Testprep Kart

Knowledge... Everywhere

Mathematics

Vectors

Table of Content

1. Introduction.

Vectors represent one of the most important mathematical systems, which is used to handle certain types of problems in Geometry, Mechanics and other branches of Applied Mathematics, Physics and Engineering.

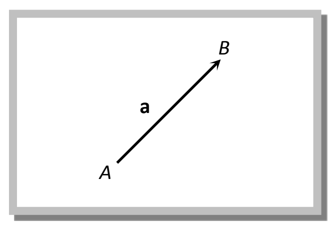
Scalar and vector quantities: Physical quantities are divided into two categories – scalar quantities and vector quantities. Those quantities which have only magnitude and which are not related to any fixed direction in space are called *scalar quantities*, or briefly scalars. Examples of scalars are mass, volume, density, work, temperature etc.

A scalar quantity is represented by a real number along with a suitable unit.

Second kind of quantities are those which have both magnitude and direction. Such quantities are called vectors. Displacement, velocity, acceleration, momentum, weight, force etc. are examples of vector quantities.

2. Representation of Vectors.

Geometrically a vector is represented by a line segment. For example, $\mathbf{a} = \overrightarrow{AB}$. Here *A* is called the initial point and *B*, the terminal point or tip.



Magnitude or modulus of $\mathbf{a}$ is expressed as $|\mathbf{a}| = |\overrightarrow{AB}| = AB$.

Note: The magnitude of a vector is always a non-negative real number.

Every vector $\overrightarrow{AB}$ has the following three characteristics:

Length: The length of $\overrightarrow{AB}$ will be denoted by $|\overrightarrow{AB}|$ or *AB*.

Support: The line of unlimited length of which *AB* is a segment is called the support of the vector $\overrightarrow{AB}$.

Sense: The sense of $\overrightarrow{AB}$ is from *A* to *B* and that of $\overrightarrow{BA}$ is from *B* to *A*. Thus, the sense of a directed line segment is from its initial point to the terminal point.

3. Types of Vector.

(1) **Zero or null vector:** A vector whose magnitude is zero is called zero or null vector and it is represented by $\vec{O}$.
The initial and terminal points of the directed line segment representing zero vector are coincident and its direction is arbitrary.

(2) **Unit vector:** A vector whose modulus is unity, is called a unit vector. The unit vector in the direction of a vector $\mathbf{a}$ is denoted by $\hat{\mathbf{a}}$, read as "*a cap*". Thus, $|\hat{\mathbf{a}}| = 1$.

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\text{Vector } a}{\text{Magnitude of } a}$$

Note: Unit vectors parallel to x-axis, y-axis and z-axis are denoted by **i**, **j** and **k** respectively.
Two unit vectors may not be equal unless they have the same direction.

(3) **Like and unlike vectors:** Vectors are said to be like when they have the same sense of direction and unlike when they have opposite directions.

(4) **Collinear or parallel vectors:** Vectors having the same or parallel supports are called collinear vectors.

(5) **Co-initial vectors:** Vectors having the same initial point are called *co-initial vectors*.

(6) **Co-planar vectors:** A system of vectors is said to be coplanar, if their supports are parallel to the same plane.

Note: Two vectors having the same initial point are always coplanar but such three or more vectors may or may not be coplanar.

(7) **Coterminous vectors:** Vectors having the same terminal point are called *coterminous vectors*.

(8) **Negative of a vector:** The vector which has the same magnitude as the vector $\mathbf{a}$ but opposite direction, is called the negative of $\mathbf{a}$ and is denoted by $-\mathbf{a}$. Thus, if $\overrightarrow{PQ} = \mathbf{a}$, then $\overrightarrow{QP} = -\mathbf{a}$.

(9) **Reciprocal of a vector:** A vector having the same direction as that of a given vector $\mathbf{a}$ but magnitude equal to the reciprocal of the given vector is known as the reciprocal of $\mathbf{a}$ and is denoted by $\mathbf{a}^{-1}$. Thus, if $|\mathbf{a}| = a, |\mathbf{a}^{-1}| = \frac{1}{a}$

Note: A unit vector is self-reciprocal.

(10) **Localized and free vectors:** A vector which is drawn parallel to a given vector through a specified point in space is called a localized vector. For example, a force acting on a rigid body is a localized vector as its effect depends on the line of action of the force. If the value of a vector depends only on its length and direction and is independent of its position in the space, it is called a free vector.

(11) **Position vectors:** The vector $\overrightarrow{OA}$ which represents the position of the point *A* with respect to a fixed point *O* (called origin) is called position vector of the point *A*. If (*x, y, z*) are co-ordinates of the point *A*, then $\overrightarrow{OA} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

(12) **Equality of vectors:** Two vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be equal, if

(i) $|\mathbf{a}| = |\mathbf{b}|$

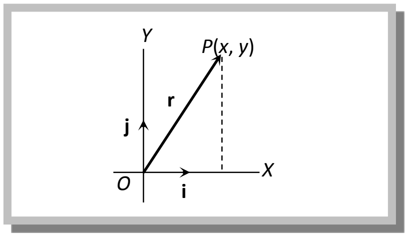
(ii) They have the same or parallel support and

(iii) The same sense.

4. Rectangular resolution of a Vector in Two and Three dimensional systems.

(1) Any vector $\mathbf{r}$ can be expressed as a linear combination of two unit vectors $\mathbf{i}$ and $\mathbf{j}$ at right angle *i.e.,* $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$

The vector $x\mathbf{i}$ and $y\mathbf{j}$ are called the perpendicular component vectors of $\mathbf{r}$. The scalars *x* and *y* are called the components or resolved parts of $\mathbf{r}$ in the directions of *x*-axis and *y*-axis respectively and the ordered pair (*x, y*) is known as co-ordinates of point whose position vector is $\mathbf{r}$.



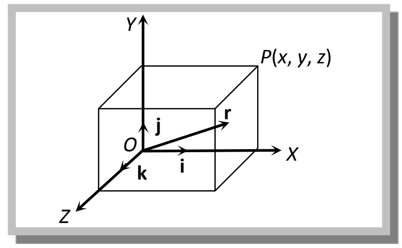
Also the magnitude of $\mathbf{r} = \sqrt{x^2 + y^2}$ and if θ be the inclination of $\mathbf{r}$ with the *x*-axis, then $\theta = \tan^{-1}(y/x)$

(2) If the coordinates of *P* are (*x, y, z*) then the position vector of $\mathbf{r}$ can be written as $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

The vectors $x\mathbf{i}, y\mathbf{j}$ and $z\mathbf{k}$ are called the right angled components of $\mathbf{r}$.

The scalars x, y, z are called the components or resolved parts of $\mathbf{r}$ in the directions of *x*-axis, *y*-axis and *z*-axis respectively and ordered triplet (*x, y, z*) is known as coordinates of *P* whose position vector is $\mathbf{r}$.

Also the magnitude or modulus of $\mathbf{r} = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$



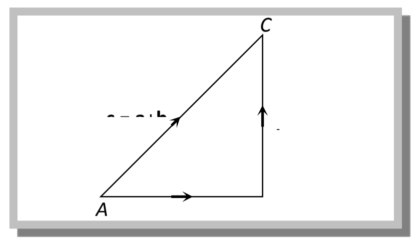
Direction cosines of $\mathbf{r}$ are the cosines of angles that the vector $\mathbf{r}$ makes with the positive direction of *x, y* and *z*-axes. $\cos\alpha = l = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{|\mathbf{r}|}$,

$\cos\beta = m = \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{y}{|\mathbf{r}|}$ and $\cos\gamma = n = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{z}{|\mathbf{r}|}$

Clearly, $l^2 + m^2 + n^2 = 1$. Here $\alpha = \angle POX$, $\beta = \angle POY$ $\gamma = \angle POZ$ and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors along OX, OY, OZ respectively.

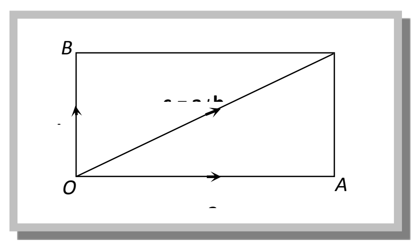
5. Properties of Vectors.

(1) **Addition of vectors**



(i) **Triangle law of addition:** If two vectors are represented by two consecutive sides of a triangle then their sum is represented by the third side of the triangle, but in opposite direction. This is known as the triangle law of addition of vectors. Thus, if $\overrightarrow{AB} = \mathbf{a}, \overrightarrow{BC} = \mathbf{b}$ and $\overrightarrow{AC} = \mathbf{c}$ then $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ *i.e.,* $\mathbf{a} + \mathbf{b} = \mathbf{c}$.

(ii) **Parallelogram law of addition:** If two vectors are represented by two adjacent sides of a parallelogram, then their sum is represented by the diagonal of the parallelogram whose initial point is the same as the initial point of the given vectors. This is known as parallelogram law of addition of vectors.



Thus, if $\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$

Then $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$ *i.e.,* $\mathbf{a} + \mathbf{b} = \mathbf{c}$, where *OC* is a diagonal of the parallelogram *OABC*.

(iii) **Addition in component form :** If the vectors are defined in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, *i.e.,* if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then their sum is defined as
$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k}$.

Properties of vector addition: Vector addition has the following properties.

(a) **Binary operation:** The sum of two vectors is always a vector.

(b) **Commutativity:** For any two vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

(c) **Associativity:** For any three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$

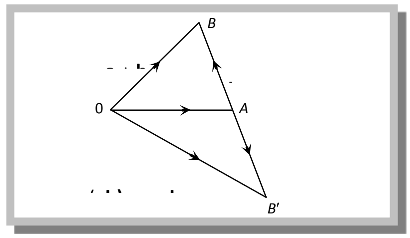
(d) **Identity:** Zero vector is the identity for addition. For any vector $\mathbf{a}$, $\mathbf{0} + \mathbf{a} = \mathbf{a} = \mathbf{a} + \mathbf{0}$

(e) **Additive inverse:** For every vector $\mathbf{a}$ its negative vector $-\mathbf{a}$ exists such that $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$ *i.e.,* $(-\mathbf{a})$ is the additive inverse of the vector $\mathbf{a}$.

(2) **Subtraction of vectors:** If $\mathbf{a}$ and $\mathbf{b}$ are two vectors, then their subtraction $\mathbf{a} - \mathbf{b}$ is defined as $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$ where $-\mathbf{b}$ is the negative of $\mathbf{b}$ having magnitude equal to that of $\mathbf{b}$ and direction opposite to $\mathbf{b}$.

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

Then $\mathbf{a} - \mathbf{b} = (a_1 - b_1)\mathbf{i} + (a_2 - b_2)\mathbf{j} + (a_3 - b_3)\mathbf{k}$.



Properties of vector subtraction

(i) $\mathbf{a} - \mathbf{b} \neq \mathbf{b} - \mathbf{a}$

(ii) $(\mathbf{a} - \mathbf{b}) - \mathbf{c} \neq \mathbf{a} - (\mathbf{b} - \mathbf{c})$

(iii) Since any one side of a triangle is less than the sum and greater than the difference of the other two sides, so for any two vectors *a* and *b*, we have

(a) $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$

(b) $|\mathbf{a} + \mathbf{b}| \geq |\mathbf{a}| - |\mathbf{b}|$

(c) $|\mathbf{a} - \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$

(d) $|\mathbf{a} - \mathbf{b}| \geq |\mathbf{a}| - |\mathbf{b}|$

(3) **Multiplication of a vector by a scalar :** If $\mathbf{a}$ is a vector and *m* is a scalar (*i.e.,* a real number) then $m\mathbf{a}$ is a vector whose magnitude is *m* times that of $\mathbf{a}$ and whose direction is the same as that of $\mathbf{a}$, if *m* is positive and opposite to that of $\mathbf{a}$, if *m* is negative.

$\therefore$ Magnitude of $m\mathbf{a} = |m\mathbf{a}| \Rightarrow m$ (magnitude of $\mathbf{a}$) = $m|\mathbf{a}|$

Again if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ then $m\mathbf{a} = (ma_1)\mathbf{i} + (ma_2)\mathbf{j} + (ma_3)\mathbf{k}$

Properties of Multiplication of vectors by a scalar: The following are properties of multiplication of vectors by scalars, for vectors $\mathbf{a}, \mathbf{b}$ and scalars *m*, *n*

(i) $m(-\mathbf{a}) = (-m)\mathbf{a} = -(m\mathbf{a})$

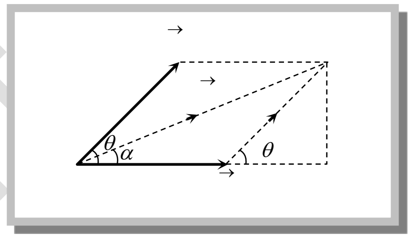
(ii) $(-m)(-\mathbf{a}) = m\mathbf{a}$

(iii) $m(n\mathbf{a}) = (mn)\mathbf{a} = n(m\mathbf{a})$

(iv) $(m+n)\mathbf{a} = m\mathbf{a} + n\mathbf{a}$

(v) $m(\mathbf{a}+\mathbf{b}) = m\mathbf{a} + m\mathbf{b}$

(4) **Resultant of two forces**



$\vec{R} = \vec{P} + \vec{Q}$

$|\vec{R}| = R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$

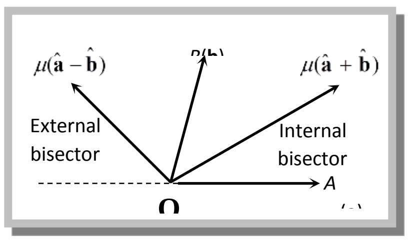
Where $|\vec{P}| = P$, $|\vec{Q}| = Q$, $\tan\alpha = \dfrac{Q\sin\theta}{P + Q\cos\theta}$

Deduction: When $|\vec{P}| = |\vec{Q}|$, *i.e.,* $P = Q$, $\tan\alpha = \dfrac{P\sin\theta}{P + P\cos\theta} = \dfrac{\sin\theta}{1+\cos\theta} = \tan\dfrac{\theta}{2}$; $\therefore$ $\alpha = \dfrac{\theta}{2}$

Hence, the angular bisector of two unit vectors $\mathbf{a}$ and $\mathbf{b}$ is along the vector sum $\mathbf{a}+\mathbf{b}$.

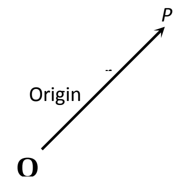
Important Tips

- *The internal bisector of the angle between any two vectors is along the vector sum of the corresponding unit vectors.*
- *The external bisector of the angle between two vectors is along the vector difference of the corresponding unit vectors.*



6. Position Vector.

If a point O is fixed as the origin in space (or plane) and P is any point, then $\overrightarrow{OP}$ is called the position vector of P with respect to O.
If we say that P is the point $\mathbf{r}$, then we mean that the position vector of P is $\mathbf{r}$ with respect to some origin O.

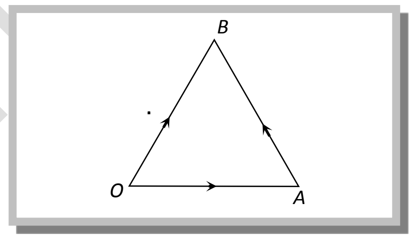


(1) $\overrightarrow{AB}$ **in terms of the position vectors of points *A* and *B*:** If $\mathbf{a}$ and $\mathbf{b}$ are position vectors of points A and B respectively. Then, $\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OB} = \mathbf{b}$

In ΔOAB, we have $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB} \Rightarrow \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$

$\Rightarrow \quad \overrightarrow{AB}$ = (Position vector of B) – (Position vector of A)

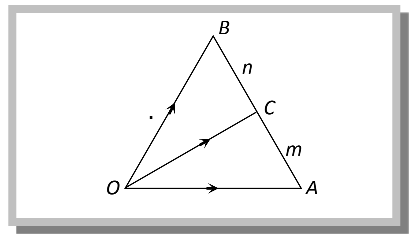
$\Rightarrow \quad \overrightarrow{AB}$ = (Position vector of head) – (Position vector of tail)



(2) **Position vector of a dividing point**

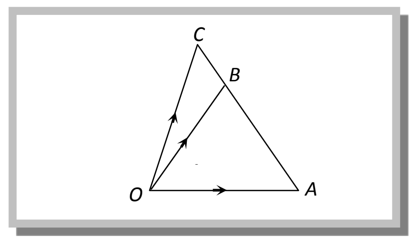
(i) **Internal division:** Let A and B be two points with position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, and let C be a point dividing AB internally in the ratio $m : n$.
Then the position vector of C is given by

$$\overrightarrow{OC} = \frac{m\mathbf{b} + n\mathbf{a}}{m+n}$$



(ii) **External division:** Let A and B be two points with position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively and let C be a point dividing AB externally in the ratio $m : n$.
Then the position vector of C is given by

$$\overrightarrow{OC} = \frac{m\mathbf{b} - n\mathbf{a}}{m-n}$$



Important Tips

- *Position vector of the mid point of AB is* $\frac{\mathbf{a}+\mathbf{b}}{2}$
- *If* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *are position vectors of vertices of a triangle, then position vector of its centroid is* $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$
- *If* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ *are position vectors of vertices of a tetrahedron, then position vector of its centroid is* $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}}{4}$.

7. Linear Combination of Vectors.

A vector $\mathbf{r}$ is said to be a linear combination of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$..... etc, if there exist scalars *x, y, z* etc., such that $\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + ...$

Examples: Vectors $\mathbf{r}_1 = 2\mathbf{a} + \mathbf{b} + 3\mathbf{c}, \mathbf{r}_2 = \mathbf{a} + 3\mathbf{b} + \sqrt{2}\mathbf{c}$ are linear combinations of the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

(1) **Collinear and Non-collinear vectors:** Let $\mathbf{a}$ and $\mathbf{b}$ be two collinear vectors and let $\mathbf{x}$ be the unit vector in the direction of $\mathbf{a}$. Then the unit vector in the direction of $\mathbf{b}$ is $\mathbf{x}$ or $-\mathbf{x}$ according as $\mathbf{a}$ and $\mathbf{b}$ are like or unlike parallel vectors. Now, $\mathbf{a} = |\mathbf{a}|\ \hat{\mathbf{x}}$ and $\mathbf{b} = \pm|\mathbf{b}|\ \hat{\mathbf{x}}$.

$$\therefore \quad \mathbf{a} = \left(\frac{|\mathbf{a}|}{|\mathbf{b}|}\right)|\mathbf{b}|\ \hat{\mathbf{x}} \Rightarrow \mathbf{a} = \left(\pm\frac{|\mathbf{a}|}{|\mathbf{b}|}\right)\mathbf{b} \Rightarrow \mathbf{a} = \lambda\mathbf{b}, \text{ where } \lambda = \pm\frac{|\mathbf{a}|}{|\mathbf{b}|}.$$

Thus, if $\mathbf{a}, \mathbf{b}$ are collinear vectors, then $\mathbf{a} = \lambda\mathbf{b}$ or $\mathbf{b} = \lambda\mathbf{a}$ for some scalar λ.

(2) **Relation between two parallel vectors**

(i) If $\mathbf{a}$ and $\mathbf{b}$ be two parallel vectors, then there exists a scalar *k* such that $\mathbf{a} = k\,\mathbf{b}$.

i.e., there exist two non-zero scalar quantities *x* and *y* so that $x\,\mathbf{a} + y\,\mathbf{b} = \mathbf{0}$.

If $\mathbf{a}$ and $\mathbf{b}$ be two non-zero, non-parallel vectors then $x\mathbf{a} + y\mathbf{b} = \mathbf{0} \Rightarrow x = 0$ and $y = 0$.

$$\text{Obviously } x\mathbf{a} + y\mathbf{b} = \mathbf{0} \Rightarrow \begin{cases} \mathbf{a} = \mathbf{0}, \mathbf{b} = \mathbf{0} \\ \quad\text{or} \\ x = 0, y = 0 \\ \quad\text{or} \\ \mathbf{a} \parallel \mathbf{b} \end{cases}$$

(ii) If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then from the property of parallel vectors, we have

$$\mathbf{a} \parallel \mathbf{b} \Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

(3) **Test of collinearity of three points:** Three points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are collinear iff there exist scalars *x, y, z* not all zero such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{0}$, where $x + y + z = 0$. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j}$, then the points with position vector $\mathbf{a}, \mathbf{b}, \mathbf{c}$ will be collinear iff

$$\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix} = 0.$$

(4) **Test of coplanarity of three vectors:** Let $\mathbf{a}$ and $\mathbf{b}$ two given non-zero non-collinear vectors. Then any vectors $\mathbf{r}$ coplanar with $\mathbf{a}$ and $\mathbf{b}$ can be uniquely expressed as $\mathbf{r} = x\mathbf{a} + y\mathbf{b}$ for some scalars *x* and *y*.

(5) **Test of coplanarity of Four points:** Four points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are coplanar iff there exist scalars *x, y, z, u* not all zero such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + u\mathbf{d} = \mathbf{0}$, where $x + y + z + u = 0$.

Four points with position vectors

$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, $\mathbf{d} = d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$

will be coplanar, *iff* $\begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{vmatrix} = 0$

8. Linear Independence and Dependence of Vectors

(1) **Linearly independent vectors:** A set of non-zero vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots\mathbf{a}_n$ is said to be linearly independent, if $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{0} \Rightarrow x_1 = x_2 = \ldots = x_n = 0$.

(2) **Linearly dependent vectors:** A set of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots\mathbf{a}_n$ is said to be linearly dependent if there exist scalars $x_1, x_2, \ldots, x_n$ not all zero such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{0}$

Three vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ will be linearly dependent vectors *iff* $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$.

Properties of linearly independent and dependent vectors

(i) Two non-zero, non-collinear vectors are linearly independent.

(ii) Any two collinear vectors are linearly dependent.

(iii) Any three non-coplanar vectors are linearly independent.

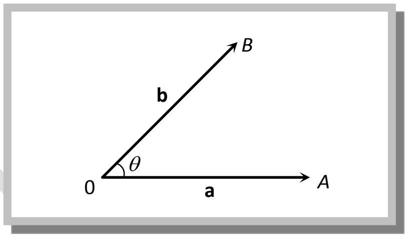
(iv) Any three coplanar vectors are linearly dependent.

(v) Any four vectors in 3-dimensional space are linearly dependent.

9. Product of Two Vectors.

Product of two vectors is processed by two methods. When the product of two vectors results is a scalar quantity, then it is called scalar product. It is also known as dot product because we are putting a dot (.) between two vectors.

When the product of two vectors results is a vector quantity then this product is called vector product. It is also known as cross product because we are putting a cross (×) between two vectors.

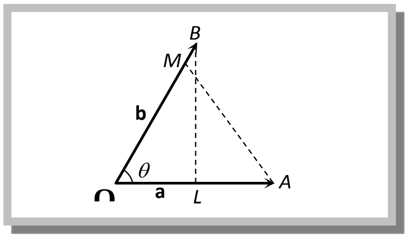


(1) **Scalar or Dot product of two vectors :** If $\mathbf{a}$ and $\mathbf{b}$ are two non-zero vectors and θ be the angle between them, then their scalar product (or dot product) is denoted by $\mathbf{a}.\mathbf{b}$ and is defined as the scalar $|\mathbf{a}||\mathbf{b}|\cos\theta$, where $|\mathbf{a}|$ and $|\mathbf{b}|$ are modulii of $\mathbf{a}$ and $\mathbf{b}$ respectively and $0 \le \theta \le \pi$.

Important Tips

- $\mathbf{a}.\mathbf{b} \in R$
- $\mathbf{a}.\mathbf{b} \le |\mathbf{a}||\mathbf{b}|$
- $\mathbf{a}.\mathbf{b} > 0 \Rightarrow$ *angle between* $\mathbf{a}$ *and* $\mathbf{b}$ *is acute.*
- $\mathbf{a}.\mathbf{b} < 0 \Rightarrow$ *angle between* $\mathbf{a}$ *and* $\mathbf{b}$ *is obtuse.*
- *The dot product of a zero and non-zero vector is a scalar zero.*

(i) **Geometrical Interpretation of scalar product**: Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively. Let θ be the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Draw $BL \perp OA$ and $AM \perp OB$.



From Δs OBL and OAM, we have $OL = OB\cos\theta$ and $OM = OA\cos\theta$. Here OL and OM are known as projection of $\mathbf{b}$ on $\mathbf{a}$ and $\mathbf{a}$ on $\mathbf{b}$ respectively.

Now $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = |\mathbf{a}|(OB\cos\theta) = |\mathbf{a}|(OL)$

$= (\text{Magnitude of } \mathbf{a})(\text{Projection of } \mathbf{b} \text{ on } \mathbf{a})$ .....(i)

Again, $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = |\mathbf{b}|(|\mathbf{a}|\cos\theta) = |\mathbf{b}|(OA\cos\theta) = |\mathbf{b}|(OM)$

a.b = (Magnitude of **b**) (Projection of $\mathbf{a}$ on $\mathbf{b}$) .....(ii)

Thus geometrically interpreted, the scalar product of two vectors is the product of modulus of either vector and the projection of the other in its direction.

(ii) **Angle between two vectors :** If $\mathbf{a}, \mathbf{b}$ be two vectors inclined at an angle θ, then, $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

$$\Rightarrow \cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right)$$

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$; $\theta = \cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}\right)$

(2) **Properties of scalar product**

(i) **Commutativity**: The scalar product of two vector is commutative *i.e.,* $\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a}$.

(ii) **Distributivity of scalar product over vector addition:** The scalar product of vectors is distributive over vector addition *i.e.,*

(a) $\mathbf{a}.(\mathbf{b}+\mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$ (Left distributivity)

(b) $(\mathbf{b}+\mathbf{c}).\mathbf{a} = \mathbf{b}.\mathbf{a} + \mathbf{c}.\mathbf{a}$ (Right distributivity)

(iii) Let $\mathbf{a}$ and $\mathbf{b}$ be two non-zero vectors $\mathbf{a}.\mathbf{b} = 0 \Leftrightarrow \mathbf{a} \perp \mathbf{b}$.

As $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are mutually perpendicular unit vectors along the co-ordinate axes, therefore

$\mathbf{i}.\mathbf{j} = \mathbf{j}.\mathbf{i} = 0$; $\mathbf{j}.\mathbf{k} = \mathbf{k}.\mathbf{j} = 0$; $\mathbf{k}.\mathbf{i} = \mathbf{i}.\mathbf{k} = 0$.

(iv) For any vector $\mathbf{a}, \mathbf{a}.\mathbf{a} = |\mathbf{a}|^2$.

As $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along the co-ordinate axes, therefore $\mathbf{i}.\mathbf{i} = |\mathbf{i}|^2 = 1$, $\mathbf{j}.\mathbf{j} = |\mathbf{j}|^2 = 1$ and $\mathbf{k}.\mathbf{k} = |\mathbf{k}|^2 = 1$

(v) If m is a scalar and $\mathbf{a}, \mathbf{b}$ be any two vectors, then $(m\mathbf{a}).\mathbf{b} = m(\mathbf{a}.\mathbf{b}) = \mathbf{a}.(m\mathbf{b})$

(vi) If *m, n* are scalars and $\mathbf{a}, \mathbf{b}$ be two vectors, then $m\mathbf{a}.n\mathbf{b} = mn(\mathbf{a}.\mathbf{b}) = (mn\mathbf{a}).\mathbf{b} = \mathbf{a}.(mn\mathbf{b})$

(vii) For any vectors $\mathbf{a}$ and $\mathbf{b}$, we have (a) $\mathbf{a}.(-\mathbf{b}) = -(\mathbf{a}.\mathbf{b}) = (-\mathbf{a}).\mathbf{b}$ (b) $(-\mathbf{a}).(-\mathbf{b}) = \mathbf{a}.\mathbf{b}$

(viii) For any two vectors $\mathbf{a}$ and $\mathbf{b}$, we have

(a) $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a}.\mathbf{b}$

(b) $|\mathbf{a}-\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a}.\mathbf{b}$

(c) $(\mathbf{a}+\mathbf{b}).(\mathbf{a}-\mathbf{b}) = |\mathbf{a}|^2 - |\mathbf{b}|^2$

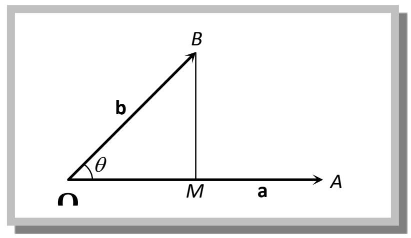
(d) $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}| + |\mathbf{b}| \Rightarrow \mathbf{a} \parallel \mathbf{b}$

(e) $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 \Rightarrow \mathbf{a} \perp \mathbf{b}$

(f) $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}| \Rightarrow \mathbf{a} \perp \mathbf{b}$

(3) **Scalar product in terms of components**. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$,
Then, $\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$. Thus, scalar product of two vectors is equal to the sum of the products of their corresponding components. In particular, $\mathbf{a}.\mathbf{a} = |\mathbf{a}|^2 = a_1^2 + a_2^2 + a_3^2$.

(4) **Components of a vector along and perpendicular to another vector:** If $\mathbf{a}$ and $\mathbf{b}$ be two vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$. Draw $BM \perp OA$. In ΔOBM, we have $\overrightarrow{OB} = \overrightarrow{OM} + \overrightarrow{MB} \Rightarrow \mathbf{b} = \overrightarrow{OM} + \overrightarrow{MB}$



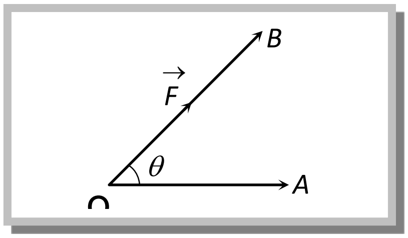
Thus, $\overrightarrow{OM}$ and $\overrightarrow{MB}$ are components of $\mathbf{b}$ along $\mathbf{a}$ and perpendicular to $\mathbf{a}$ respectively.

Now, $\overrightarrow{OM} = (OM)\,\hat{\mathbf{a}} = (OB\cos\theta)\,\hat{\mathbf{a}} = (|\mathbf{b}|\cos\theta)\hat{\mathbf{a}} = \left(|\mathbf{b}|\dfrac{(\mathbf{a}.\mathbf{b})}{|\mathbf{a}||\mathbf{b}|}\right)\hat{\mathbf{a}} = \left(\dfrac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|}\right)\hat{\mathbf{a}} = \left(\dfrac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = \left(\dfrac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$

$\therefore\ \mathbf{b} = \overrightarrow{OM} + \overrightarrow{MB} \Rightarrow \overrightarrow{MB} = \mathbf{b} - \overrightarrow{OM} = \mathbf{b} - \left(\dfrac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$

Thus, the components of $\mathbf{b}$ along and perpendicular to $\mathbf{a}$ are $\left(\dfrac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$ and $\mathbf{b} - \left(\dfrac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$ respectively.

(5) **Work done by a force:** A force acting on a particle is said to do work if the particle is displaced in a direction which is not perpendicular to the force.



The work done by a force is a scalar quantity and its measure is equal to the product of the magnitude of the force and the resolved part of the displacement in the direction of the force.

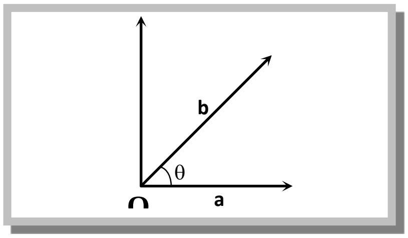
If a particle be placed at *O* and a force $\vec{F}$ represented by $\overrightarrow{OB}$ be acting on the particle at *O*. Due to the application of force $\vec{F}$ the particle is displaced in the direction of $\overrightarrow{OA}$. Let $\overrightarrow{OA}$ be the displacement. Then the component of $\overrightarrow{OA}$ in the direction of the force $\vec{F}$ is $|\overrightarrow{OA}|\cos\theta$.

$\therefore$ Work done = $|\vec{F}||\overrightarrow{OA}|\cos\theta = \vec{F}.\overrightarrow{OA} = \vec{F}.\mathbf{d}$, where $\mathbf{d} = \overrightarrow{OA}$ Or Work done = (Force) . (Displacement)

If a number of forces are acting on a particle, then the sum of the works done by the separate forces is equal to the work done by the resultant force.

10. Vector or Cross product of Two Vectors

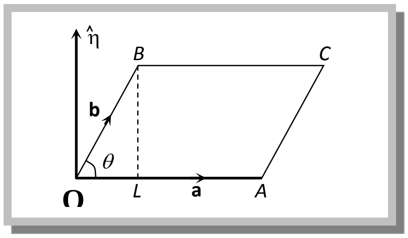
Let $\mathbf{a},\mathbf{b}$ be two non-zero, non-parallel vectors. Then the vector product $\mathbf{a}\times\mathbf{b}$, in that order, is defined as a vector whose magnitude is $|\mathbf{a}||\mathbf{b}|\sin\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ whose direction is perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ in such a way that $\mathbf{a},\mathbf{b}$ and this direction constitute a right handed system.



In other words, $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{\eta}}$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $\hat{\mathbf{\eta}}$ is a unit vector perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a},\mathbf{b},\hat{\mathbf{\eta}}$ form a right handed system.

(1) **Geometrical interpretation of vector product:** If $\mathbf{a},\mathbf{b}$ be two non-zero, non-parallel vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively and let θ be the angle between them. Complete the parallelogram *OACB*. Draw $BL \perp OA$.

In ΔOBL, $\sin\theta = \dfrac{BL}{OB} \Rightarrow BL = OB\sin\theta = |\mathbf{b}|\sin\theta$ ......(i)



Now, $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{\eta}} = (OA)(BL)\hat{\mathbf{\eta}}$

$= (\text{Base}\times\text{Height})\hat{\mathbf{\eta}} = (\text{area of parallelogram } OACB)\,\hat{\mathbf{\eta}}$

= Vector area of the parallelogram *OACB*

Thus, $\mathbf{a} \times \mathbf{b}$ is a vector whose magnitude is equal to the area of the parallelogram having $\mathbf{a}$ and $\mathbf{b}$ as its adjacent sides and whose direction $\hat{\boldsymbol{\eta}}$ is perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}, \mathbf{b}, \hat{\boldsymbol{\eta}}$ form a right handed system. Hence $\mathbf{a} \times \mathbf{b}$ represents the vector area of the parallelogram having adjacent sides along $\mathbf{a}$ and $\mathbf{b}$.

Thus, area of parallelogram *OACB* = $|\mathbf{a} \times \mathbf{b}|$.

Also, area of $\Delta OAB = \frac{1}{2}$ area of parallelogram *OACB* = $\frac{1}{2}|\mathbf{a} \times \mathbf{b}| = \frac{1}{2}|\overrightarrow{OA} \times \overrightarrow{OB}|$

(2) **Properties of vector product**

(i) Vector product is not commutative *i.e.,* if $\mathbf{a}$ and $\mathbf{b}$ are any two vectors, then $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$, however, $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$

(ii) If $\mathbf{a}, \mathbf{b}$ are two vectors and *m* is a scalar, then $m\mathbf{a} \times \mathbf{b} = m(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times m\mathbf{b}$

(iii) If $\mathbf{a}, \mathbf{b}$ are two vectors and *m, n* are scalars, then $m\mathbf{a} \times n\mathbf{b} = mn(\mathbf{a} \times \mathbf{b}) = m(\mathbf{a} \times n\mathbf{b}) = n(m\,\mathbf{a} \times \mathbf{b})$

(iv) Distributivity of vector product over vector addition.
Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be any three vectors. Then

(a) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (Left distributivity)

(b) $(\mathbf{b} + \mathbf{c}) \times \mathbf{a} = \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a}$ (Right distributivity)

(v) For any three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ we have $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c}$

(vi) The vector product of two non-zero vectors is zero vector *iff* they are parallel (Collinear) *i.e.,* $\mathbf{a} \times \mathbf{b} = \mathbf{0} \Leftrightarrow \mathbf{a} \parallel \mathbf{b}, \mathbf{a}, \mathbf{b}$ are non-zero vectors.

It follows from the above property that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for every non-zero vector $\mathbf{a}$, which in turn implies that $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$

(vii) Vector product of orthonormal triad of unit vectors **i, j, k** using the definition of the vector product, we obtain $\mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{j} \times \mathbf{k} = \mathbf{i}, \mathbf{k} \times \mathbf{i} = \mathbf{j}$, $\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \mathbf{i} \times \mathbf{k} = -\mathbf{j}$

(viii) Lagrange's identity: If **a**, **b** are any two vector then $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}.\mathbf{b})^2$ or $|\mathbf{a} \times \mathbf{b}|^2 + (\mathbf{a}.\mathbf{b})^2 = |\mathbf{a}|^2|\mathbf{b}|^2$

(3) **Vector product in terms of components:** If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$.

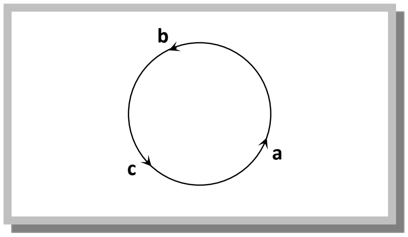
Then, $\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$.

(4) **Angle between two vectors:** If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then $\sin\theta = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$

Expression for $\sin\theta$ **:** If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ and θ be angle between $\mathbf{a}$ and $\mathbf{b}$, then

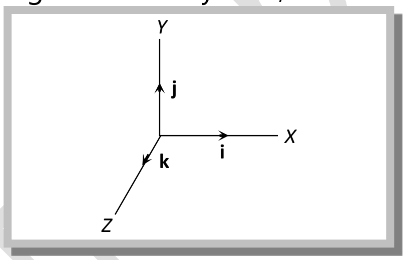
$$\sin^2\theta = \frac{(a_2b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2}{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}$$

(5) (i) **Right handed system of vectors:** Three mutually perpendicular vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form a right handed system of vector *iff* $\mathbf{a} \times \mathbf{b} = \mathbf{c}$, $\mathbf{b} \times \mathbf{c} = \mathbf{a}$, $\mathbf{c} \times \mathbf{a} = \mathbf{b}$



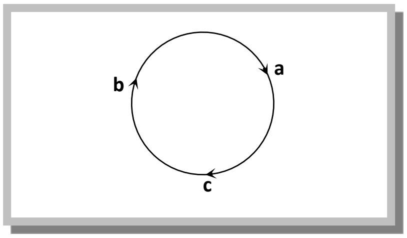
Example: The unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ form a right-handed system,

$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{j} \times \mathbf{k} = \mathbf{i}, \mathbf{k} \times \mathbf{i} = \mathbf{j}$



(ii) **Left handed system of vectors**: The vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, mutually perpendicular to one another form a left handed system of vector *iff*

$\mathbf{c} \times \mathbf{b} = \mathbf{a}, \mathbf{a} \times \mathbf{c} = \mathbf{b}, \mathbf{b} \times \mathbf{a} = \mathbf{c}$



(6) **Vector normal to the plane of two given vectors:** If $\mathbf{a}, \mathbf{b}$ be two non-zero, nonparallel vectors and let θ be the angle between them. $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \, \hat{\boldsymbol{\eta}}$ Where $\hat{\boldsymbol{\eta}}$ is a unit vector $\perp$ to the plane of $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}, \mathbf{b}, \boldsymbol{\eta}$ from a right-handed system.

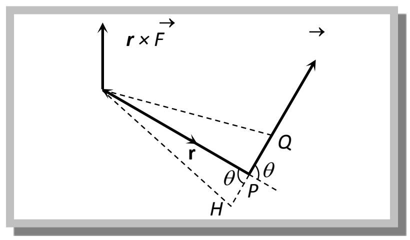
$$\Rightarrow \quad (\mathbf{a} \times \mathbf{b}) = |\mathbf{a} \times \mathbf{b}| \hat{\boldsymbol{\eta}} \Rightarrow \hat{\boldsymbol{\eta}} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$$

Thus, $\frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$ is a unit vector $\perp$ to the plane of $\mathbf{a}$ and $\mathbf{b}$. Note that $-\frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$ is also a unit vector $\perp$ to the plane of $\mathbf{a}$ and $\mathbf{b}$. Vectors of magnitude 'λ' normal to the plane of $\mathbf{a}$ and $\mathbf{b}$ are given by $\pm \frac{\lambda(\mathbf{a} \times \mathbf{b})}{|\mathbf{a} \times \mathbf{b}|}$.

11. Moment of a Force and Couple.

(1) **Moment of a force**

(i) **About a point:** Let a force $\vec{F}$ be applied at a point P of a rigid body. Then the moment of $\vec{F}$ about a point O measures the tendency of $\vec{F}$ to turn the body about point O. If this tendency of rotation about O is in anticlockwise direction, the moment is positive, otherwise it is negative.



Let $\mathbf{r}$ be the position vector of P relative to O. Then the moment or torque of $\vec{F}$ about the point O is defined as the vector $\vec{M} = \mathbf{r} \times \vec{F}$.

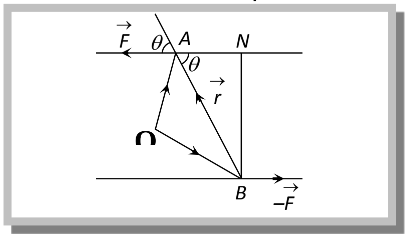
If several forces are acting through the same point P, then the vector sum of the moment of the separate forces about O is equal to the moment of their resultant force about O.

(ii) **About a line:** The moment of a force $\vec{F}$ acting at a point P about a line L is a scalar given by $(\mathbf{r} \times \vec{F}).\hat{\mathbf{a}}$ where $\hat{\mathbf{a}}$ is a unit vector in the direction of the line, and $\overrightarrow{OP} = \mathbf{r}$, where O is any point on the line.

Thus, the moment of a force $\vec{F}$ about a line is the resolved part (component) along this line, of the moment of $\vec{F}$ about any point on the line.

Note: The moment of a force about a point is a vector while the moment about a straight line is a scalar quantity.

(2) **Moment of a couple:** A system consisting of a pair of equal unlike parallel forces is called a couple. The vector sum of two forces of a couple is always zero vector.



The moment of a couple is a vector perpendicular to the plane of couple and its magnitude is the product of the magnitude of either force with the perpendicular distance between the lines of the forces.

$\vec{M} = \mathbf{r} \times \vec{F}$, where $\mathbf{r} = \overrightarrow{BA}$

$|\vec{M}| = |\overrightarrow{BA} \times \vec{F}| = |\vec{F}||\overrightarrow{BA}| \sin\theta$, where θ is the angle between $\overrightarrow{BA}$ and $\vec{F}$

$= |\vec{F}|(BN) = |\vec{F}| a$

Where $a = BN$ is the arm of the couple and $+ve$ or $-ve$ sign is to be taken according as the forces indicate a counter-clockwise rotation or clockwise rotation.