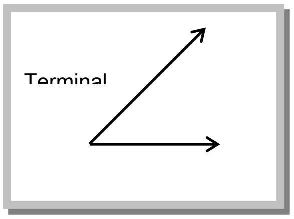
Mathematics

Trigonometrical Ratio, Functions & Identities

Table of Content

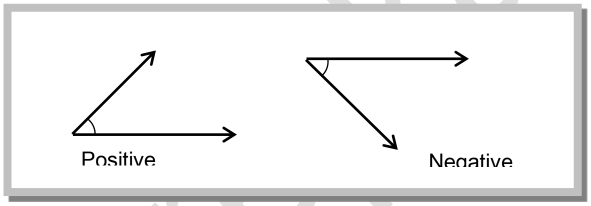
1. Definition.

(1) **Angle:** The motion of any revolving line in a plane from its initial position (initial side) to the final position (terminal side) is called angle. The end point O about which the line rotates is called the vertex of the angle.



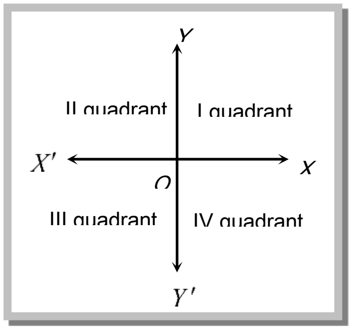
(2) **Measure of an angle:** The measure of an angle is the amount of rotation from the initial side to the terminal side.

(3) **Sense of an angle:** The sense of an angle is determined by the direction of rotation of the initial side into the terminal side. The sense of an angle is said to be positive or negative according as the initial side rotates in anticlockwise or clockwise direction to get the terminal side.



(4) **Right angle:** If the revolving ray starting from its initial position to final position describes one quarter of a circle. Then we say that the measure of the angle formed is a right angle.

(5) **Quadrants:** Let $X'OX$ and YOY' be two lines at right angles in the plane of the paper. These lines divide the plane of paper into four equal parts. Which are known as quadrants. The lines $X'OX$ and YOY' are known as x-axis and y-axis. These two lines taken together are known as the co-ordinate axes.



(6) **Angle in standard position:** An angle is said to be in standard position if its vertex concides with the origin O and the initial side concides with OX i.e., the positive direction of x-axis.

(7) **Angle in a quadrant:** An angle is said to be in a particular quadrant if the terminal side of the angle in standard position lies in that quadrant.

(8) **Quadrant angle:** An angle is said to be a quadrant angle if the terminal side concides with one of the axes.

2. System of Measurement of Angles

There are three system for measuring angles

(1) **Sexagesimal or English system:** Here a right angle is divided into 90 equal parts known as degrees. Each degree is divided into 60 equal parts called minutes and each minute is further divided into 60 equal parts called seconds. Therefore, 1 right angle = 90 degree $(=90^o)$

$1^o = 60$ minutes $(=60')$

$1' = 60$ second $(=60'')$

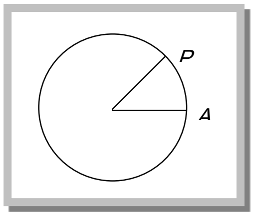
(2) **Centesimal or French system :** It is also known as French system, here a right angle is divided into 100 equal parts called grades and each grade is divided into 100 equal parts, called minutes and each minute is further divided into 100 seconds. Therefore,

1 right angle = 100 grades $(=100^g)$

1 grade = 100 minutes $(=100')$

1 minute = 100 seconds $(=100'')$

(3) **Circular system:** In this system the unit of measurement is radian. One radian, written as 1^c, is the measure of an angle subtended at the centre of a circle by an arc of length equal to the radius of the circle.



Consider a circle of radius r having centre at O. Let A be a point on the circle. Now cut off an arc AP whose length is equal to the radius r of the circle. Then by the definition the measure of $\angle AOP$ is 1 radian $(=1^c)$.

3. Relation between Three Systems of Measurement of an Angle.

Let D be the number of degrees, R be the number of radians and G be the number of grades in an angleθ.

Now, 90^o = 1 right angle $\Rightarrow 1^o = \frac{1}{90}$ right angle

$\Rightarrow$ $D^o = \frac{D}{90}$ right angles $\Rightarrow \theta = \frac{D}{90}$ right angles ....(i)

Again, π radians = 2 right angles $\Rightarrow$ 1 radian $= \frac{2}{\pi}$ right angles

$\Rightarrow$ R radians $= \frac{2R}{\pi}$ right angles $\Rightarrow \theta = \frac{2R}{\pi}$ right angles ....(ii)

and 100 grades = 1 right angle $\Rightarrow$ 1 grade $= \frac{1}{100}$ right angle

$\Rightarrow$ G grades $= \frac{G}{100}$ right angles $\Rightarrow \theta = \frac{G}{100}$ right angles ....(iii)

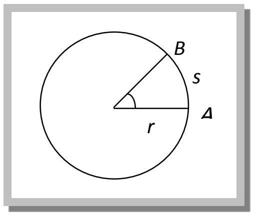
From (i), (ii) and (iii) we get, $\boxed{\frac{D}{90} = \frac{G}{100} = \frac{2R}{\pi}}$

This is the required relation between the three systems of measurement of an angle.

Note: One radian $= \frac{180^o}{\pi} \Rightarrow \pi$ radians $= 180^o$ $\Rightarrow$ 1 radian = $57^o\ 17'44.8'' \approx 57^o 17'45''$.

4. Relation between an Arc and an Angle.

If s is the length of an arc of a circle of radius r, then the angle θ (in radians) subtended by this arc at the centre of the circle is given by $\theta = \frac{s}{r}$ or $\boxed{s = r\theta}$ i.e., arc = radius × angle in radians



Sectorial area: Let OAB be a sector having central angle θ^C and radius r. Then area of the sector OAB is given by $\boxed{\frac{1}{2}r^2\theta}$.

Important Tips

- The angle between two consecutive digits in a clock is 30° (= π/6 radians). The hour hand rotates through an angle of 30° in one hour.
- The minute hand rotate through an angle of 6° in one minute.

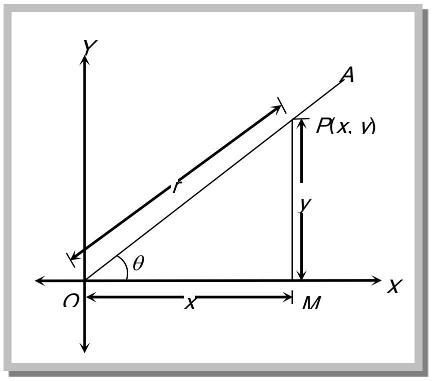
5. Trigonometrical Ratios or Functions.

In the right angled triangle OMP, we have base = OM = x, perpendicular =PM = y and hypotenuse = OP =r. We define the following trigonometric ratio which are also known as trigonometric function.

$\sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenues}} = \frac{y}{r}$ $\cos\theta = \frac{\text{Base}}{\text{Hypotenues}} = \frac{x}{r}$

$\tan\theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{y}{x}$ $\cot\theta = \frac{\text{Base}}{\text{Perpendicular}} = \frac{x}{y}$,

$\sec\theta = \frac{\text{Hypotenues}}{\text{Base}} = \frac{r}{x}$ $\text{cosec}\,\theta = \frac{\text{Hypotenues}}{\text{Perpendicular}} = \frac{r}{y}$



(1) **Relation between trigonometric ratios (function)**

(i) $\sin\theta.cosec\,\theta = 1$

(ii) $\tan\theta.\cot\theta = 1$

(iii) $\cos\theta.\sec\theta = 1$

(iv) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

(v) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

(2) **Fundamental trigonometric identities**

(i) $\sin^2\theta + \cos^2\theta = 1$

(ii) $1 + \tan^2\theta = \sec^2\theta$

(iii) $1 + \cot^2\theta = \text{cosec}^2\theta$

Important Tips

☞ If $x = \sec\theta + \tan\theta$, then $\dfrac{1}{x} = \sec\theta - \tan\theta$.

☞ If $x = \text{coesc}\theta + \cot\theta$, then $\dfrac{1}{x} = \text{cosec}\theta - \cot\theta$.

(3) **Sign of trigonometrical ratios or functions:** Their signs depends on the quadrant in which the terminal side of the angle lies.

(i) **In first quadrant:** $x > 0, y > 0 \Rightarrow \sin\theta = \dfrac{y}{r} > 0, \cos\theta = \dfrac{x}{r} > 0, \tan\theta = \dfrac{y}{x} > 0,\ \ \text{cosec}\,\theta = \dfrac{r}{y} > 0$,

$\sec\theta = \dfrac{r}{x} > 0$ and $\cot\theta = \dfrac{x}{y} > 0$. Thus, in the first quadrant all trigonometric functions are positive.

(ii) **In second quadrant:** $x<0, y>0 \Rightarrow \sin\theta=\frac{y}{r}>0, \cos\theta=\frac{x}{r}<0, \tan\theta=\frac{y}{x}<0, \operatorname{cosec}\theta=\frac{r}{y}>0,$

$\sec\theta=\frac{r}{x}<0$ and $\cot\theta=\frac{x}{y}<0$. Thus, in the second quadrant sin and cosec function are positive and all others are negative.

(iii) **In third quadrant:** $x<0, y<0 \Rightarrow \sin\theta=\frac{y}{r}<0, \cos\theta=\frac{x}{r}<0, \tan\theta=\frac{y}{x}>0, \operatorname{cosec}\theta=\frac{r}{y}<0,$

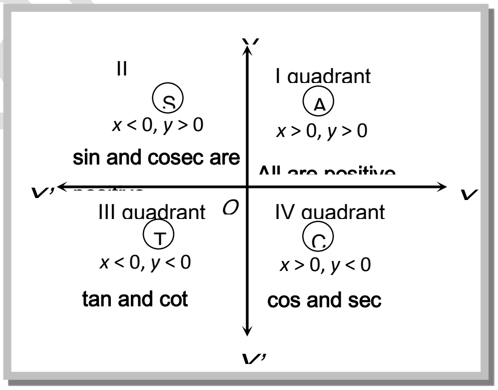
$\sec\theta=\frac{r}{x}<0$ and $\cot\theta=\frac{x}{y}>0$. Thus, in the third quadrant all trigonometric functions are negative except tangent and cotangent.

(iv) **In fourth quadrant:** $x>0, y<0 \Rightarrow \sin\theta=\frac{y}{r}<0, \quad \cos\theta=\frac{x}{r}>0,$

$\tan\theta=\frac{y}{x}<0, \quad \operatorname{cosec}\theta=\frac{r}{y}<0$, $\sec\theta=\frac{r}{x}>0$ and $\cot\theta=\frac{x}{y}<0$

Thus, in the fourth quadrant all trigonometric functions are negative except cos and sec.

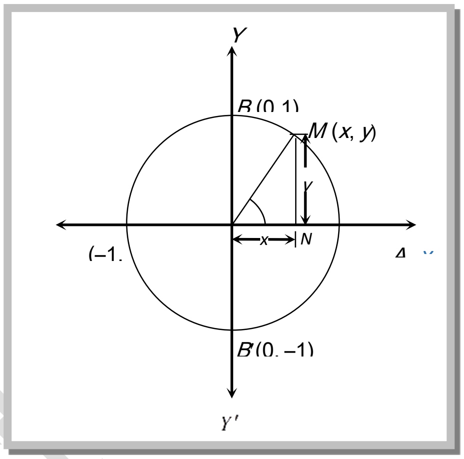
In brief : A crude aid to memorise the signs of trigonometrical ratio in different quadrant. "**A**dd **S**ugar **T**o **C**offee".



Important Tips

- First determine the sign of the trigonometric function.
- If θ is measured from $X'OX$ i.e., $\{(\pi \pm \theta, 2\pi - \theta)\}$ then retain the original name of the function.
- If θ is measured from $Y'OY$ i.e., $\left\{\frac{\pi}{2} \pm \theta, \frac{3\pi}{2} \pm \theta\right\}$, then change sine to cosine, cosine to sine, tangent to cotangent, cot to tan, sec to cosec and cosec to sec.

(4) **Variations in values of trigonometric functions in different quadrants:** Let $X'OX$ and YOY' be the coordinate axes. Draw a circle with center at origin O and radius unity.



Let $M(x, y)$ be a point on the circle such that $\angle AOM = \theta$ then $x = \cos\theta$ and $y = \sin\theta$; $-1 \le \cos\theta \le 1$ and $-1 \le \sin\theta \le 1$ for all values of θ.

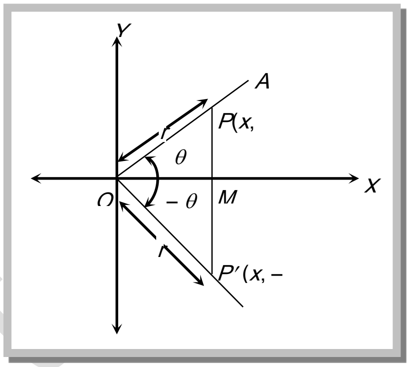
II-Quadrant (S)	**I-Quadrant (A)**
$\sin\theta \to$ decreases from 1 to 0	$\sin\theta \to$ increases from 0 to 1
$\cos\theta \to$ decreases from 0 to – 1	$\cos\theta \to$ decreases from 1 to 0
$\tan\theta \to$ increases from – ∞ to 0	$\tan\theta \to$ increases from 0 to ∞
$\cot\theta \to$ decreases from 0 to – ∞	$\cot\theta \to$ decreases from ∞ to 0
$\sec\theta \to$ increases from – ∞ to – 1	$\sec\theta \to$ increases from 1 to ∞
$\text{cosec}\,\theta \to$ increases from 1 to ∞	$\text{cosec}\,\theta \to$ decreases from ∞ to 1
III-Quadrant (T)	**IV-Quadrant (C)**
$\sin\theta \to$ decreases from 0 to – 1	$\sin\theta \to$ increases from – 1 to 0
$\cos\theta \to$ increases from – 1 to 0	$\cos\theta \to$ increases from 0 to 1
$\tan\theta \to$ increases from 0 to ∞	$\tan\theta \to$ increases from – ∞ to 0
$\cot\theta \to$ decreases from ∞ to 0	$\cot\theta \to$ decreases from 0 to – ∞
$\sec\theta \to$ decreases from – 1 to – ∞	$\sec\theta \to$ decreases from ∞ to 1
$\text{cosec}\,\theta \to$ increases from – ∞ to – 1	$\text{cosec}\,\theta \to$ decreases from – 1 to – ∞

Note: $+\infty$ and – ∞ are two symbols. These are not real number. When we say that tanθ increases from 0 to ∞ for as θ varies from 0 to $\frac{\pi}{2}$ it means that $\tan\theta$ increases in the interval $\left(0, \frac{\pi}{2}\right)$ and it attains large positive values as θ tends to $\frac{\pi}{2}$. Similarly for other trigonometric functions.

6. Trigonometrical Ratios of Allied Angles.

Two angles are said to be allied when their sum or difference is either zero or a multiple of 90^o .

(1) **Trigonometric ratios of (–θ):** Let a revolving ray starting from its initial position OX, trace out an angle $\angle XOA = \theta$. Let P(x, y) be a point on OA such that OP = r. Draw PM ⊥ from P on x-axis. Angle $\angle XOA' = -\theta$ in the clockwise sense. Let P' be a point on OA' such that $OP' = OP$. Clearly M and M' coincide and ΔOMP is congruent to $\Delta OMP'$ then P' are (x, – y).



$\sin(-\theta) = \frac{-y}{r} \Rightarrow \frac{-y}{r} = -\sin\theta$; $\cos(-\theta) = \frac{x}{r} = \cos\theta$; $\tan(-\theta) = \frac{-y}{x} = -\tan\theta$

Taking the reciprocal of these trigonometric ratios;

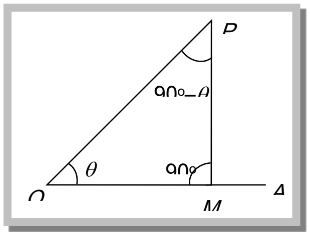
$\text{cosec}(-\theta) = -\text{cosec}\,\theta$, $\sec(-\theta) = \sec\theta$ and $\cot(-\theta) = -\cot\theta$

Note: A function $f(x)$ is said to be an even function if $f(-x) = f(x)$ for all x in its domain.

A function $f(x)$ is said to be an odd function if $f(-x) = -f(x)$ for all x in its domain.

$\sin\theta, \tan\theta, \cot\theta, \text{cosec}\,\theta$ are odd functions and $\cos\theta, \sec\theta$ are even functions.

(2) **Trigonometric function of (90 – θ):** Let the revolving line, starting from OA, trace out any acute angle AOP, equal to θ. From any point P, draw PM ⊥ to OA. Three angles of a triangle are together equal to two right angles, and since OMP is a right angle, the sum of the two angles MOP and OPM is right angle.



$\angle OPM = 90^o - \theta$

[When the angle OPM is consider, the line PM is the ‘base’ and MO is the ‘perpendicular’]

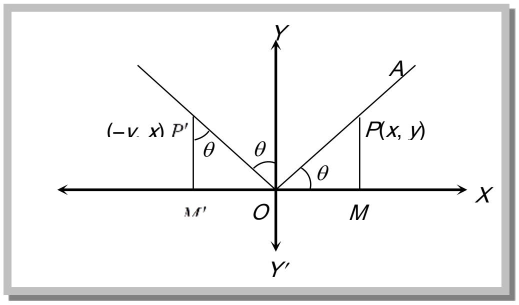
$\sin(90^o - \theta) = \sin MPO = \frac{MO}{PO} = \cos AOP = \cos\theta$, $\cos(90^o - \theta) = \cos MPO = \frac{PM}{PO} = \sin AOP = \sin\theta$

$\tan(90^o - \theta) = \tan MPO = \frac{MO}{PM} = \cot AOP = \cot\theta$, $\cot(90^o - \theta) = \cot MPO = \frac{PM}{MO} = \tan AOP = \tan\theta$

$\text{cosec}(90^o - \theta) = \text{cosec}\ MPO = \frac{PO}{MO} = \sec AOP = \sec\theta$,

$\sec(90^o - \theta) = \sec MPO = \frac{PO}{PM} = \text{cosec}\ AOP = \text{cosec}\,\theta$

(3) **Trigonometric function of (90+θ):** Let a revolving ray OA starting from its initial position OX, trace out an angle $\angle XOA = \theta$ and let another revolving ray OA' starting from the same initial position OX, first trace out an angle θ. So as to coincide with OA and then it revolves through an angle of 90^o in anticlockwise direction to form an angle $\angle XOA' = 90^o + \theta$.



Let P and P' be points on OA and OA' respectively such that $OP = OP' = r$.

Draw perpendicular PM and PM' from P and P' respectively on OX. Let the coordinates of P be (x, y). Then $OM = x$ and $PM = y$

clearly, $OM' = PM = y$ and $P'M' = OM = x$.

So the coordinates of P' are –y, x

$$\sin(90+\theta) = \frac{M'P'}{OP'} = \frac{x}{r} = \cos\theta, \qquad \cos(90+\theta) = \frac{OM'}{OP'} = \frac{-y}{r} = -\sin\theta$$

$$\tan(90+\theta) = \frac{M'P'}{OM'} = \frac{x}{-y} = \frac{-x}{y} = -\cot\theta, \cot(90+\theta) = -\tan\theta, \sec(90+\theta) = -\mathrm{cosec}\,\theta, \quad \mathrm{cosec}(90+\theta) = \sec\theta$$

Allied angles → / **Trigo. Ratio** ↓	$(-\theta)$	**(90 – θ)** or $\left(\frac{\pi}{2}-\theta\right)$	$(90+\theta)$ or $\left(\frac{\pi}{2}+\theta\right)$	$(180-\theta)$ or $(\pi-\theta)$	$(180+\theta)$ or $(\pi+\theta)$	$(270-\theta)$ or $\left(\frac{3\pi}{2}-\theta\right)$	$(270+\theta)$ or $\left(\frac{3\pi}{2}+\theta\right)$	$(360-\theta)$ or $(2\pi-\theta)$
sinθ	– sinθ	cosθ	cos θ	sinθ	– sinθ	– cosθ	– cos θ	– sinθ
cosθ	cosθ	sinθ	– sinθ	– cosθ	– cosθ	– sinθ	sinθ	cosθ
tan θ	– tanθ	cotθ	– cotθ	– tanθ	tanθ	cotθ	– cotθ	– tanθ

Important Tips

- $\sin n\pi = 0, \quad \cos n\pi = (-1)^n$
- $\sin(n\pi+\theta) = (-1)^n \sin\theta$, $\cos(n\pi+\theta) = (-1)^n \cos\theta$
- $\sin\left(\frac{n\pi}{2}+\theta\right) = (-1)^{\frac{n-1}{2}} \cos\theta$, if n is odd

$= (-1)^{n/2} \sin\theta$, if n is even

- $\cos\left(\frac{n\pi}{2}+\theta\right) = (-1)^{\frac{n+1}{2}} \sin\theta, \text{ if } n \text{ is odd}$

  $= (-1)^{n/2} \cos\theta, \quad \text{if } n \text{ is even}$

7. Trigonometrical Ratios for Various Angles.

θ	0	π/6	π/4	π/3	π/2	π	3π/2	2π
sinθ	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0	–1	0
cosθ	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0	–1	0	1
tanθ	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞	0	∞	0

8. Trigonometrical Ratios in terms of Each other.

	sinθ	cosθ	tanθ	cotθ	secθ	cosecθ
sinθ	sinθ	$\sqrt{1-\cos^2\theta}$	$\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$	$\frac{1}{\sqrt{1+\cot^2\theta}}$	$\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\frac{1}{\operatorname{cosec}\theta}$
cosθ	$\sqrt{1-\sin^2\theta}$	cosθ	$\frac{1}{\sqrt{1+\tan^2\theta}}$	$\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$	$\frac{1}{\sec\theta}$	$\frac{\sqrt{\operatorname{cosec}^2\theta-1}}{\operatorname{cosec}\theta}$
tanθ	$\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	tanθ	$\frac{1}{\cot\theta}$	$\sqrt{\sec^2\theta-1}$	$\frac{1}{\sqrt{\operatorname{cosec}^2\theta-1}}$
cot θ	$\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\frac{1}{\tan\theta}$	cotθ	$\frac{1}{\sqrt{\sec^2\theta-1}}$	$\sqrt{\operatorname{cosec}^2\theta-1}$
secθ	$\frac{1}{\sqrt{1-\sin^2\theta}}$	$\frac{1}{\cos\theta}$	$\sqrt{1+\tan^2\theta}$	$\frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$	secθ	$\frac{\operatorname{cosec}\theta}{\sqrt{\operatorname{cosec}^2\theta-1}}$
cosecθ	$\frac{1}{\sin\theta}$	$\frac{1}{\sqrt{1-\cos^2\theta}}$	$\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$\sqrt{1+\cot^2\theta}$	$\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	cosecθ

Important Tips

☞ Values for some standard angles

$\sin 15^o = \cos 75^o = \frac{\sqrt{3}-1}{2\sqrt{2}}$; $\cos 15^o = \sin 75^o = \frac{\sqrt{3}+1}{2\sqrt{2}}$; $\tan 15^o = \cot 75^o = 2-\sqrt{3}$;

$\sin 18^o = \cos 72^o = \frac{\sqrt{5}-1}{4}$; $\cos 36^o = \sin 54^o = \frac{\sqrt{5}+1}{4}$; $\tan 75^o = \cot 15^o = 2+\sqrt{3}$

$\sin 22\frac{1}{2}^o = \cos 67\frac{1}{2}^o = \frac{\sqrt{2-\sqrt{2}}}{2}$, $\cos 22\frac{1}{2}^o = \sin 67\frac{1}{2}^o = \frac{\sqrt{2+\sqrt{2}}}{2}$; $\cot 22\frac{1}{2}^o = \tan 67\frac{1}{2}^o = \sqrt{2}+1$

$\tan 22\frac{1}{2}^o = \cot 67\frac{1}{2}^o = \sqrt{2}-1$

9. Formulae for the Trigonometric Ratios of Sum and Differences of Two Angles.

(1) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

(2) $\sin(A-B) = \sin A \cos B - \cos A \sin B$

(3) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

(4) $\cos(A-B) = \cos A \cos B + \sin A \sin B$

(5) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(6) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(7) $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$

(8) $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

(9) $\sin(A+B).\sin(A-B)=\sin^2 A-\sin^2 B=\cos^2 B-\cos^2 A$

(10) $\cos(A+B).\cos(A-B)=\cos^2 A-\sin^2 B=\cos^2 B-\sin^2 A$

(11) $\tan A \pm \tan B = \frac{\sin A}{\cos A} \pm \frac{\sin B}{\cos B} = \frac{\sin A \cos B \pm \cos A \sin B}{\cos A \cos B} = \frac{\sin(A \pm B)}{\cos A.\cos B}$ $\left(A \neq n\pi + \frac{\pi}{2}, B \neq m\pi\right)$

(12) $\cot A \pm \cot B = \frac{\sin(B \pm A)}{\sin A.\sin B}$ $\left(A \neq n\pi, B \neq m\pi + \frac{\pi}{2}\right)$

10. Formulae for the Trigonometric Ratios of Sum and Differences of Three Angles.

(1) $\sin(A+B+C)=\sin A\cos B\cos C+\cos A\sin B\cos C+\cos A\cos B\sin C-\sin A\sin B\sin C$

or sin $(A+B+C)=\cos A\cos B\cos C(\tan A+\tan B+\tan C-\tan A.\tan B.\tan C)$

(2) $\cos(A+B+C)=\cos A\cos B\cos C-\sin A\sin B\cos C-\sin A\cos B\sin C-\cos A\sin B\sin C$

$\cos(A+B+C)=\cos A\cos B\cos C(1-\tan A\tan B-\tan B\tan C-\tan C\tan A)$

(3) $\tan(A+B+C)=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$

(4) $\cot(A+B+C)=\frac{\cot A\cot B\cot C-\cot A-\cot B-\cot C}{\cot A\cot B+\cot B\cot C+\cot C.\cot A-1}$

In general;

(5) $\sin(A_1+A_2+......+A_n) = \cos A_1\cos A_2.....\cos A_n(S_1-S_3+S_5-S_7+...)$

(6) $\cos(A_1+A_2+....+A_n)=\cos A_1\cos A_2...\cos A_n(1-S_2+S_4-S_6....)$

(7) $\tan(A_1+A_2+.....+A_n)=\frac{S_1-S_3+S_5-S_7+....}{1-S_2+S_4-S_6+....}$

Where; $S_1 = \tan A_1 + \tan A_2 + ... + \tan A_n$ = The sum of the tangents of the separate angles.

$S_2 = \tan A_1 \tan A_2 + \tan A_1 \tan A_3 + ...$ = The sum of the tangents taken two at a time.

$S_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + ...$ = Sum of tangents three at a time, and so on.

If $A_1 = A_2 = ... = A_n = A$, then $S_1 = n \tan A$, $S_2 = {}^nC_2 \tan^2 A$, $S_3 = {}^nC_3 \tan^3 A, ...$

(8) $\sin nA = \cos^n A({}^nC_1 \tan A - {}^nC_3 \tan^3 A + {}^nC_5 \tan^5 A - ....)$

(9) $\cos nA = \cos^n A(1 - {}^nC_2 \tan^2 A + {}^nC_4 \tan^4 A - ...)$

(10) $\tan nA = \dfrac{{}^nC_1 \tan A - {}^nC_3 \tan^3 A + {}^nC_5 \tan^5 A - ....}{1 - {}^nC_2 \tan^2 A + {}^nC_4 \tan^4 A - {}^nC_6 \tan^6 A + ...}$

(11) $\sin nA + \cos nA = \cos^n A(1 + {}^nC_1 \tan A - {}^nC_2 \tan^2 A - {}^nC_3 \tan^3 A + {}^nC_4 \tan^4 A + {}^nC_5 \tan^5 A - {}^nC_6 \tan^6 A - .....)$

(12)

$\sin nA - \cos nA = \cos^n A(-1 + {}^nC_1 \tan A + {}^nC_2 \tan^2 A - {}^nC_3 \tan^3 A - {}^nC_4 \tan^4 A + {}^nC_5 \tan^5 A + {}^nC_6 \tan^6 A ...)$

(13) $\sin(\alpha) + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + ... + \sin(\alpha + (n-1)\beta) = \dfrac{\sin\{\alpha + (n-1)(\beta/2)\}.\sin(n\beta/2)}{\sin(\beta/2)}$

(14) $\cos(\alpha) + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + ... + \cos(\alpha + (n-1)\beta) = \dfrac{\cos\left\{\alpha + (n-1)\left(\frac{\beta}{2}\right)\right\}.\sin\left\{n\left(\frac{\beta}{2}\right)\right\}}{\sin\left(\frac{\beta}{2}\right)}$

11. Formulae to Transform the Product into Sum or Difference.

(1) $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

(2) $2 \cos A \sin B = \sin(A+B) - \sin(A-B)$

(3) $2 \cos A \cos B = \cos(A+B) + \cos(A-B)$

(4) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

Let $A+B=C$ and $A-B=D$

Then, $A = \frac{C+D}{2}$ and $B = \frac{C-D}{2}$

Therefore, we find out the formulae to transform the sum or difference into product.

(5) $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

(6) $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

(7) $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

(8) $\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2} = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

Important Tips

- $\sin(60^o-\theta).\sin\theta\sin(60^o+\theta) = \frac{1}{4}\sin 3\theta$
- $\cos(60-\theta).\cos\theta\cos(60^o+\theta) = \frac{1}{4}\cos 3\theta$
- $\tan(60^o-\theta).\tan\theta\tan(60^o+\theta) = \tan 3\theta$
- $\cos A.\cos 2A.\cos 2^2A.\cos 2^3A....... \cos 2^{n-1}A = \frac{\sin 2^nA}{2^n\sin A}, \text{if } A = n\pi$

= 1, if $A = 2n\pi$

= 1, if $A = (2n+1)\pi$

12. Trigonometric Ratio of Multiple of an Angle.

(1) $\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$

(2) $\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \cos^2 A - \sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A}$; where $A \neq (2n+1)\frac{\pi}{4}$.

(3) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

(4) $\sin 3A = 3\sin A - 4\sin^3 A = 4\sin(60^o - A).\sin A.\sin(60^o + A)$

(5) $\cos 3A = 4\cos^3 A - 3\cos A = 4\cos(60^o - A).\cos A.\cos(60^o + A)$

(6) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} = \tan(60^o - A).\tan A.\tan(60^o + A)$, where $A \neq n\pi + \pi/6$

(7) $\sin 4\theta = 4\sin\theta.\cos^3\theta - 4\cos\theta\sin^3\theta$ (8) $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

(9) $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ (10) $\sin 5A = 16\sin^5 A - 20\sin^3 A + 5\sin A$

(11) $\cos 5A = 16\cos^5 A - 20\cos^3 A + 5\cos A$

13. Trigonometric Ratio of Sub-multiple of an Angle.

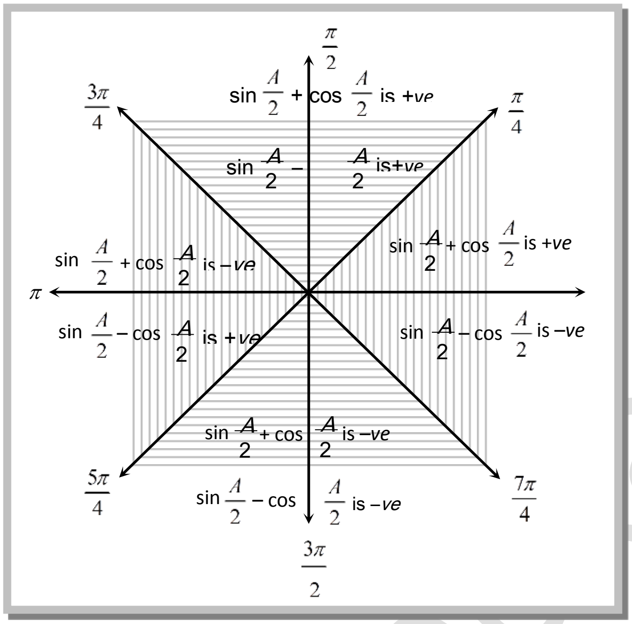
(1) $\left|\sin\dfrac{A}{2} + \cos\dfrac{A}{2}\right| = \sqrt{1 + \sin A}$ or $\sin\dfrac{A}{2} + \cos\dfrac{A}{2} = \pm\sqrt{1 + \sin A}$ i.e., $\begin{cases} +, \text{If } 2n\pi - \pi/4 \le A/2 \le 2n\pi + \dfrac{3\pi}{4} \\ -, \text{otherwise} \end{cases}$

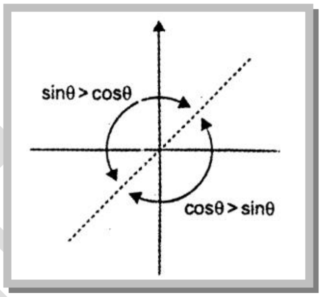
(2) $\left|\sin\dfrac{A}{2} - \cos\dfrac{A}{2}\right| = \sqrt{1 - \sin A}$ or $(\sin\dfrac{A}{2} - \cos\dfrac{A}{2}) = \pm\sqrt{1 - \sin A}$ i.e., $\begin{cases} +, \text{If } 2n\pi + \pi/4 \le A/2 \le 2n\pi + \dfrac{5\pi}{4} \\ -, \text{otherwise} \end{cases}$

(3) (i) $\tan\dfrac{A}{2} = \dfrac{\pm\sqrt{\tan^2 A + 1} - 1}{\tan A} = \pm\sqrt{\dfrac{1 - \cos A}{1 + \cos A}} = \dfrac{1 - \cos A}{\sin A}$, where $A \neq (2n+1)\pi$

(ii) $\cot\dfrac{A}{2} = \pm\sqrt{\dfrac{1 + \cos A}{1 - \cos A}} = \dfrac{1 + \cos A}{\sin A}$, where $A \neq 2n\pi$

The ambiguities of signs are removed by locating the quadrants in which $\frac{A}{2}$ lies or you can follow the following figure,





(4) $\tan^2\frac{A}{2}=\frac{1-\cos A}{1+\cos A}$; where $A\neq(2n+1)\pi$

(5) $\cot^2\frac{A}{2}=\frac{1+\cos A}{1-\cos A}$; where $A\neq 2n\pi$

Important Tips

- Any formula that gives the value of $\sin\frac{A}{2}$ in terms of sin A shall also give the value of sine of $\frac{n\pi+(-1)^n A}{2}$.
- Any formula that gives the value of $\cos\frac{A}{2}$ in terms of cos A shall also give the value of cos of $\frac{2n\pi\pm A}{2}$.
- Any formula that gives the value of $\tan\frac{A}{2}$ in terms of tan A shall also give the value of tan of $\frac{n\pi\pm A}{2}$.

14. Maximum and Minimum Value of a cosθ + b sinθ.

Let $a = r\cos\alpha$ ....(i) and $b = r\sin\alpha$ ....(ii)

Squaring and adding (i) and (ii), then $a^2 + b^2 = r^2$ or, $r = \sqrt{a^2 + b^2}$

$\therefore \quad a\sin\theta + b\cos\theta = r(\sin\theta\cos\alpha + \cos\theta\sin\alpha) = r\sin(\theta+\alpha)$

But $-1 \le \sin\theta < 1$ So, $-1 \le \sin(\theta+\alpha) \le 1$; Then $-r \le r\sin(\theta+\alpha) \le r$

Hence, $-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$

Then the greatest and least values of $a\sin\theta + b\cos\theta$ are respectively $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$.

Note: $\sin^2 x + \text{cosec}^2 x \ge 2$, for every real x.

$\cos^2 x + \sec^2 x \ge 2$, for every real x.

$\tan^2 x + \cot^2 x \ge 2$, for every real x.

Important Tips

Use of Σ (Sigma) and Π (Pie) notation

- $\sin(A+B+C) = \Sigma\sin A\cos B\cos C - \Pi\sin A$, $\cos(A+B+C) = \Pi\cos A - \Sigma\cos A\sin B\sin C$,

  $\tan(A+B+C) = \dfrac{\Sigma\tan A - \Pi\tan A}{1 - \Sigma\tan A\tan B}$. ($\because$ Σ denotes summation)

- $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + ...$ n terms ($\because$ Π denotes product)

  $= \dfrac{\sin[\alpha+(n-1)\beta/2]\sin[n\beta/2]}{\sin(\beta/2)}$ or $\sum_{r=1}^{n}\sin(A+\overline{r-1}B) = \dfrac{\sin\left(A+\dfrac{n-1}{2}B\right)\sin\dfrac{nB}{2}}{\sin\dfrac{B}{2}}$.

- $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + ...$ n terms $= \dfrac{\cos[\alpha+(n-1)\beta/2]\sin[n\beta/2]}{\sin[\beta/2]}$ or

  $\sum_{r=1}^{n}\cos(A+\overline{r-1}B) = \dfrac{\cos\left(A+\dfrac{n-1}{2}B\right)\sin\dfrac{nB}{2}}{\sin\dfrac{B}{2}}$.

- $\sin A/2 \pm \cos A/2 = \sqrt{2}\sin[\pi/4 \pm A] = \sqrt{2}\cos[A \mp \pi/4]$.

- $\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma) = 4\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\beta+\gamma}{2}\cos\dfrac{\gamma+\alpha}{2}$.

- $\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha+\beta+\gamma) = 4\sin\dfrac{\alpha+\beta}{2}\sin\dfrac{\beta+\gamma}{2}\sin\dfrac{\gamma+\alpha}{2}$.

- $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = \cot\alpha$.

15. Conditional Trigonometrical Identities.

We have certain trigonometric identities. Like, $\sin^2\theta+\cos^2\theta=1$ and $1+\tan^2\theta=\sec^2\theta$ etc.

Such identities are identities in the sense that they hold for all value of the angles which satisfy the given condition among them and they are called conditional identities.

If A, B, C denote the angles of a triangle ABC, then the relation A + B + C = π enables us to establish many important identities involving trigonometric ratios of these angles.

(1) If A + B + C = π, then A + B = π – C, B + C = π – A and C + A = π – B.

(2) If A + B + C = π, then $\sin(A+B)=\sin(\pi-C)=\sin C$

Similarly, $\sin(B+C)=\sin(\pi-A)=\sin A$ and $\sin(C+A)=\sin(\pi-B)=\sin B$

(3) If $A+B+C=\pi$, then $\cos(A+B)=\cos(\pi-C)=-\cos C$

Similarly, $\cos(B+C)=\cos(\pi-A)=-\cos A$ and $\cos(C+A)=\cos(\pi-B)=-\cos B$

(4) If A + B + C = π, then $\tan(A+B)=\tan(\pi-C)=-\tan C$

Similarly, $\tan(B+C)=\tan(\pi-A)=-\tan A$ and $\tan(C+A)=\tan(\pi-B)=-\tan B$

(5) If $A+B+C=\pi$, then $\frac{A+B}{2}=\frac{\pi}{2}-\frac{C}{2}$ and $\frac{B+C}{2}=\frac{\pi}{2}-\frac{A}{2}$ and $\frac{C+A}{2}=\frac{\pi}{2}-\frac{B}{2}$

$$\sin\left(\frac{A+B}{2}\right)=\sin\left(\frac{\pi}{2}-\frac{C}{2}\right)=\cos\left(\frac{C}{2}\right),\qquad \cos\left(\frac{A+B}{2}\right)=\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)=\sin\left(\frac{C}{2}\right),$$

$$\tan\left(\frac{A+B}{2}\right)=\tan\left(\frac{\pi}{2}-\frac{C}{2}\right)=\cot\left(\frac{C}{2}\right)$$

All problems on conditional identities are broadly divided into the following three types

1. **Identities involving sine and cosine of the multiple or sub-multiple of the angles involved**

Working Method

Step (i)**:** Use $C \pm D$ formulae.

Step (ii)**:** Use the given relation (A + B + C = π) in the expression obtained in step-(i) such that a factor can be taken common after using multiple angles formulae in the remaining term.

Step (iii)**:** Take the common factor outside.

Step (iv)**:** Again use the given relation (A + B + C =π) within the bracket in such a manner so that we can apply $C \pm D$ formulae.

Step (v)**:** Find the result according to the given options.

2. **Identities involving squares of sine and cosine of multiple or sub-multiples of the angles involved**

Working Method

Step (i)**:** Arrange the terms of the identity such that either $\sin^2 A - \sin^2 B = \sin(A+B).\sin(A-B)$ or $\cos^2 A - \sin^2 B = \cos(A+B).\cos(A-B)$ can be used.

Step (ii)**:** Take the common factor outside.

Step (iii)**:** Use the given relation $(A+B+C=\pi)$ within the bracket in such a manner so that we can apply $C \pm D$ formulae.

Step (iv)**:** Find the result according to the given options.

3. **Identities for tangent and cotangent of the angles**

Working Method

Step (i)**:** Express the sum of the two angles in terms of third angle by using the given relation $(A+B+C=\pi)$.

Step (ii)**:** Taking tangent or cotangent of the angles of both the sides.

Step (iii)**:** Use sum and difference formulae in the left hand side.

Step (iv)**:** Use cross multiplication in the expression obtained in the step (iii).

Step (v)**:** Arrange the terms as per the result required.

Important Tips

☞ **Method of componendo and dividendo**

If $\frac{p}{q} = \frac{a}{b}$, then by componendo and dividendo

We can write $\frac{p+q}{p-q} = \frac{a+b}{a-b}$ or $\frac{q+p}{q-p} = \frac{b+a}{b-a}$ or $\frac{p-q}{p+q} = \frac{a-b}{a+b}$ or $\frac{q-p}{q+p} = \frac{b-a}{b+a}$.