$	$a - y$

4. Special form of Parabola (y – k)^2 = 4a(x – h).

The equation of a parabola with its vertex at (h, k) and axis as parallel to x-axis is $(y-k)^2 = 4a(x-h)$

If the vertex of the parabola is (p, q) and its axis is parallel to y-axis, then the equation of the parabola is $(x-p)^2 = 4b(y-q)$

When origin is shifted at $A'(h, k)$ without changing the direction of axes, its equation becomes $(y-k)^2 = 4a(x-h)$ or $(x-p)^2 = 4b(y-q)$



Equation of Parabola	**Vertex**	**Axis**	**Focus**	**Directrix**	**Equation of L.R.**	**Length of L.R.**
$(y-K)^2 = 4a(x-h)$	(h, k)	$y = k$	$(h+a, k)$	$x + a - h = 0$	$x = a + h$	$4a$
$(x-p)^2 = 4b(y-q)$	(p, q)	$x = p$	$(p, b+q)$	$y + b - q = 0$	$y = b + q$	$4b$

Important Tips

- $y^2 = 4a(x+a)$ is the equation of the parabola whose focus is the origin and the axis is x-axis.
- $y^2 = 4a(x-a)$ is the equation of parabola whose axis is x-axis and y-axis is directrix.
- $x^2 = 4a(y+a)$ is the equation of parabola whose focus is the origin and the axis is y-axis.
- $x^2 = 4a(y-a)$ is the equation of parabola whose axis is y-axis and the directrix is x-axis.
  - The equation to the parabola whose vertex and focus are on x-axis at a distance a and a' respectively from the origin is $y^2 = 4(a'-a)(x-a)$.
  - The equation of parabola whose axis is parallel to x-axis is $x = Ay^2 + By + C$ and $y = Ax^2 + Bx + C$ is a parabola with its axis parallel to y-axis.

5. Parametric equations of a Parabola.

The simplest and the best form of representing the coordinates of a point on the parabola $y^2 = 4ax$ is $(at^2 2at)$ because these coordinates satisfy the equation $y^2 = 4ax$ for all values of t. The equations $x = at^2, y = 2at$ taken together are called the parametric equations of the parabola $y^2 = 4ax$, t being the parameter.

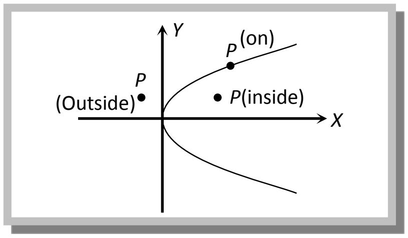
The following table gives the parametric coordinates of a point on four standard forms of the parabola and their parametric equation.

Parabola	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Parametric Coordinates	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$
Parametric Equations	$x = at^2$ $y = 2at$	$x = -at^2$ $y = 2at$	$x = 2at$ $y = at^2$	$x = 2at$, $y = -at^2$

Note: The parametric equation of parabola $(y-k)^2 = 4a(x-h)$ are $x = h + at^2$ and $y = k + 2at$

6. Position of a point and a Line with respect to a Parabola.

(1) **Position of a point with respect to a parabola**: The point $P(x_1, y_1)$ lies outside on or inside the parabola $y^2 = 4ax$ according as $y_1^2 - 4ax_1 >, =, \text{or} < 0$



(2) **Intersection of a line and a parabola:** Let the parabola be $y^2 = 4ax$ .....(i)

And the given line be $y = mx + c$ .....(ii)

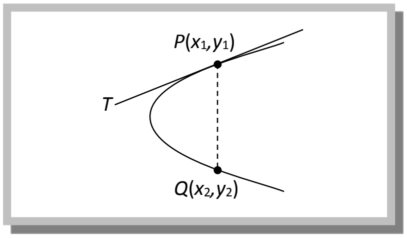
Eliminating y from (i) and (ii) then $(mx+c)^2 = 4ax$ or $m^2x^2 + 2x(mc - 2a) + c^2 = 0$ .....(iii)

This equation being quadratic in x, gives two values of x. It shows that every straight line will cut the parabola in two points, may be real, coincident or imaginary, according as discriminate of (iii) >, = or < 0

$\therefore$ The line $y = mx + c$ does not intersect, touches or intersect a parabola $y^2 = 4ax$, according as $c >, =, < \frac{a}{m}$

7. Equation of Tangent in Different forms.

(1) **Point Form:** The equation of the tangent to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is $yy_1 = 2a(x + x_1)$



Equation of tangent of all other standard parabolas at (x_1, y_1)	
Equation of parabolas	**Tangent at (x_1, y_1)**
$y^2 = -4ax$	$yy_1 = -2a(x + x_1)$
$x^2 = 4ay$	$xx_1 = 2a(y + y_1)$
$x^2 = -4ay$	$xx_1 = -2a(y + y_1)$

Note: The equation of tangent at (x_1, y_1) to a curve can also be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{x + x_1}{2}$, y by $\frac{y + y_1}{2}$ and xy by $\frac{xy_1 + x_1y}{2}$ provided the equation of curve is a polynomial of second degree in x and y.

(2) **Parametric form:** The equation of the tangent to the parabola $y^2 = 4ax$ at $(at^2, 2at)$ is $ty = x + at^2$

Equations of tangent of all other standard parabolas at 't'		
Equations of parabolas	**Parametric co-ordinates 't'**	**Tangent at 't'**
$y^2 = -4ax$	$(-at^2, 2at)$	$ty = -x + at^2$
$x^2 = 4ay$	$(2at, at^2)$	$tx = y + at^2$
$x^2 = -4ay$	$(2at, -at^2)$	$tx = -y + at^2$

(3) **Slope Form:** The equation of a tangent of slope m to the parabola $y^2 = 4ax$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ is

$$y = mx + \frac{a}{m}$$

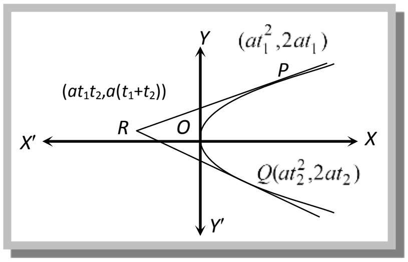
Equation of parabolas	Point of contact in terms of slope (m)	Equation of tangent in terms of slope (m)	Condition of Tangency
$y^2 = 4ax$	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	$y = mx + \frac{a}{m}$	$c = \frac{a}{m}$
$y^2 = -4ax$	$\left(-\frac{a}{m^2}, -\frac{2a}{m}\right)$	$y = mx - \frac{a}{m}$	$c = -\frac{a}{m}$
$x^2 = 4ay$	$(2am, am^2)$	$y = mx - am^2$	$c = -am^2$
$x^2 = -4ay$	$(-2am, -am^2)$	$y = mx + am^2$	$c = am^2$

Important Tips

- If the straight line $lx + my + n = 0$ touches the parabola $y^2 = 4ax$ then $ln = am^2$.
- If the line $x\cos\alpha + y\sin\alpha = p$ touches the parabola $y^2 = 4ax$, then $P\cos\alpha + a\sin^2\alpha = 0$ and point of contact is $(a\tan^2\alpha, -2a\tan\alpha)$
- If the line $\frac{x}{l} + \frac{y}{m} = 1$ touches the parabola $y^2 = 4a(x+b)$, then $m^2(l+b) + al^2 = 0$

8. Point of intersection of Tangents at any two points on the Parabola.

The point of intersection of tangents at two points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is $(at_1t_2, a(t_1 + t_2))$.



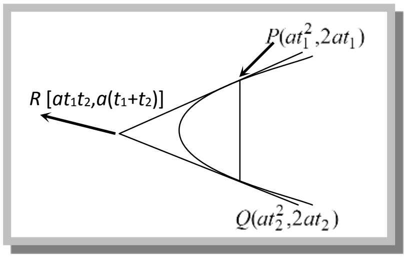
The locus of the point of intersection of tangents to the parabola $y^2 = 4ax$ which meet at an angle α is $(x+a)^2\tan^2\alpha = y^2 - 4ax$.

Director circle: The locus of the point of intersection of perpendicular tangents to a conic is known as its director circle. The director circle of a parabola is its directrix.

Note: Clearly, x-coordinates of the point of intersection of tangents at P and Q on the parabola is the G.M of the x-coordinate of P and Q and y-coordinate is the A.M. of y-coordinate of P and Q.

The equation of the common tangents to the parabola $y^2 = 4ax$ and $x^2 = 4by$ is

$$a^{\frac{1}{3}}x + b^{\frac{1}{3}}y + a^{\frac{2}{3}}b^{\frac{2}{3}} = 0$$



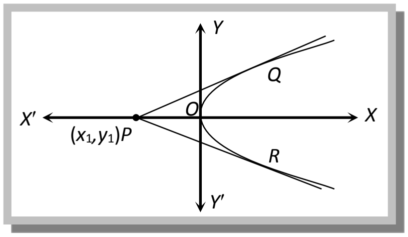
The tangents to the parabola $y^2 = 4ax$ at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ intersect at R.

Then the area of triangle PQR is $\frac{1}{2}a^2(t_1 - t_2)^3$

9. Equation of Pair of Tangents from a point to a Parabola.

If $y_1^2 - 4ax_1 > 0$, then any point $P(x_1, y_1)$ lies out side the parabola and a pair of tangents PQ, PR can be drawn to it from P

The combined equation of the pair of the tangents drawn from a point to a parabola is $SS' = T^2$ where $S = y^2 - 4ax; S' = y_1^2 - 4ax_1$ and $T = yy_1 - 2a(x + x_1)$



Note: The two tangents can be drawn from a point to a parabola. The two tangent are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the parabola.

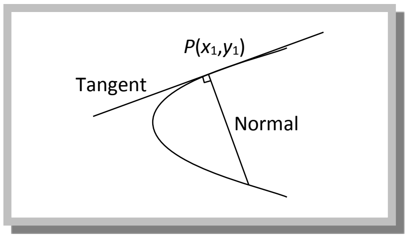
Important Tips

- Tangents at the extremities of any focal chord of a parabola meet at right angles on the directrix.
- Area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.
- If the tangents at the points P and Q on a parabola meet in T, then ST is the geometric mean between SP and SQ, i.e. $ST^2 = SP.SQ$
- Tangent at one extremity of the focal chord of a parabola is parallel to the normal at the other extremity.
- The angle of intersection of two parabolas $y^2 = 4ax$ and $x^2 = 4by$ is given by $\tan^{-1}\frac{3a^{1/3}b^{1/3}}{2(a^{2/3} + b^{2/3})}$

10. Equations of Normal in Different forms.

(1) **Point form:** The equation of the normal to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$



Equation of normals of all other standard parabolas at (x_1, y_1)	
Equation of parabolas	**Normal at (x_1, y_1)**
$y^2 = -4ax$	$y - y_1 = \frac{y_1}{2a}(x - x_1)$
$x^2 = 4ay$	$y - y_1 = -\frac{2a}{x_1}(x - x_1)$
$x^2 = -4ay$	$y - y_1 = \frac{2a}{x_1}(x - x_1)$

(2) **Parametric form**: The equation of the normal to the parabola $y^2 = 4ax$ at $(at^2, 2at)$ is $y + tx = 2at + at^3$

Equations of normal of all other standard parabola at 't'		
Equations of parabolas	**Parametric co-ordinates**	**Normals at 't'**
$y^2 = -4ax$	$(-at^2, 2at)$	$y - tx = 2at + at^3$
$x^2 = 4ay$	$(2at, at^2)$	$x + ty = 2at + at^3$
$x^2 = -4ay$	$(2at, -at^2)$	$x - ty = 2at + at^3$

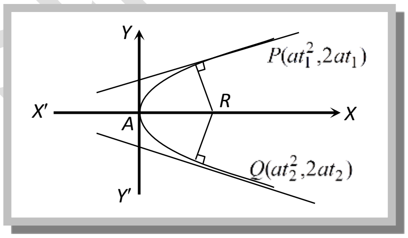
(3) **Slope form:** The equation of normal of slope m to the parabola $y^2 = 4ax$ is $y = mx - 2am - am^3$ at the point $(am^2, -2am)$.

Equations of normal, point of contact, and condition of normality in terms of slope (m)			
Equations of parabola	**Point of contact in terms of slope (m)**	**Equations of normal in terms of slope (m)**	**Condition of normality**
$y^2 = 4ax$	$(am^2, -2am)$	$y = mx - 2am - am^3$	$c = -2am - am^3$
$y^2 = -4ax$	$(-am^2, 2am)$	$y = mx + 2am + am^3$	$c = 2am + am^3$
$x^2 = 4ay$	$\left(-\frac{2a}{m}, \frac{a}{m^2}\right)$	$y = mx + 2a + \frac{a}{m^2}$	$c = 2a + \frac{a}{m^2}$
$x^2 = -4ay$	$\left(\frac{2a}{m}, -\frac{a}{m^2}\right)$	$y = mx - 2a - \frac{a}{m^2}$	$c = -2a - \frac{a}{m^2}$

Note: The line $lx + my + n = 0$ is a normal to the parabola $y^2 = 4ax$ if $al(l^2 + 2m^2) + m^2 n = 0$

11. Point of intersection of normals at any two points on the Parabola.

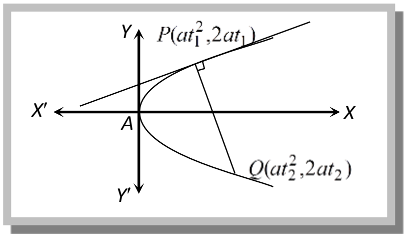
If R is the point of intersection then point of intersection of normals at any two points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is $R[2a + a(t_1^2 + t_2^2 + t_1 t_2), -at_1 t_2(t_1 + t_2)]$



12. Relation between 't_1' and 't_2' if Normal at 't_1' meets the Parabola again at 't_2'

If the normal at the point $P(at_1^2, 2at_1)$ meets the parabola $y^2 = 4ax$ again at $Q(at_2^2, 2at_2)$, then

$$t_2 = -t_1 - \frac{2}{t_1}$$

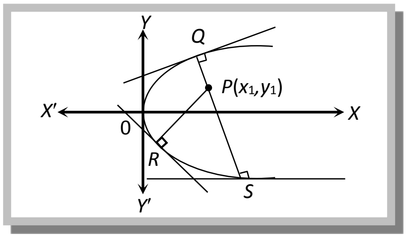


Important Tips

- If the normals at points $(at_1^2, 2at)$ and $(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ meet on the parabola then $t_1 t_2 = 2$
- If the normal at a point $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ subtends a right angle at the vertex of the parabola then $t^2 = 2$.
- If the normal to a parabola $y^2 = 4ax$, makes an angle ϕ with the axis, then it will cut the curve again at an angle $\tan^{-1}\left(\frac{1}{2}\tan\phi\right)$.
- The normal chord to a parabola $y^2 = 4ax$ at the point whose ordinate is equal to abscissa subtends a right angle at the focus.
  - If the normal at two points P and Q of a parabola $y^2 = 4ax$ intersect at a third point R on the curve. Then the product of the ordinate of P and Q is $8a^2$.

13. Co-normal Points.

The points on the curve at which the normals pass through a common point are called co-normal points. Q, R, S are co-normal points. The co- normal points are also called the feet of the normals.



If the normal passes through point $P(x_1, y_1)$ which is not on parabola, then

$y_1 = mx_1 - 2am - am^3 \Rightarrow am^3 + (2a - x_1)m + y_1 = 0$ ......(i)

Which gives three values of m. Let three values of m are m_1, m_2 and m_3, which are the slopes of the normals at Q, R and S respectively, then the coordinates of Q, R and S are $(am_1^2, -2am_1), (am_2^2, -2am_2)$ and $(am_{3,}^2 - 2am_3)$ respectively. These three points are called the feet of the normals.

Now $m_1 + m_2 + m_3 = 0$, $m_1m_2 + m_2m_3 + m_3m_1 = \frac{(2a - x_1)}{a}$ and $m_1m_2m_3 = \frac{-y_1}{a}$

In general, three normals can be drawn from a point to a parabola.

(1) The algebraic sum of the slopes of three concurrent normals is zero.

(2) The sum of the ordinates of the co-normal points is zero.

(3) The centroid of the triangle formed by the co-normal points lies on the axis of the parabola.

(4) The centroid of a triangle formed by joining the foots of the normal of the parabola lies on its axis and is given by $\left(\frac{am_1^2 + am_2^2 + am_3^2}{3}, \frac{2am_1 + 2am_2 + 2am_3}{3}\right) = \left(\frac{am_1^2 + am_2^2 + am_3^2}{3}, 0\right)$

(5) If three normals drawn to any parabola $y^2 = 4ax$ from a given point (h, k) be real, then $h > 2a$ for $a = 1$, normals drawn to the parabola $y^2 = 4x$ from any point (h, k) are real, if $h > 2$.

(6) Out of these three at least one is real, as imaginary normals will always occur in pairs.

14. Circle through Co-normal points.

Equation of the circle passing through the three (co-normal) points on the parabola $y^2 = 4ax$, normal at which pass through a given point (α, β); is $x^2 + y^2 - (2a + \alpha)x - \frac{\beta}{2}y = 0$

(1) The algebraic sum of the ordinates of the four points of intersection of a circle and a parabola is zero.

(2) The common chords of a circle and a parabola are in pairs, equally inclined to the axis of parabola.

(3) The circle through co-normal points passes through the vertex of the parabola.

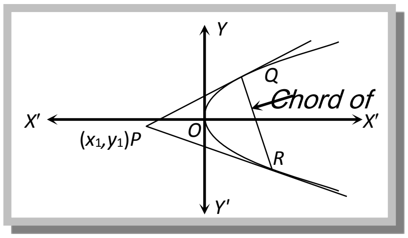
(4) The centroid of four points; in which a circle intersects a parabola, lies on the axis of the parabola.

15. Equation of the Chord of contact of Tangents to a Parabola.

Let PQ and PR be tangents to the parabola $y^2 = 4ax$ drawn from any external point $P(x_1, y_1)$ then QR is called the 'Chord of contact' of the parabola $y^2 = 4ax$.

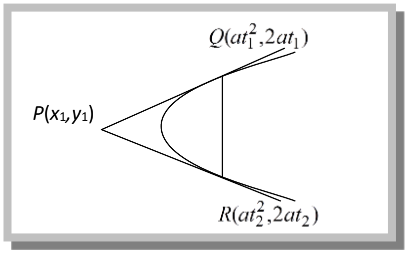
The chord of contact of tangents drawn from a point (x_1, y_1) to the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$

The equation is same as equation of the tangents at the point (x_1, y_1).



Note: The chord of contact joining the point of contact of two perpendicular tangents always passes through focus.

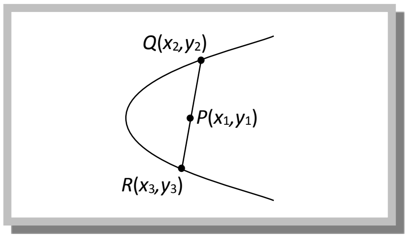
If tangents are drawn from the point (x_1, y_1) to the parabola $y^2 = 4ax$, then the length of their chord of contact is $\frac{1}{|a|}\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}$



The area of the triangle formed by the tangents drawn from (x_1, y_1) to $y^2 = 4ax$ and their chord of contact is $\frac{(y_1^2 - 4ax_1)^{3/2}}{2a}$.

16. Equation of the Chord of the Parabola which is bisected at a given point.

The equation of the chord at the parabola $y^2 = 4ax$ bisected at the point (x_1, y_1) is given by $T = S_1$, where $T = yy_1 - 2a(x + x_1)$ and $S_1 = y_1^2 - 4ax_1$. i.e., $yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$



17. Equation of the Chord joining any two points on the Parabola.

Let $P(at_1^2, 2at_1), Q(at_2^2, 2at_2)$ be any two points on the parabola $y^2 = 4ax$. Then, the equation of the chord joining these points is, $y - 2at_1 = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2}(x - at_1^2)$ or $y - 2at_1 = \frac{2}{t_1 + t_2}\left(x - at_1^2\right)$ or

$y(t_1 + t_2) = 2x + 2at_1t_2$

(1) **Condition for the chord joining points having parameters t_1 and t_2 to be a focal chord:** If the chord joining points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ on the parabola passes through its focus, then $(a, 0)$ satisfies the equation $y(t_1 + t_2) = 2x + 2at_1t_2 \Rightarrow 0 = 2a + 2at_1t_2 \Rightarrow t_1t_2 = -1$ or $t_2 = -\frac{1}{t_1}$

(2) **Length of the focal chord:** The length of a focal chord having parameters t_1 and t_2 for its end points is $a(t_2 - t_1)^2$.

Note: If one extremity of a focal chord is $(at_1^2, 2at_1)$, then the other extremity $(at_2^2, 2at_2)$ becomes $\left(\frac{a}{t_1^2}, \frac{-2a}{t_1}\right)$ by virtue of relation $t_1t_2 = -1$.

If one end of the focal chord of parabola is $(at^2, 2at)$, then other end will be $\left(\frac{a}{t^2}, -2at\right)$ and length of chord $= a\left(t + \frac{1}{t}\right)^2$.

The length of the chord joining two point 't_1' and 't_2' on the parabola $y^2 = 4ax$ is $a(t_1 - t_2)\sqrt{(t_1 + t_2)^2 + 4}$

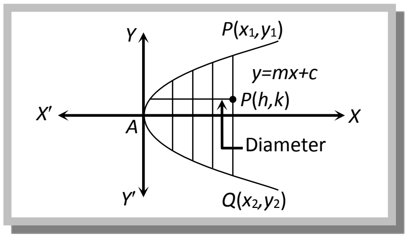
The length of intercept made by line $y = mx + c$ between the parabola $y^2 = 4ax$ is $\frac{4}{m^2}\sqrt{a(1 + m^2)(a - mc)}$.

Important Tips

- The focal chord of parabola $y^2 = 4ax$ making an angle α with the x-axis is of length $4a\operatorname{cosec}^2\alpha$.
- The length of a focal chord of a parabola varies inversely as the square of its distance from the vertex.
- If l_1 and l_2 are the length of segments of a focal chord of a parabola, then its latus-rectum is $\frac{4l_1l_2}{l_1+l_2}$
- The semi latus rectum of the parabola $y^2 = 4ax$ is the harmonic mean between the segments of any focal chord of the parabola.

18. Diameter of a Parabola.

The locus of the middle points of a system of parallel chords is called a diameter and in case of a parabola this diameter is shown to be a straight line which is parallel to the axis of the parabola.



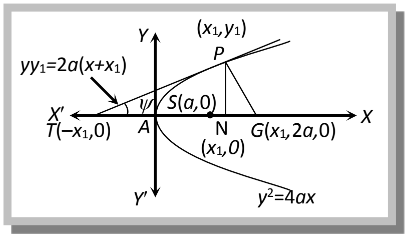
The equation of the diameter bisecting chords of the parabola $y^2 = 4ax$ of slope m is $y = \frac{2a}{m}$

Note: Every diameter of a parabola is parallel to its axis.
The tangent at the end point of a diameter is parallel to corresponding system of parallel chords.
The tangents at the ends of any chord meet on the diameter which bisects the chord.

19. Length of Tangent, Sub tangent, Normal and Subnormal.

Let the parabola $y^2 = 4ax$. Let the tangent and normal at $P(x_1, y_1)$ meet the axis of parabola at T and G respectively, and tangent at $P(x_1, y_1)$ makes angle ψ with the positive direction of x-axis.

$A(0,0)$ is the vertex of the parabola and $PN = y$. Then,



(1) Length of tangent $= PT = PN \operatorname{cosec}\psi = y_1 \operatorname{cosec}\psi$

(2) Length of normal $= PG = PN\operatorname{cosec}(90^o - \psi) = y_1 \sec\psi$

(3) Length of subtangent $= TN = PN\cot\psi = y_1\cot\psi$

(4) Length of subnormal $= NG = PN\cot(90^o - \psi) = y_1\tan\psi$

where , $\tan\psi = \frac{2a}{y_1} = m$, [slope of tangent at P(x, y)]

Length of tangent, subtangent, normal and subnormal to $y^2 = 4ax$ at $(at^2, 2at)$

(1) Length of tangent at $(at^2, 2at) = 2at \operatorname{cosec}\psi \ = 2at\sqrt{(1+\cot^2\psi)} \ = 2at\sqrt{1+t^2}$

(2) Length of normal at $(at^2, 2at) = 2at\sec\psi = 2at\sqrt{(1+\tan^2\psi)} = 2a\sqrt{t^2 + t^2\tan^2\psi} \ = 2a\sqrt{(t^2+1)}$

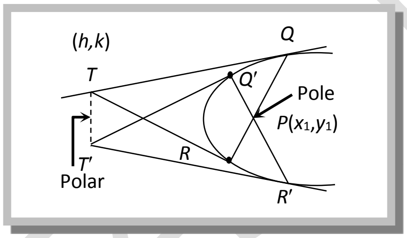
(3) Length of subtangent at $(at^2, 2at) = 2at\cot\psi = 2at^2$

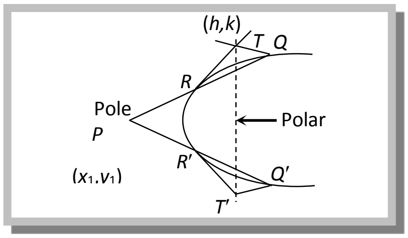
(4) Length of subnormal at $(at^2, 2at) = 2at\tan\psi = 2a$

20. Pole and Polar.

The locus of the point of intersection of the tangents to the parabola at the ends of a chord drawn from a fixed point P is called the polar of point P and the point P is called the pole of the polar.

Equation of polar: Equation of polar of the point (x_1, y_1) with respect to parabola $y^2 = 4ax$ is same as chord of contact and is given by $yy_1 = 2a(x + x_1)$

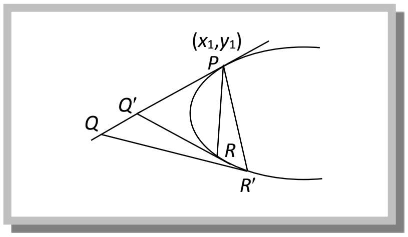




(1) **Polar of the focus is directrix:** Since the focus is $(a, 0)$

$\therefore$ Equation of polar of $y^2 = 4ax$ is $y.0 = 2a(x + a) \Rightarrow x + a = 0,$ which is the directrix of the parabola $y^2 = 4ax$.

(2) **Any tangent is the polar of its point of contact:** If the point $P(x_1 y_1)$ be on the parabola. Its polar and tangent at P are identical. Hence the tangent is the polar of its own point of contact.



Coordinates of pole: The pole of the line $lx + my + n = 0$ with respect to the parabola $y^2 = 4ax$ is $\left(\frac{n}{l}, \frac{-2am}{l}\right)$.

(i) Pole of the chord joining (x_1, y_1) and (x_2, y_2) is $\left(\frac{y_1 y_2}{4a}, \frac{y_1 + y_2}{2}\right)$ which is the same as the point of intersection of tangents at (x_1, y_1) and (x_2, y_2).

(ii) The point of intersection of the polar of two points Q and R is the pole of QR.

21. Characteristics of Pole and Polar.

(1) **Conjugate points:** If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$, then the polar of $Q(x_2, y_2)$ goes through $P(x_1, y_1)$ and such points are said to be conjugate points.

Two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ are conjugate points with respect to the parabola $y^2 = 4ax$, if $y_1 y_2 = 2a(x_1 + x_2)$.

(2) **Conjugate lines:** If the pole of a line $ax + by + c = 0$ lies on the line $a_1 x + b_1 y + c_1 = 0$, then the pole of the second line will lie on the first and such lines are said to be conjugate lines.

Two lines $l_1 x + m_1 y + n_1 = 0$ and $l_2 x + m_2 y + n_2 = 0$ are conjugate lines with respect to parabola $y^2 = 4ax$, if $(l_1 n_2 + l_2 n_1) = 2am_1 m_2$

Note: The chord of contact and polar of any point on the directrix always passes through focus.

The pole of a focal chord lies on directrix and locus of poles of focal chord is the directrix.

The poplars of all points on directrix always pass through a fixed point and this fixed point is focus.