Testprep Kart

Knowledge... Everywhere

Mathematics

3D – Co-ordinate Geometry

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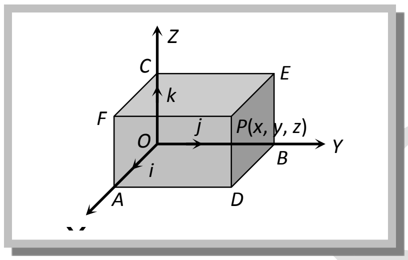
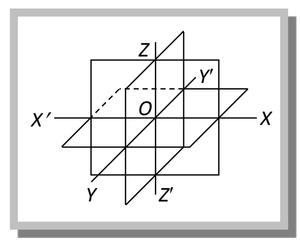
System of Co-ordinates

1. Co-ordinates of a Point in Space.

(1) **Cartesian Co-ordinates:** Let O be a fixed point, known as origin and let OX, OY and OZ be three mutually perpendicular lines, taken as x-axis, y-axis and z-axis respectively, in such a way that they form a right-handed system.



The planes XOY, YOZ and ZOX are known as xy-plane, yz-plane and zx-plane respectively.



Let P be a point in space and distances of P from yz, zx and xy-planes be x, y, z respectively (with proper signs), then we say that co-ordinates of P are (x, y, z).

Also OA = x, OB = y, OC = z.

The three co-ordinate planes (XOY, YOZ and ZOX) divide space into eight parts and these parts are called octants.

Signs of co-ordinates of a point: The signs of the co-ordinates of a point in three dimension follow the convention that all distances measured along or parallel to OX, OY, OZ will be positive and distances moved along or parallel to OX′, OY′, OZ′ will be negative.

The following table shows the signs of co-ordinates of points in various octant:

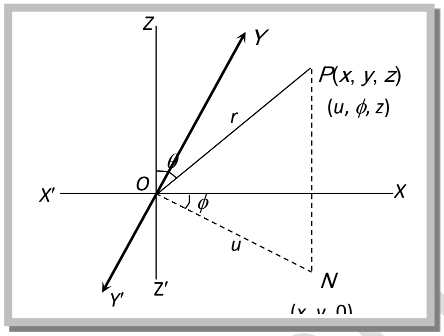
Octant co-ordinate	OXYZ	OX′YZ	OXY′Z	OX′Y′Z	OXYZ′	OX′YZ′	OXY′Z′	OX′Y′Z′
x	+	–	+	–	+	–	+	–
y	+	+	–	–	+	+	–	–
z	+	+	+	+	–	–	–	–

(2) **Other methods of defining the position of any point P in space:**

(i) **Cylindrical co-ordinates:** If the rectangular cartesian co-ordinates of P are (x, y, z), then those of N are (x, y, 0) and we can easily have the following relations : x = u cosϕ, y = u sinϕ and z = z.

Hence, $u^2 = x^2 + y^2$ and $\phi = \tan^{-1}(y/x)$.

Cylindrical co-ordinates of P ≡ (u, ϕ, z)



(ii) **Spherical polar co-ordinates:** The measures of quantities r, θ, ϕ are known as spherical or three dimensional polar co-ordinates of the point P. If the rectangular cartesian co-ordinates of P are (x, y, z) then

z = r cosθ, u = r sinθ ∴ x = u cosϕ = r sinθ cosϕ, y = u sinϕ = r sinθ sinϕ and z = r cosθ

Also $r^2 = x^2 + y^2 + z^2$ and $\tan\theta = \frac{u}{z} = \frac{\sqrt{x^2+y^2}}{z}$; $\tan\phi = \frac{y}{x}$

Note: The co-ordinates of a point on xy-plane is (x, y, 0), on yz-plane is (0, y, z) and on zx-plane is (x, 0, z)

The co-ordinates of a point on x-axis is (x, 0, 0), on y-axis is (0, y, 0) and on z-axis is (0, 0, z)

Position vector of a point : Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be unit vectors along OX, OY and OZ respectively. Then position vector of a point P(x, y, z) is $\overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

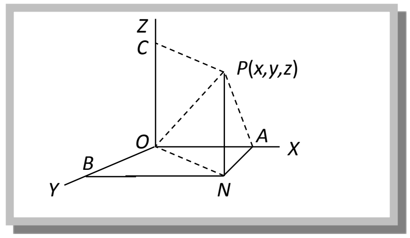
2. Distance Formula.

(1) **Distance formula:** The distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by

$$AB = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$$

(2) **Distance from origin:** Let O be the origin and P(x, y, z) be any point, then $OP = \sqrt{(x^2 + y^2 + z^2)}$.

(3) **Distance of a point from co-ordinate axes:** Let P(x, y, z) be any point in the space. Let PA, PB and PC be the perpendiculars drawn from P to the axes OX, OY and OZ respectively.



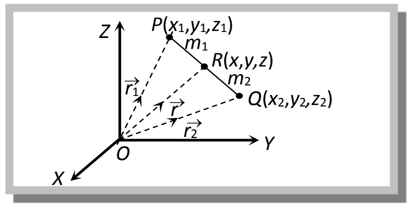
Then, $PA = \sqrt{(y^2 + z^2)}$

$$PB = \sqrt{(z^2 + x^2)}$$

$$PC = \sqrt{(x^2 + y^2)}$$

3. Section Formulas.

(1) **Section formula for internal division:** Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points. Let R be a point on the line segment joining P and Q such that it divides the join of P and Q internally in the ratio $m_1 : m_2$. Then the co-ordinates of R are



$$\left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}, \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}\right).$$

(2) **Section formula for external division:** Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points, and let R be a point on PQ produced, dividing it externally in the ratio $m_1 : m_2$ $(m_1 \neq m_2)$. Then the co-ordinates of R are $\left(\frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}, \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2}, \frac{m_1 z_2 - m_2 z_1}{m_1 - m_2}\right)$.

Note: **Co-ordinates of the midpoint:** When division point is the mid-point of PQ then ratio will be 1 : 1, hence co-ordinates of the midpoint of PQ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

Co-ordinates of the general point: The co-ordinates of any point lying on the line joining points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ may be taken as $\left(\frac{kx_2 + x_1}{k+1}, \frac{ky_2 + y_1}{k+1}, \frac{kz_2 + z_1}{k+1}\right)$, which divides PQ in the ratio k : 1. This is called general point on the line PQ.

4. Triangle.

(1) **Co-ordinates of the centroid**

(i) If $(x_1,y_1,z_1),(x_2,y_2,z_2)$ and (x_3,y_3,z_3) are the vertices of a triangle, then co-ordinates of its centroid are $\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3},\frac{z_1+z_2+z_3}{3}\right)$.

(ii) If (x_r,y_r,z_r); r = 1, 2, 3, 4, are vertices of a tetrahedron, then co-ordinates of its centroid are $\left(\frac{x_1+x_2+x_3+x_4}{4},\frac{y_1+y_2+y_3+y_4}{4},\frac{z_1+z_2+z_3+z_4}{4}\right)$.

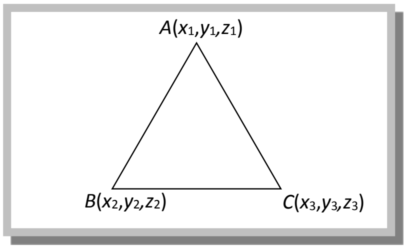
(iii) If G (α, β, γ) is the centroid of ΔABC, where A is (x_1,y_1,z_1), B is (x_2,y_2,z_2), then C is $(3\alpha-x_1-x_2,3\beta-y_1-y_2,3\gamma-z_1-z_2)$.

(2) **Area of triangle**: Let $A(x_1,y_1,z_1)$, $B(x_2,y_2,z_2)$ and $C(x_3,y_3,z_3)$ be the vertices of a triangle, then

$$\Delta_x=\frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1\\ y_2 & z_2 & 1\\ y_3 & z_3 & 1\end{vmatrix},\ \Delta_y=\frac{1}{2}\begin{vmatrix} x_1 & z_1 & 1\\ x_2 & z_2 & 1\\ x_3 & z_3 & 1\end{vmatrix},\ \Delta_z=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}$$

Now, area of ΔABC is given by the relation $\Delta=\sqrt{\Delta_x^2+\Delta_y^2+\Delta_z^2}$.

Also, $\Delta=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\frac{1}{2}\left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ x_2-x_1 & y_2-y_1 & z_2-z_1\\ x_3-x_1 & y_3-y_1 & z_3-z_1\end{matrix}\right\|$



(3) **Condition of collinearity**: Points $A(x_1,y_1,z_1),B(x_2,y_2,z_2)$ and $C(x_3,y_3,z_3)$ are collinear

If $\frac{x_1-x_2}{x_2-x_3}=\frac{y_1-y_2}{y_2-y_3}=\frac{z_1-z_2}{z_2-z_3}$

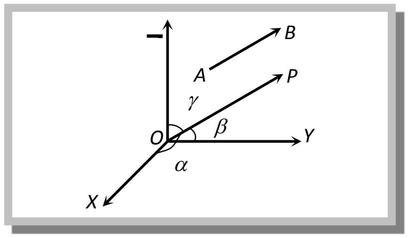
5. Volume of Tetrahedron.

Volume of tetrahedron with vertices (x_r, y_r, z_r); r = 1, 2, 3, 4, is $V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$

6. Direction cosines and Direction ratio.

(1) **Direction cosines**

(i) The cosines of the angle made by a line in anticlockwise direction with positive direction of co-ordinate axes are called the direction cosines of that line.



If α, β, γ be the angles which a given directed line makes with the positive direction of the x, y, z co-ordinate axes respectively, then cosα, cosβ, cosγ are called the direction cosines of the given line and are generally denoted by l, m, n respectively.

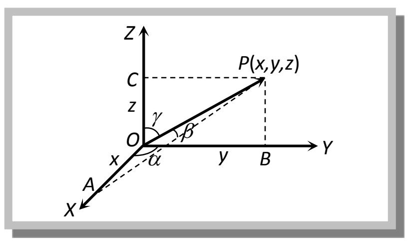
Thus, $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$.

By definition, it follows that the direction cosine of the axis of x are respectively $\cos 0^o$, $\cos 90^o$, $\cos 90^o$ i.e. (1, 0, 0). Similarly direction cosines of the axes of y and z are respectively (0, 1, 0) and (0, 0, 1).

Relation between the direction cosines: Let OP be any line through the origin O which has direction cosines l, m, n. Let P = (x, y, z) and OP = r. Then $OP^2 = x^2 + y^2 + z^2 = r^2$ .....(i)

From P draw PA, PB, PC perpendicular on the co-ordinate axes, so that

OA = x, OB = y, OC = z. Also, $\angle POA = \alpha, \angle POB = \beta$ and $\angle POC = \gamma$.



From triangle AOP, $l = \cos\alpha = \frac{x}{r} \Rightarrow x = lr$

Similarly $y = mr$ and $z = nr$.

Hence from (i), $r^2(l^2 + m^2 + n^2) = x^2 + y^2 + z^2 = r^2 \Rightarrow l^2 + m^2 + n^2 = 1$

or, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, or, $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

Note: If OP = r and the co-ordinates of point P be (x, y, z), then d.c.'s of line OP are x/r, y/r, z/r.

Direction cosines of $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ are $\frac{a}{|\mathbf{r}|}, \frac{b}{|\mathbf{r}|}, \frac{c}{|\mathbf{r}|}$.

Since $-1 \le \cos x \le 1$, $\forall x \in R$, hence values of l, m, n are such real numbers which are not less than – 1 and not greater than 1. Hence $\text{d.c.'s} \in [-1, 1]$.

The direction cosines of a line parallel to any co-ordinate axis are equal to the direction cosines of the co-ordinate axis.

The number of lines which are equally inclined to the co-ordinate axes is 4.

If l, m, n are the d.c.'s of a line, then the maximum value of $lmn = \frac{1}{3\sqrt{3}}$.

Important Tips

- The angles α, β, γ are called the direction angles of line AB.
- The d.c.'s of line BA are $\cos(\pi - \alpha)$, $\cos(\pi - \beta)$ and $\cos(\pi - \gamma)$ i.e., $-\cos\alpha, -\cos\beta, -\cos\gamma$.
- Angles α, β, γ are not coplanar.
- $\alpha + \beta + \gamma$ is not equal to 360° as these angles do not lie in same plane.
- If P(x, y,z) be a point in space such that $\mathbf{r} = \overrightarrow{OP}$ has d.c.'s l, m, n then $x = l|\mathbf{r}|,\ y = m|\mathbf{r}|,\ z = n|\mathbf{r}|$.
- Projection of a vector **r** on the co-ordinate axes are $l|\mathbf{r}|, m|\mathbf{r}|, n|\mathbf{r}|$.
- $\mathbf{r} = |\mathbf{r}|(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})$ and $\hat{\mathbf{r}} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$

(2) **Direction ratio**

(i) Three numbers which are proportional to the direction cosines of a line are called the direction ratio of that line. If a, b, c are three numbers proportional to direction cosines l, m, n of a line, then a, b, c are called its direction ratios. They are also called direction numbers or direction components.

Hence by definition, we have $\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k$ (say) $\Rightarrow$ l = ak, m = bk, n = ck

$$\Rightarrow l^2 + m^2 + n^2 = (a^2 + b^2 + c^2) = k^2 \Rightarrow k = \pm \frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

$$l = \pm \frac{a}{\sqrt{a^2 + b^2 + c^2}},\ m = \pm \frac{b}{\sqrt{a^2 + b^2 + c^2}},\ n = \pm \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

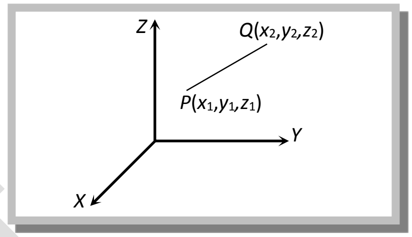
where the sign should be taken all positive or all negative.

Note: Direction ratios are not uniques, whereas d.c.'s are unique. i.e., $a^2 + b^2 + c^2 \ne 1$

(ii) Let $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ be a vector. Then its d.r.'s are a, b, c

If a vector **r** has d.r.'s a, b, c then $\mathbf{r} = \dfrac{|\mathbf{r}|}{\sqrt{a^2 + b^2 + c^2}}(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$

(iii) **D.c.'s and d.r.'s of a line joining two points:** The direction ratios of line PQ joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $x_2 - x_1 = a$, $y_2 - y_1 = b$ and $z_2 - z_1 = c$ (say).



Then direction cosines are,

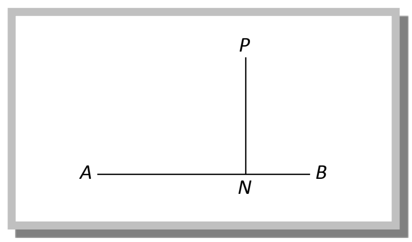
$$l = \frac{(x_2 - x_1)}{\sqrt{\Sigma(x_2 - x_1)^2}}, m = \frac{(y_2 - y_1)}{\sqrt{\Sigma(x_2 - x_1)^2}}, n = \frac{(z_2 - z_1)}{\sqrt{\Sigma(x_2 - x_1)^2}}$$

i.e., $l = \dfrac{x_2 - x_1}{PQ}, m = \dfrac{y_2 - y_1}{PQ}, n = \dfrac{z_2 - z_1}{PQ}$.

7. Projection.

(1) **Projection of a point on a line:** The projection of a point P on a line AB is the foot N of the perpendicular PN from P on the line AB.

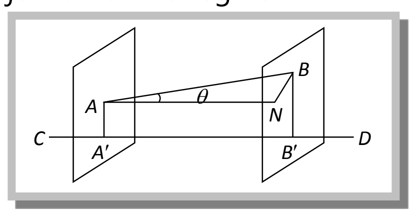
N is also the same point where the line AB meets the plane through P and perpendicular to AB.



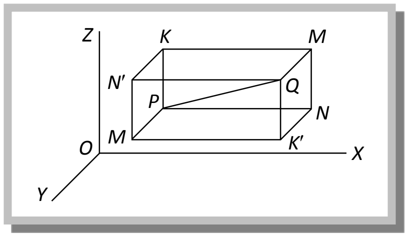
(2) **Projection of a segment of a line on another line and its length:** The projection of the segment AB of a given line on another line CD is the segment A′B′ of CD where A′ and B′ are the projections of the points A and B on the line CD.

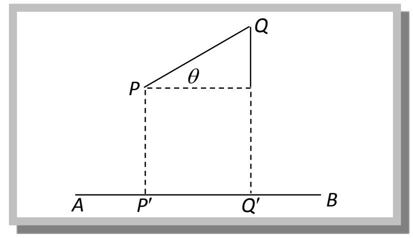
The length of the projection A′ B′.

$A'B' = AN = AB\cos\theta$



(3) **Projection of a line joining the points P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) on another line whose direction cosines are l, m and n :** Let PQ be a line segment where $P \equiv (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ and AB be a given line with d.c.'s as l, m, n. If the line segment PQ makes angle θ with the line AB, then





Projection of PQ is P′Q′ = PQ cosθ $= (x_2 - x_1)\cos\alpha + (y_2 - y_1)\cos\beta + (z_2 - z_1)\cos\gamma$
$= (x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n$

Important Tips

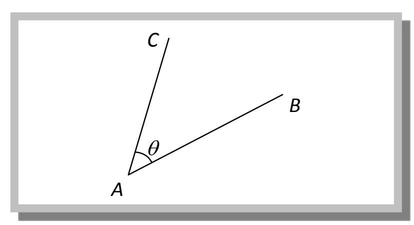
- For x-axis, l = 1, m =0, n=0.

Hence, projection of PQ on x-axis = $x_2 - x_1$, Projection of PQ on y-axis = $y_2 - y_1$ and Projection of PQ on z-axis = $z_2 - z_1$

- If P is a point (x_1, y_1, z_1), then projection of OP on a line whose direction cosines are l, m, n, is $l_1x_1 + m_1y_1 + n_1z_1$, where O is the origin.
- If l_1, m_1, n_1 and l_2, m_2, n_2 are the d.c.'s of two concurrent lines, then the d.c.'s of the lines bisecting the angles between them are proportional to $l_1 \pm l_2$, $m_1 \pm m_2$, $n_1 \pm n_2$.

8. Angle between Two lines.

(1) **Cartesian form:** Let θ be the angle between two straight lines AB and AC whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 respectively, is given by $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$.



If direction ratios of two lines a_1, b_1, c_1 and a_2, b_2, c_2 are given, then angle between two lines is given by $\cos\theta = \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}.\sqrt{a_2^2 + b_2^2 + c_2^2}}$.

Particular results: We have, $\sin^2\theta = 1-\cos^2\theta = (l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_2^2)-(l_1l_2+m_1m_2+n_1n_2)^2$

$=(l_1m_2-l_2m_1)^2+(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2$

$\Rightarrow \sin\theta = \pm\sqrt{\Sigma(l_1m_2-l_2m_1)^2}$, which is known as Lagrange's identity.

The value of sinθ can easily be obtained by the following form. $\sin\theta = \sqrt{\begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}^2 + \begin{vmatrix} m_1 & n_1 \\ n_2 & n_2 \end{vmatrix}^2 + \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}^2}$

When d.r.'s of the lines are given if a_1, b_1, c_1 and a_2, b_2, c_2 are d.r.'s of given two lines, then angle θ between them is given by $\sin\theta = \dfrac{\sqrt{\Sigma(a_1b_2-a_2b_1)^2}}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$

Condition of perpendicularity: If the given lines are perpendicular, then $\theta = 90°$ i.e. cosθ = 0

$\Rightarrow l_1l_2+m_1m_2+n_1n_2=0$ or $a_1a_2+b_1b_2+c_1c_2=0$

Condition of parallelism: If the given lines are parallel, then $\theta = 0^o$ i.e. sinθ = 0

$\Rightarrow (l_1m_2-l_2m_1)^2+(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2=0$, which is true, only when

$l_1m_2-l_2m_1=0$, $m_1n_2-m_2n_1=0$ and $n_1l_2-n_2l_1=0$

$\Rightarrow \dfrac{l_1}{l_2}=\dfrac{m_1}{m_2}=\dfrac{n_1}{n_2}$.

Similarly, $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=\dfrac{c_1}{c_2}$.

Note: The angle between any two diagonals of a cube is $\cos^{-1}\left(\dfrac{1}{3}\right)$.

The angle between a diagonal of a cube and the diagonal of a faces of the cube is $\cos^{-1}\left(\sqrt{\dfrac{2}{3}}\right)$.

If a straight line makes angles α, β, γ, δ with the diagonals of a cube, then

$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\dfrac{4}{3}$

If the edges of a rectangular parallelopiped be a, b, c, then the angles between the two diagonals are

$\cos^{-1}\left[\dfrac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right]$

(2) **Vector form:** Let the vector equations of two lines be $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \lambda\mathbf{b}_2$

As the lines are parallel to the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ respectively, therefore angle between the lines is same as the angle between the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$. Thus if θ is the angle between the given lines, then

$$\cos\theta = \frac{\mathbf{b}_1.\mathbf{b}_2}{|\mathbf{b}_1||\mathbf{b}_2|}.$$

Note: If the lines are perpendicular, then $\mathbf{b}_1.\mathbf{b}_2 = 0$.

If the lines are parallel, then $\mathbf{b}_1$ and $\mathbf{b}_2$ are parallel, therefore $\mathbf{b}_1 = \lambda\mathbf{b}_2$ for some scalar λ

The Straight Line

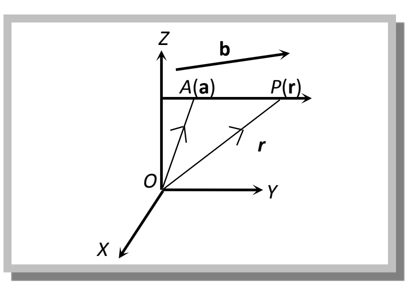
9. Straight line in Space.

Every equation of the first degree represents a plane. Two equations of the first degree are satisfied by the co-ordinates of every point on the line of intersection of the planes represented by them. Therefore, the two equations together represent that line. Therefore $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$ together represent a straight line.

(1) **Equation of a line passing through a given point**

(i) **Cartesian form or symmetrical form:** Cartesian equation of a straight line passing through a fixed point (x_1, y_1, z_1) and having direction ratios a, b, c is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$.

(ii) **Vector form:** Vector equation of a straight line passing through a fixed point with position vector **a** and parallel to a given vector **b** is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$.



Important Tips

- The parametric equations of the line $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ are $x=x_1+a\lambda, y=y_1+b\lambda, z=z_1+c\lambda$, where λ is the parameter.
- The co-ordinates of any point on the line $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ are $(x_1+a\lambda, y_1+b\lambda, z_1+c\lambda)$, where $\lambda \in$ R.
- Since the direction cosines of a line are also direction ratios, therefore equation of a line passing through (x_1, y_1, z_1) and having direction cosines l, m, n is $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$.
- Since x, y and z-axes pass through the origin and have direction cosines 1, 0, 0; 0, 1, 0 and 0, 0, 1 respectively. Therefore, the equations are x-axis : $\frac{x-0}{1}=\frac{y-0}{0}=\frac{z-0}{0}$ or y = 0 and z = 0.
y-axis : $\frac{x-0}{0}=\frac{y-0}{1}=\frac{z-0}{0}$ or x = 0 and z = 0; z-axis : $\frac{x-0}{0}=\frac{y-0}{0}=\frac{z-0}{1}$ or x = 0 and y = 0.
- In the symmetrical form of equation of a line, the coefficients of x, y, z are unity.

10. Equation of Line passing through two given points.

(i) **Cartesian form:** If $A(x_1, y_1, z_1), B(x_2, y_2, z_2)$ be two given points, the equations to the line AB are

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

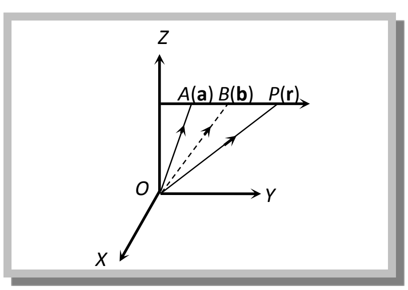
The co-ordinates of a variable point on AB can be expressed in terms of a parameter λ in the form

$$x=\frac{\lambda x_2+x_1}{\lambda+1}, y=\frac{\lambda y_2+y_1}{\lambda+1}, z=\frac{\lambda z_2+z_1}{\lambda+1}$$

λ being any real number different from –1. In fact, (x, y, z) are the co-ordinates of the point which divides the join of A and B in the ratio λ : 1.

(ii) **Vector form :** The vector equation of a line passing through two points with position vectors **a** and **b** is

$$\mathbf{r}=\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})$$



11. Changing Unsymmetrical form to Symmetrical form.

The unsymmetrical form of a line $ax+by+cz+d=0, a'x+b'y+c'z+d'=0$

Can be changed to symmetrical form as follows : $\dfrac{x-\dfrac{bd'-b'd}{ab'-a'b}}{bc'-b'c}=\dfrac{y-\dfrac{da'-d'a}{ab'-a'b}}{ca'-c'a}=\dfrac{z}{ab'-a'b}$

12. Angle between Two lines.

Let the cartesian equations of the two lines be

$\dfrac{x-x_1}{a_1}=\dfrac{y-y_1}{b_1}=\dfrac{z-z_1}{c_1}$ .....(i) and $\dfrac{x-x_2}{a_2}=\dfrac{y-y_2}{b_2}=\dfrac{z-z_2}{c_2}$ .....(ii)

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Condition of perpendicularity: If the lines are perpendicular, then $a_1a_2+b_1b_2+c_1c_2=0$

Condition of parallelism: If the lines are parallel, then $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=\dfrac{c_1}{c_2}$.

13. Reduction of Cartesian form of the Equation of a line to Vector form and vice versa.

Cartesian to vector: Let the Cartesian equation of a line be $\dfrac{x-x_1}{a}=\dfrac{y-y_1}{b}=\dfrac{z-z_1}{c}$ ......(i)

This is the equation of a line passing through the point $A(x_1,y_1,z_1)$ and having direction ratios a, b, c. In vector form this means that the line passes through point having position vector $\mathbf{a}=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}$ and is parallel to the vector $\mathbf{m}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$. Thus, the vector form of (i) is $\mathbf{r}=\mathbf{a}+\lambda\mathbf{m}$ or $\mathbf{r}=(x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k})+\lambda(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})$, where λ is a parameter.

Vector to cartesian: Let the vector equation of a line be $\mathbf{r}=\mathbf{a}+\lambda\mathbf{m}$ ......(ii)

Where $\mathbf{a}=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}, \mathbf{m}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ and λ is a parameter.

To reduce (ii) to Cartesian form we put $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and equate the coefficients of **i**, **j** and **k** as discussed below.

Putting $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and $\mathbf{m} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ in (ii), we obtain

$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}) + \lambda(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$

Equating coefficients of **i**, **j** and **k**, we get $x = x_1 + a\lambda, y = y_1 + b\lambda, z = z_1 + c\lambda$ or

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda$$

14. Intersection of Two lines.

Determine whether two lines intersect or not. In case they intersect, the following algorithm is used to find their point of intersection.

Algorithm for cartesian form: Let the two lines be $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ ......(i)

And $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ ......(ii)

Step I: Write the co-ordinates of general points on (i) and (ii). The co-ordinates of general points on (i) and (ii) are given by $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1} = \lambda$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2} = \mu$ respectively.

i.e., $(a_1\lambda + x_1, b_1\lambda + y_1 + c_1\lambda + z_1)$ and $(a_2\mu + x_2, b_2\mu + y_2, c_2\mu + z_2)$

Step II: If the lines (i) and (ii) intersect, then they have a common point.

$a_1\lambda + x_1 = a_2\mu + x_2, b_1\lambda + y_1 = b_2\mu + y_2$ and $c_1\lambda + z_1 = c_2\mu + z_2$.

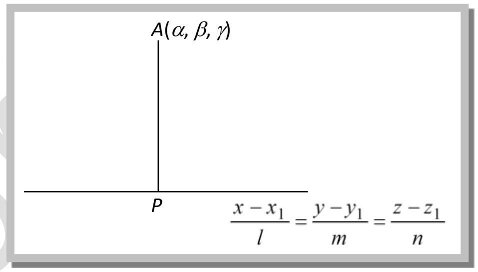
Step III: Solve any two of the equations in λ and μ obtained in step II. If the values of λ and μ satisfy the third equation, then the lines (i) and (ii) intersect, otherwise they do not intersect.

Step IV: To obtain the co-ordinates of the point of intersection, substitute the value of λ (or μ) in the co-ordinates of general point (s) obtained in step I.

15. Foot of perpendicular from a point A (α, β, γ) to the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$.

(1) **Cartesian form**

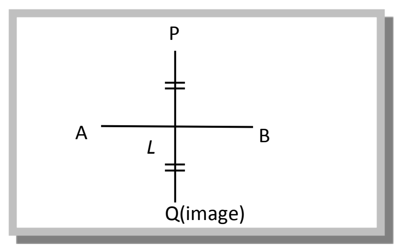
Foot of perpendicular from a point A(α, β, γ) to the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ **:** If P be the foot of perpendicular, then P is $(lr+x_1, mr+y_1, nr+z_1)$. Find the direction ratios of AP and apply the condition of perpendicularity of AP and the given line. This will give the value of r and hence the point P which is foot of perpendicular.



Length and equation of perpendicular: The length of the perpendicular is the distance AP and its equation is the line joining two known points A and P.

Note: The length of the perpendicular is the perpendicular distance of given point from that line.

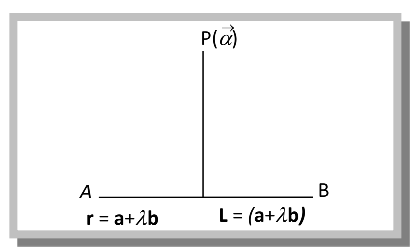
Reflection or image of a point in a straight line: If the perpendicular PL from point P on the given line be produced to Q such that PL = QL, then Q is known as the image or reflection of P in the given line. Also, L is the foot of the perpendicular or the projection of P on the line.



(2) **Vector form**

Perpendicular distance of a point from a line: Let L is the foot of perpendicular drawn from $P(\vec{\alpha})$ on the line $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$. Since $\mathbf{r}$ denotes the position vector of any point on the line $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$. So, let the position vector of L be $\mathbf{a}+\lambda\mathbf{b}$.

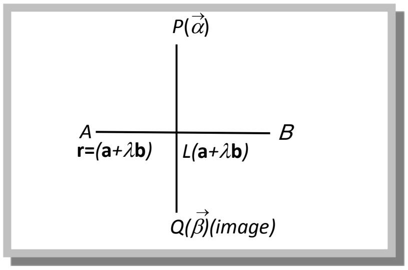
Then $\overrightarrow{PL}=\mathbf{a}-\vec{\alpha}+\lambda\mathbf{b}=(\mathbf{a}-\vec{\alpha})-\left(\frac{(\mathbf{a}-\vec{\alpha})\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}$



The length PL, is the magnitude of $\overrightarrow{PL}$, and required length of perpendicular.

Image of a point in a straight line : Let $Q(\vec{\beta})$ is the image of P in $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$

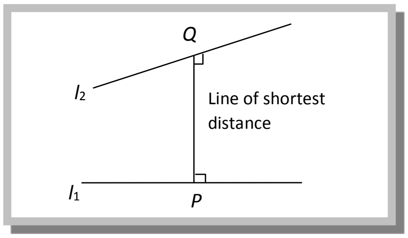
Then, $\vec{\beta} = 2\mathbf{a} - \left(\dfrac{2(\mathbf{a} - \vec{\alpha}).\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}.\alpha$



16. Shortest distance between two straight lines.

(1) **Skew lines:** Two straight lines in space which are neither parallel nor intersecting are called skew lines.

Thus, the skew lines are those lines which do not lie in the same plane.



(2) **Line of shortest distance:** If l_1 and l_2 are two skew lines, then the straight line which is perpendicular to each of these two non-intersecting lines is called the "line of shortest distance."

Note: There is one and only one line perpendicular to each of lines l_1 and l_2.

(3) **Shortest distance between two skew lines**

(i) **Cartesian form:** Let two skew lines be $\dfrac{x - x_1}{l_1} = \dfrac{y - y_1}{m_1} = \dfrac{z - z_1}{n_1}$ and $\dfrac{x - x_2}{l_2} = \dfrac{y - y_2}{m_2} = \dfrac{z - z_2}{n_2}$

Therefore, the shortest distance between the lines is given by

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{(m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - l_1 n_2)^2 + (l_1 m_2 - l_2 m_1)^2}}$$

(ii) **Vector form:** Let l_1 and l_2 be two lines whose equations are $l_1 : \mathbf{r} = \mathbf{a}_1 + \lambda \mathbf{b}_1$ and $l_2 : \mathbf{r} = \mathbf{a}_2 + \mu \mathbf{b}_2$ respectively. Then, shortest distance $PQ = \left| \frac{(\mathbf{b}_1 \times \mathbf{b}_2).(\mathbf{a}_2 - \mathbf{a}_1)}{|\mathbf{b}_1 \times \mathbf{b}_2|} \right| = \frac{|[\mathbf{b}_1\ \mathbf{b}_2\ (\mathbf{a}_2 - \mathbf{a}_1)]}{|\mathbf{b}_1 \times \mathbf{b}_2|}$

(4) **Shortest distance between two parallel lines:** The shortest distance between the parallel lines $\mathbf{r} = \mathbf{a}_1 + \lambda \mathbf{b}$ and $\mathbf{r} = \mathbf{a}_2 + \mu \mathbf{b}$ is given by $d = \frac{|(\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{b}|}{|\mathbf{b}|}$.

(5) **Condition for two lines to be intersecting i.e. coplanar**

(i) **Cartesian form:** If the lines $\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$ and $\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$ intersect, then

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 .$$

(ii) **Vector form:** If the lines $\mathbf{r} = \mathbf{a}_1 + \lambda \mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \lambda \mathbf{b}_2$ intersect, then the shortest distance between them is zero. Therefore, $[\mathbf{b}_1 \mathbf{b}_2 (\mathbf{a}_2 - \mathbf{a}_1)] = 0 \Rightarrow [(\mathbf{a}_2 - \mathbf{a}_1)\ \mathbf{b}_1 \mathbf{b}_2] = 0 \Rightarrow (\mathbf{a}_2 - \mathbf{a}_1).(\mathbf{b}_1 \times \mathbf{b}_2) = 0$

Important Tips

- Skew lines are non-coplanar lines.
- Parallel lines are not skew lines.
- If two lines intersect, the shortest distance (SD) between them is zero.
- Length of shortest distance between two lines is always taken to be positive.
- Shortest distance between two skew lines is perpendicular to both the lines.

(6) **To determine the equation of line of shortest distance:** To find the equation of line of shortest distance, we use the following procedure:

(i) From the given equations of the straight lines,

i.e. $\frac{x - a_1}{l_1} = \frac{y - b_1}{m_1} = \frac{z - c_1}{n_1} = \lambda$ (say) ......(i) and $\dfrac{x-a_2}{l_2}=\dfrac{y-b_2}{m_2}=\dfrac{z-c_2}{n_2}=\mu$ (say) ......(ii)

Find the co-ordinates of general points on straight lines (i) and (ii) as $(a_1+\lambda l_1, b_1+\lambda m_1, c_1+\lambda n_1)$ and $(a_2+\mu l_2, b_2+\mu m_2, c_2+\mu n_2)$.

(ii) Let these be the co-ordinates of P and Q, the two extremities of the length of shortest distance. Hence, find the direction ratios of PQ as $(a_2+l_2\mu)-(a_1+l_1\lambda),(b_2+m_2\mu)-(b_1+m_1\lambda),(c_2+m_2\mu)-(c_1+n_1\lambda)$.

(iii) Apply the condition of PQ being perpendicular to straight lines (i) and (ii) in succession and get two equations connecting λ and μ. Solve these equations to get the values of λ and μ.

(iv) Put these values of λ and μ in the co-ordinates of P and Q to determine points P and Q.

(v) Find out the equation of the line passing through P and Q, which will be the line of shortest distance.

Note: The same algorithm may be observed to find out the position vector of P and Q, the two extremities of the shortest distance, in case of vector equations of straight lines. Hence, the line of shortest distance, which passes through P and Q, can be obtained.

17. Definition of plane and its equations.

If point P(x, y, z) moves according to certain rule, then it may lie in a 3-D region on a surface or on a line or it may simply be a point. Whatever we get, as the region of P after applying the rule, is called locus of P. Let us discuss about the plane or curved surface. If Q be any other point on it's locus and all points of the straight line PQ lie on it, it is a plane. In other words if the straight line PQ, however small and in whatever direction it may be, lies completely on the locus, it is a plane, otherwise any curved surface.

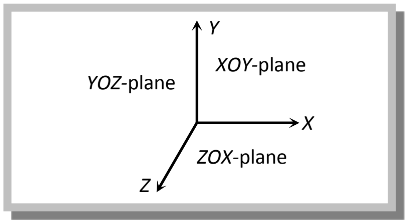
(1) **General equation of plane:** Every equation of first degree of the form $Ax + By + Cz + D = 0$ represents the equation of a plane. The coefficients of x, y and z i.e. A, B, C are the direction ratios of the normal to the plane.

(2) **Equation of co-ordinate planes**

XOY-plane: z = 0

YOZ -plane: x = 0

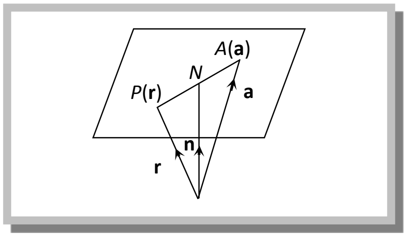
ZOX-plane: y = 0



(3) **Vector equation of plane**

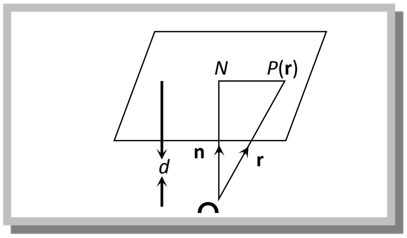
(i) Vector equation of a plane through the point $A(\mathbf{a})$ and perpendicular to the vector **n** is $(\mathbf{r}-\mathbf{a}).\mathbf{n} = 0$ or $\mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$

Note: The above equation can also be written as $\mathbf{r}.\mathbf{n} = d$, where $d = \mathbf{a}.\mathbf{n}$. This is known as the scalar product form of a plane.

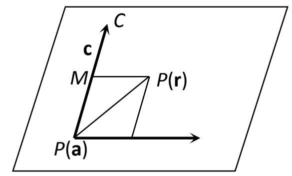


(4) **Normal form:** Vector equation of a plane normal to unit vector $\hat{\mathbf{n}}$ and at a distance d from the origin is $\mathbf{r}.\hat{\mathbf{n}} = d$.

Note: If **n** is not a unit vector, then to reduce the equation $\mathbf{r}.\mathbf{n} = d$ to normal form we divide both sides by |**n**| to obtain $\mathbf{r} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} = \frac{d}{|\mathbf{n}|}$ or $\mathbf{r}.\hat{\mathbf{n}} = \frac{d}{|\mathbf{n}|}$.



(5) **Equation of a plane passing through a given point and parallel to two given vectors:** The equation of the plane passing through a point having position vector **a** and parallel to **b** and **c** is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, where λ and μ are scalars.



(6) **Equation of plane in various forms**

(i) **Intercept form**: If the plane cuts the intercepts of length a, b, c on co-ordinate axes, then its equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

(ii) **Normal form:** Normal form of the equation of plane is $lx + my + nz = p$,

where l, m, n are the d.c.'s of the normal to the plane and p is the length of perpendicular from the origin.

(7) **Equation of plane in particular cases**

(i) Equation of plane through the origin is given by $Ax + By + Cz = 0$.

i.e. if D = 0, then the plane passes through the origin.

(8) **Equation of plane parallel to co-ordinate planes or perpendicular to co-ordinate axes**

(i) Equation of plane parallel to YOZ-plane (or perpendicular to x-axis) and at a distance 'a' from it is x = a.

(ii) Equation of plane parallel to ZOX-plane (or perpendicular to y-axis) and at a distance 'b' from it is y = b.

(iii) Equation of plane parallel to XOY-plane (or perpendicular to z-axis) and at a distance 'c' from it is z = c.

Important Tips

- Any plane perpendicular to co-ordinate axis is evidently parallel to co-ordinate plane and vice versa.
- A unit vector perpendicular to the plane containing three points A, B, C is $\frac{\overrightarrow{AB}\times\overrightarrow{AC}}{|\overrightarrow{AB}\times\overrightarrow{AC}|}$.

(9) **Equation of plane perpendicular to co-ordinate planes or parallel to co-ordinate axes**

(i) Equation of plane perpendicular to YOZ-plane or parallel to x-axis is $By + Cz + D = 0$.

(ii) Equation of plane perpendicular to ZOX-plane or parallel to y axis is $Ax + Cz + D = 0$.

(iii) Equation of plane perpendicular to XOY-plane or parallel to z-axis is $Ax + By + D = 0$.

(10) **Equation of plane passing through the intersection of two planes**

(i) **Cartesian form:** Equation of plane through the intersection of two planes $P = a_1x + b_1y + c_1z + d_1 = 0$ and $Q = a_2x + b_2y + c_2z + d_2 = 0$ is $P + \lambda Q = 0$, where λ is the parameter.

(ii) **Vector form:** The equation of any plane through the intersection of planes $\mathbf{r}.\mathbf{n}_1 = d_1$ and $\mathbf{r}.\mathbf{n}_2 = d_2$ is $\mathbf{r}.(\mathbf{n}_1 + \lambda\mathbf{n}_2) = d_1 + \lambda d_2$, where λ is an arbitrary constant.

(11) **Equation of plane parallel to a given plane**

(i) **Cartesian form:** Plane parallel to a given plane $ax + by + cz + d = 0$ is $ax + by + cz + d' = 0$, i.e. only constant term is changed.

(ii) **Vector form:** Since parallel planes have the common normal, therefore equation of plane parallel to plane $\mathbf{r}.\mathbf{n} = d_1$ is $\mathbf{r}.\mathbf{n} = d_2$, where d_2 is a constant determined by the given condition.

18. Equation of plane passing through the given point.

(1) **Equation of plane passing through a given point:** Equation of plane passing through the point (x_1, y_1, z_1) is $A(x-x_1)+B(y-y_1)+C(z-z_1)=0$, where A, B and C are d.r.'s of normal to the plane.

(2) **Equation of plane through three points:** The equation of plane passing through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is $\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}=0$ or $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$.

19. Foot of perpendicular from a point A (α, β, γ) to a given plane ax + by + cz + d = 0.

If AP be the perpendicular from A to the given plane, then it is parallel to the normal, so that its equation is

$$\frac{x-\alpha}{a}=\frac{y-\beta}{b}=\frac{z-\gamma}{c}=r \quad \text{(say)}$$

Any point P on it is $(ar+\alpha, br+\beta, cr+\gamma)$. It lies on the given plane and we find the value of r and hence the point P.

(1) **Perpendicular distance**

(i) **Cartesian form :** The length of the perpendicular from the point $P(x_1, y_1, z_1)$ to the plane $ax+by+cz+d=0$ is $\left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$.

Note: The distance between two parallel planes is the algebraic difference of perpendicular distances on the planes from origin.

Distance between two parallel planes $Ax+By+Cz+D_1=0$ and $Ax+By+Cz+D_2=0$ is $\frac{D_2 \sim D_1}{\sqrt{A^2+B^2+C^2}}$.

(ii) **Vector form:** The perpendicular distance of a point having position vector **a** from the plane $\mathbf{r}.\mathbf{n} = d$ is given by $p = \frac{|\,\mathbf{a}.\mathbf{n} - d\,|}{|\,\mathbf{n}\,|}$

(2) **Position of two point's w.r.t. a plane:** Two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ lie on the same or opposite sides of a plane $ax + by + cz + d = 0$ according to $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are of same or opposite signs. The plane divides the line joining the points P and Q externally or internally according to P and Q are lying on same or opposite sides of the plane.

20. Angle between two planes.

(1) **Cartesian form:** Angle between the planes is defined as angle between normals to the planes drawn from any point. Angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is

$$\cos^{-1}\left(\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)}}\right)$$

Note: If $a_1a_2 + b_1b_2 + c_1c_2 = 0$, then the planes are perpendicular to each other.

If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then the planes are parallel to each other.

(2) **Vector form:** An angle θ between the planes $\mathbf{r}_1.\mathbf{n}_1 = d_1$ and $\mathbf{r}_2.\mathbf{n}_2 = d_2$ is given by

$$\cos\theta = \pm\frac{\mathbf{n}_1.\mathbf{n}_2}{|\,\mathbf{n}_1\,||\,\mathbf{n}_2\,|}.$$

21. Equation of planes bisecting angle between two given planes.

(1) **Cartesian form:** Equations of planes bisecting angles between the planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d=0$ are $\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{(a_1^2+b_1^2+c_1^2)}}=\pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{(a_2^2+b_2^2+c_2^2)}}$.

Note: If angle between bisector plane and one of the plane is less than 45°, then it is acute angle bisector otherwise it is obtuse angle bisector.

If $a_1a_2+b_1b_2+c_1c_2$ is negative, then origin lies in the acute angle between the given planes provided d_1 and d_2 are of same sign and if $a_1a_2+b_1b_2+c_1c_2$ is positive, then origin lies in the obtuse angle between the given planes.

(2) **Vector form:** The equation of the planes bisecting the angles between the planes $\mathbf{r}_1.\mathbf{n}_1=d_1$ and $\mathbf{r}_2.\mathbf{n}_2=d_2$ are $\frac{|\mathbf{r}.\mathbf{n}_1-d_1|}{|\mathbf{n}_1|}=\frac{|\mathbf{r}.\mathbf{n}_2-d_2|}{|\mathbf{n}_2|}$ or $\frac{\mathbf{r}.\mathbf{n}_1-d_1}{|\mathbf{n}_1|}=\pm\frac{\mathbf{r}.\mathbf{n}_2-d_2}{|\mathbf{n}_2|}$ or $\mathbf{r}.(\hat{\mathbf{n}}_1\pm\hat{\mathbf{n}}_2)=\frac{d_1}{|\mathbf{n}_1|}\pm\frac{d_2}{|\mathbf{n}_2|}$.

22. Image of a point in a plane.

Let P and Q be two points and let π be a plane such that

(i) Line PQ is perpendicular to the plane π, and

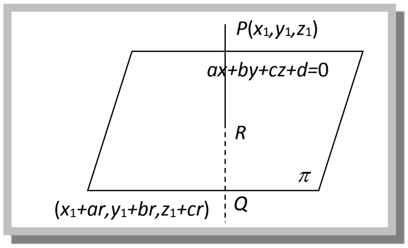
(ii) Mid-point of PQ lies on the plane π.

Then either of the point is the image of the other in the plane π.

To find the image of a point in a given plane, we proceed as follows

(i) Write the equations of the line passing through P and normal to the given plane as

$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$.



(ii) Write the co-ordinates of image Q as $(x_1+ar, y_1,+br, z_1+cr)$.

(iii) Find the co-ordinates of the mid-point R of PQ.

(iv) Obtain the value of r by putting the co-ordinates of R in the equation of the plane.

(v) Put the value of r in the co-ordinates of Q.

23. Coplanar lines.

Lines are said to be coplanar if they lie in the same plane or a plane can be made to pass through them.

(1) **Condition for the lines to be coplanar**

(i) **Cartesian form:** If the lines $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ are coplanar

Then $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$.

The equation of the plane containing them is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$ or $\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$

.

(ii) **Vector form:** If the lines $\mathbf{r}=\mathbf{a}_1+\lambda\mathbf{b}_1$ and $\mathbf{r}=\mathbf{a}_2+\lambda\mathbf{b}_2$ are coplanar, then $[\mathbf{a}_1\mathbf{b}_1\mathbf{b}_2]=[\mathbf{a}_2\mathbf{b}_1\mathbf{b}_2]$ and the equation of the plane containing them is $[\mathbf{r}\,\mathbf{b}_1\,\mathbf{b}_2]=[\mathbf{a}_1\,\mathbf{b}_1\,\mathbf{b}_2]$ or $[\mathbf{r}\,\mathbf{b}_1\,\mathbf{b}_2]=[\mathbf{a}_2\,\mathbf{b}_1\,\mathbf{b}_2]$.

Note: Every pair of parallel lines is coplanar.

Two coplanar lines are either parallel or intersecting.

The three sides of a triangle are coplanar.

Important Tips

- **Division by plane :** The ratio in which the line segment PQ, joining P(x_1, y_1, z_1) and Q(x_2, y_2, z_2), is divided by plane $ax+by+cz+d=0$ is $=-\left(\frac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d}\right)$.
- **Division by co-ordinate planes :** The ratio in which the line segment PQ, joining P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) is divided by co-ordinate planes are as follows :

(i) By yz-plane : $-x_1/x_2$ (ii) By zx-plane : $-y_1/y_2$ (ii) By xy-plane : $-z_1/z_2$

Line and plane

24. Equation of plane through a given line.

(1) If equation of the line is given in symmetrical form as $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$, then equation of plane is

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ ...... (i)

where a, b, c are given by $al+bm+cn=0$ ......(ii)

(2) If equation of line is given in general form as $a_1x+b_1y+c_1z+d_1=0=a_2x+b_2y+c_2z+d_2$, then the equation of plane passing through this line is $(a_1x+b_1y+c_1z+d_1)+\lambda(a_2x+b_2y+c_2z+d_2)=0$.

(3) **Equation of plane through a given line parallel to another line:** Let the d.c.'s of the other line be l_2, m_2, n_2. Then, since the plane is parallel to the given line, normal is perpendicular.

$\therefore\ al_2+bm_2+cn_2=0$ ......(iii)

Hence, the plane from (i), (ii) and (iii) is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$.

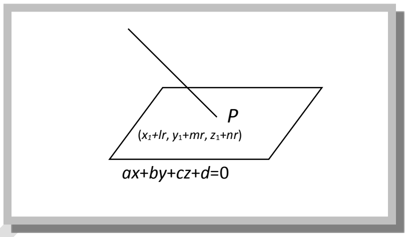
25. Transformation from unsymmetric form of the equation of line to the symmetric form.

If $P\equiv a_1x+b_1y+c_1z+d_1=0$ and $Q\equiv a_2x+b_2y+c_2z+d_2=0$ are equations of two non-parallel planes, then these two equations taken together represent a line. Thus the equation of straight line can be written as $P=0=Q$. This form is called unsymmetrical form of a line.

To transform the equations to symmetrical form, we have to find the d.r.'s of line and co-ordinates of a point on the line.

26. Intersection point of a line and plane.

To find the point of intersection of the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ and the plane $ax+by+cz+d=0$.



The co-ordinates of any point on the line

$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ are given by

$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r$ (say) or (x_1+lr, y_1+mr, z_1+nr) .....(i)

If it lies on the plane $ax+by+cz+d=0$, then

$a(x_1+lr)+b(y_1+mr)+c(z_1+n\,r)+d=0 \Rightarrow (ax_1+by_1+cz_1+d)+r(al+bm+cn)=0$

$\therefore\ r=-\frac{(ax_1+by_1+cz_1+d)}{al+bm+cn}$.

Substituting the value of r in (i), we obtain the co-ordinates of the required point of intersection.

Algorithm for finding the point of intersection of a line and a plane

Step I: Write the co-ordinates of any point on the line in terms of some parameters r (say).

Step II: Substitute these co-ordinates in the equation of the plane to obtain the value of r.

Step III: Put the value of r in the co-ordinates of the point in step I.

27. Angle between line and plane.

(1) **Cartesian form:** The angle θ between the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$, and the plane $ax+by+cz+d=0$, is given by $\sin\theta=\frac{al+bm+cn}{\sqrt{(a^2+b^2+c^2)}\sqrt{(l^2+m^2+n^2)}}$.

(i) The line is perpendicular to the plane if and only if $\frac{a}{l}=\frac{b}{m}=\frac{c}{n}$.

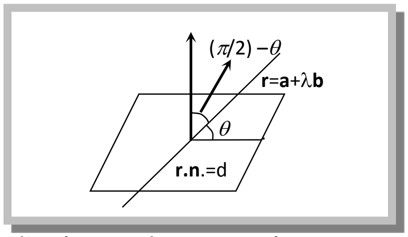
(ii) The line is parallel to the plane if and only if $al+bm+cn=0$.

(iii) The line lies in the plane if and only if $al+bm+cn=0$ and $a\alpha+b\beta+c\gamma+d=0$.

(2) **Vector form:** If θ is the angle between a line $\mathbf{r}=(\mathbf{a}+\lambda\mathbf{b})$ and the plane $\mathbf{r}.\mathbf{n}=d$, then $\sin\theta=\frac{\mathbf{b}.\mathbf{n}}{|\mathbf{b}||\mathbf{n}|}$

.

(i) **Condition of perpendicularity:** If the line is perpendicular to the plane, then it is parallel to the normal to the plane. Therefore **b** and **n** are parallel.
So, $\mathbf{b}\times\mathbf{n}=0$ or **b** = λ**n** for some scalar λ.



(ii) **Condition of parallelism:** If the line is parallel to the plane, then it is perpendicular to the normal to the plane. Therefore **b** and **n** are perpendicular. So, **b**.**n** = 0.

(iii) If the line $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$ lies in the plane **r**.**n** = d, then (i) **b**.**n** = 0 and (ii) **a**.**n** = d.

28. Projection of a line on a plane.

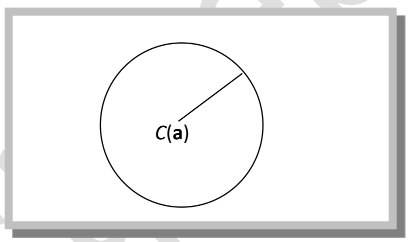
If P be the point of intersection of given line and plane and Q be the foot of the perpendicular from any point on the line to the plane then PQ is called the projection of given line on the given plane.

Image of line about a plane: Let line is $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$, plane is $a_2x+b_2y+c_2z+d=0$.

Find point of intersection (say P) of line and plane. Find image (say Q) of point (x_1, y_1, z_1) about the plane. Line PQ is the reflected line.

Sphere

A sphere is the locus of a point which moves in space in such a way that its distance from a fixed point always remains constant.
The fixed point is called the center and the constant distance is called the radius of the sphere.



29. General equation of sphere.

The general equation of a sphere is $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ with centre (–u, –v, –w)

i.e. (–(1/2) coeff. of x, –(1/2) coeff. of y, –(1/2) coeff. of z) and, radius $=\sqrt{u^2+v^2+w^2-d}$

From the above equation, we note the following characteristics of the equation of a sphere :

(i) It is a second degree equation in x, y, z;

(ii) The coefficients of x^2, y^2, z^2 are all equal;

(iii) The terms containing the products xy, yz and zx are absent.

Note: The equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ represents,

(i) A real sphere, if $u^2 + v^2 + w^2 - d > 0$.

(ii) A point sphere, if $u^2 + v^2 + w^2 - d = 0$.

(iii) An imaginary sphere, if $u^2 + v^2 + w^2 - d < 0$.

Important Tips

- If $u^2 + v^2 + w^2 - d < 0$, then the radius of sphere is imaginary, whereas the centre is real. Such a sphere is called "pseudo-sphere" or a "virtual sphere.
- The equation of the sphere contains four unknown constants u, v, w and d and therefore a sphere can be found to satisfy four conditions.

30. Equation in sphere in various forms

(1) **Equation of sphere with given center and radius**

(i) **Cartesian form :** The equation of a sphere with center (a, b, c) and radius R is

$(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$ ......(i)

If the centre is at the origin, then equation (i) takes the form $x^2 + y^2 + z^2 = R^2$, which is known as the standard form of the equation of the sphere.

(ii) **Vector form:** The equation of sphere with center at C(**c**) and radius 'a' is $|\mathbf{r} - \mathbf{c}| = a$.

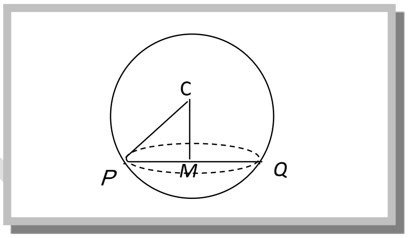
(2) **Diameter form of the equation of a sphere**

(i) **Cartesian form:** If (x_1, y_1, z_1) and (x_2, y_2, z_2) are the co-ordinates of the extremities of a diameter of a sphere, then its equation is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$.

(ii) **Vector form:** If the position vectors of the extremities of a diameter of a sphere are **a** and **b**, then its equation is $(\mathbf{r}-\mathbf{a}).(\mathbf{r}-\mathbf{b})=0$ or $|\mathbf{r}|^2 - \mathbf{r}.(\mathbf{a}-\mathbf{b})+\mathbf{a}.\mathbf{b}=0$.

31. Section of a sphere by a plane.

Consider a sphere intersected by a plane. The set of points common to both sphere and plane is called a plane section of a sphere. The plane section of a sphere is always a circle. The equations of the sphere and the plane taken together represent the plane section.



Let C be the centre of the sphere and M be the foot of the perpendicular from C on the plane. Then M is the centre of the circle and radius of the circle is given by $PM=\sqrt{CP^2-CM^2}$

The centre M of the circle is the point of intersection of the plane and line CM which passes through C and is perpendicular to the given plane.

Centre: The foot of the perpendicular from the centre of the sphere to the plane is the centre of the circle.

(radius of circle)^2 = (radius of sphere)^2 – (perpendicular from centre of spheres on the plane)^2

Great circle: The section of a sphere by a plane through the centre of the sphere is a great circle. Its centre and radius are the same as those of the given sphere.

32. Condition of tangency of a plane to a sphere.

A plane touches a given sphere if the perpendicular distance from the centre of the sphere to the plane is equal to the radius of the sphere.

(1) **Cartesian form:** The plane $lx+my+nz=p$ touches the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$, if $(ul+vm+wn-p)^2=(l^2+m^2+n^2)(u^2+v^2+w^2-d)$

(2) **Vector form:** The plane $\mathbf{r}.\mathbf{n}=d$ touches the sphere $|\mathbf{r}-\mathbf{a}|=R$ if $\frac{|\mathbf{a}.\mathbf{n}-d|}{|\mathbf{n}|}=R$.

Important Tips

- Two spheres S_1 and S_2 with centres C_1 and C_2 and radii r_1 and r_2 respectively

(i) Do not meet and lies farther apart iff $|C_1C_2| > r_1 + r_2$

(ii) Touch internally iff $|C_1C_2| = |r_1 - r_2|$

(iii) Touch externally iff $|C_1C_2| = r_1 + r_2$

(iv) Cut in a circle iff $|r_1 - r_2| < |C_1C_2| < r_1 + r_2$

(v) One lies within the other if $|C_1C_2| < |r_1 - r_2|$.

When two spheres touch each other the common tangent plane is $S_1 - S_2 = 0$ and when they cut in a circle, the plane of the circle is $S_1 - S_2 = 0$; coefficients of x^2, y^2, z^2 being unity in both the cases.

- Let p be the length of perpendicular drawn from the centre of the sphere $x^2 + y^2 + z^2 = r^2$ to the plane $Ax + By + Cz + D = 0$, then

(i) The plane cuts the sphere in a circle iff p < r and in this case, the radius of circle is $\sqrt{r^2 - p^2}$.

(ii) The plane touches the sphere iff $p = r$.

(iii) The plane does not meet the sphere iff p > r.

- **Equation of concentric sphere :** Any sphere concentric with the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + \lambda = 0$, where λ is some real which makes it a sphere.

33. Intersection of straight line and a sphere.

Let the equations of the sphere and the straight line be $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ .....(i)

And $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$ (say) .....(ii)

Any point on the line (ii) is $(\alpha+lr, \beta+mr, \gamma+nr)$.

If this point lies on the sphere (i) then we have,

$(\alpha+lr)^2+(\beta+mr)^2+(\gamma+nr)^2+2u(\alpha+lr)+2v(\beta+mr)+2w(\gamma+nr)+d=0$

or, $r^2[l^2+m^2+n^2]+2r[l(u+\alpha)+m(v+\beta)]+n(w+\gamma)]+(\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d)=0$ ....(iii)

This is a quadratic equation in r and so gives two values of r and therefore the line (ii) meets the sphere (i) in two points which may be real, coincident and imaginary, according as root of (iii) are so.

Note: If l, m, n are the actual d.c.'s of the line, then $l^2+m^2+n^2=1$ and then the equation (iii) can be simplified.

34. Angle of intersection of two spheres.

The angle of intersection of two spheres is the angle between the tangent planes to them at their point of intersection. As the radii of the spheres at this common point are normal to the tangent planes so this angle is also equal to the angle between the radii of the spheres at their point of intersection.

If the angle of intersection of two spheres is a right angle, the spheres are said to be orthogonal.

Condition for orthogonality of two spheres

Let the equation of the two spheres be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ .....(i)

and $x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0$ .....(ii)

If the sphere (i) and (ii) cut orthogonally, then $2uu'+2vv'+2ww'=d+d'$, which is the required condition.

Note: If the spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ cut orthogonally, then $d=a^2$.

Two spheres of radii r_1 and r_2 cut orthogonally, then the radius of the common circle is $\frac{r_1r_2}{\sqrt{r_1^2+r_2^2}}$