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Knowledge... Everywhere

Mathematics

Hyperbolic Functions

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1. Definition.

We know that parametric co-ordinates of any point on the unit circle $x^2 + y^2 = 1$ is $(\cos\theta, \sin\theta)$; so that these functions are called circular functions and co-ordinates of any point on unit hyperbola $x^2 - y^2 = 1$ is $\left(\frac{e^{\theta} + e^{-\theta}}{2}, \frac{e^{\theta} - e^{-\theta}}{2}\right)$ i.e., $(\cosh\theta, \sinh\theta)$. It means that the relation which exists amongst $\cos\theta, \sin\theta$ and unit circle, that relation also exist amongst $\cosh\theta, \sinh\theta$ and unit hyperbola. Because of this reason these functions are called as Hyperbolic functions.

For any (real or complex) variable quantity x,

(1) $\sinh x = \frac{e^x - e^{-x}}{2}$ [Read as 'hyperbolic sine x']

(2) $\cosh x = \frac{e^x + e^{-x}}{2}$ [Read as 'hyperbolic cosine x']

(3) $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

(4) $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

(5) $\text{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$

(6) $\sec\text{h} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$

Note: $\sinh 0 = 0$, $\cosh 0 = 1$, $\tanh 0 = 0$

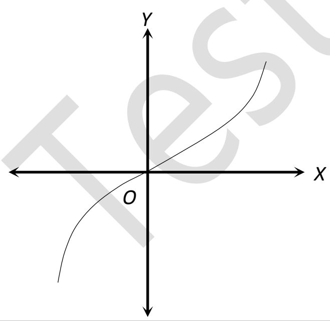
2. Domain and Range of Hyperbolic Functions.

Let x is any real number

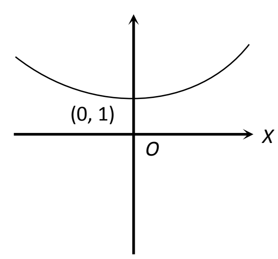
Function	Domain	Range
$\sinh x$	**R**	**R**
$\cosh x$	**R**	$[1, \infty)$
$\tanh x$	**R**	$(-1, 1)$
$\coth x$	R_0	$R - [-1, 1]$
$\text{sech } x$	**R**	$(0, 1]$
$\text{cosech } x$	R_0	R_0

3. Graph of Real Hyperbolic Functions.

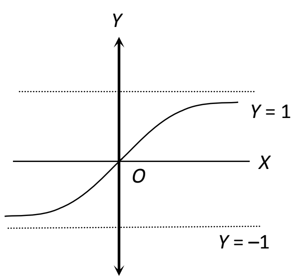
(1) $\sinh x$



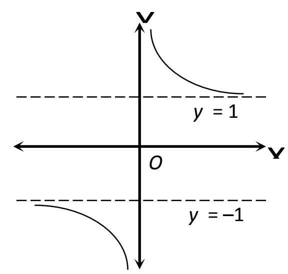
(2) $\cosh x$

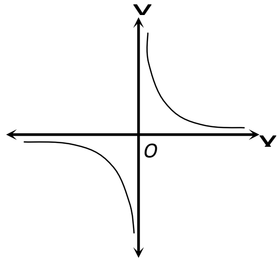
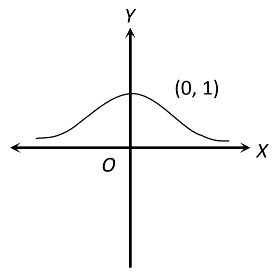


(3) $\tanh x$



(4) $\coth x$



(5) cosech x 	(6) sech x 

4. Formulae for Hyperbolic Functions.

The following formulae can easily be established directly from above definitions

(1) **Reciprocal formulae**

(i) $\text{cosech}\, x = \dfrac{1}{\sinh x}$ (ii) $\text{sech}\, x = \dfrac{1}{\cosh x}$ (iii) $\coth x = \dfrac{1}{\tanh x}$

(iv) $\tanh x = \dfrac{\sinh x}{\cosh x}$ (v) $\coth x = \dfrac{\cosh x}{\sinh x}$

(2) **Square formulae**

(i) $\cosh^2 x - \sinh^2 x = 1$ (ii) $\text{sech}^2 x + \tanh^2 x = 1$

(iii) $\coth^2 x - \text{cosech}^2 x = 1$ (iv) $\cosh^2 x + \sinh^2 x = \cosh 2x$

(3) **Expansion or Sum and difference formulae**

(i) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$ (ii) $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

(iii) $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

(4) **Formulae to transform the product into sum or difference**

(i) $\sinh x + \sinh y = 2\sinh\frac{x+y}{2}\cosh\frac{x-y}{2}$

(ii) $\sinh x - \sinh y = 2\cos\text{h}\frac{x+y}{2}\sin\text{h}\frac{x-y}{2}$

(iii) $\cosh x + \cosh y = 2\cosh\frac{x+y}{2}\cosh\frac{x-y}{2}$

(iv) $\cosh x - \cosh y = 2\sinh\frac{x+y}{2}\sinh\frac{x-y}{2}$

(v) $2\sinh x\cosh y = \sinh(x+y) + \sinh(x-y)$

(vi) $2\cosh x\sinh y = \sinh(x+y) - \sinh(x-y)$

(vii) $2\cosh x\cosh y = \cosh(x+y) + \cosh(x-y)$

(viii) $2\sinh x\sinh y = \cosh(x+y) - \cosh(x-y)$

(ix) $\cosh x + \sinh x = e^x$

(x) $\cosh x - \sinh x = e^{-x}$

(xi) $(\cosh x + \sinh x)^n = \cosh\ nx + \sinh nx$

(5) **Trigonometric ratio of multiple of an angle**

(i) $\sinh 2x = 2\sinh x\cosh x = \frac{2\tanh x}{1-\tanh^2 x}$

(ii) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x = \frac{1+\tanh^2 x}{1-\tanh^2 x}$

(iii) $2\cosh^2 x = \cosh 2x + 1$

(iv) $2\sinh^2 x = \cosh 2x - 1$

(v) $\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$

(vi) $\sinh 3x = 3\sinh x + 4\sinh^3 x$

(vii) $\cosh 3x = 4\cosh^3 x - 3\cosh x$

(viii) $\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$

(6)

(i) $\cosh x + \sinh x = e^x$

(ii) $\cosh x - \sinh x = e^{-x}$

(iii) $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$

5. Transformation of a Hyperbolic Functions

Since, $\cosh^2 x - \sinh^2 x = 1$

$\Rightarrow \sinh x = \sqrt{\cosh^2 x - 1}$ $\quad\Rightarrow \sinh x = \dfrac{\sqrt{1 - \operatorname{sech}^2 x}}{\sec h\, x}$

$\Rightarrow \sinh x = \dfrac{\tanh x}{\sqrt{1 - \tanh^2 x}}$ $\quad\Rightarrow \sinh x = \dfrac{1}{\sqrt{\coth^2 x - 1}}$

Also, $\sinh x = \dfrac{1}{\operatorname{cosech} x}$

In a similar manner we can express $\cosh x$, $\tanh x, \coth x$,................ in terms of other hyperbolic functions.

6. Expansion of Hyperbolic Functions.

(1) $\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + ...$

(2) $\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ...$

(3) $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = x - \frac{x^3}{3} + 2x^5 - \frac{17}{315}x^7 + ...$

The expansion of $\coth x, \operatorname{cosech} x$ does not exist because $\coth(0) = \infty, \operatorname{cosech}(0) = \infty$.

7. Relation between Hyperbolic and Circular Functions.

We have from Euler formulae,

$e^{ix} = \cos x + i\sin x$ ....(i) and $e^{-ix} = \cos x - i\sin x$ ....(ii)

Adding (i) and (ii) $\Rightarrow$ $\cos x = \frac{e^{ix} + e^{-ix}}{2}$

Subtracting (ii) from (i) $\Rightarrow$ $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

Replacing x by ix in these values, we get $\cos(ix) = \frac{e^{-x} + e^{x}}{2} = \cosh x$

$\therefore \cos(ix) = \cosh x$

$\sin(ix) = \frac{e^{-x} - e^{x}}{2i} = i\left(\frac{e^x - e^{-x}}{2}\right)$

$\therefore \sin(ix) = i\sinh x$

Also $\tan(ix) = \frac{\sin(ix)}{\cos(ix)} = \frac{i\sinh x}{\cosh x}$

$\tan(ix) = i\tanh x$

Similarly replacing x by ix in the definitions of $\sinh x$ and $\cosh x$, we get $\cosh(ix) = \dfrac{e^{ix} + e^{-ix}}{2} = \cos x$

Also, $\tanh(ix) = \dfrac{\sinh(ix)}{\cosh(ix)} = \dfrac{i\sin x}{\cos x} = i\tan x$

Thus, we obtain the following relations between hyperbolic and trigonometrical functions.

(1) $\sin(ix) = i\sinh x$ $\sinh(ix) = i\sin x$ $\sinh x = -i\sin(ix)$ $\sin x = -i\sin h(ix)$	(2) $\cos(ix) = \cosh x$ $\cosh(ix) = \cos x$ $\cosh x = \cos(ix)$ $\cos x = \cos h(ix)$
(3) $\tan(ix) = i\tanh x$ $\tanh(ix) = i\tan x$ $\tanh x = -i\tan(ix)$ $\tan x = -i\tanh(ix)$	(4) $\cot(ix) = -i\coth x$ $\coth(ix) = -i\cot x$ $\coth x = i\cot(ix)$ $\cot x = i\coth(ix)$
(5) $\sec(ix) = \text{sech} x$ $\sec h(ix) = \sec x$ $\sec hx = \sec(ix)$ $\sec x = \text{sech}(ix)$	(6) $\text{cosec}(ix) = -i\,\text{cosech}\, x$ $\text{cosech}(ix) = i\,\text{cosec} x$ $\text{cosech}\, x = i\,\text{cosec}(ix)$ $\text{cosec} x = i\,\text{cosech}(ix)$

Important Tips

☞ For obtaining any formula given in (5)[th] article, use the following substitutions in the corresponding formula for trigonometric functions.

$\sin x \longrightarrow i\sinh x$ $\quad$ $\cos x \longrightarrow \cosh x$ $\quad$ $\tan x \longrightarrow i\tanh x$

$\sin^2 x \longrightarrow -\sinh^2 x$ $\quad$ $\cos^2 x \longrightarrow \cosh^2 x$ $\quad$ $\tan^2 x \longrightarrow -\tanh^2 x$

For example,

For finding out the formula for $\cosh 2x$ in terms of $\tanh x$, replace $\tan x$ by $i\tanh x$ and $\tan^2 x$ by $\tan^2 x$ by $-\tanh^2 x$ in the following formula of trigonometric function of $\cos 2x$:

$\cos 2x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$ we get, $\cosh 2x = \dfrac{1+\tanh^2 x}{1-\tanh^2 x}$

8. Period of Hyperbolic Functions.

If for any function $f(x), f(x+T)=f(x),$ then $f(x)$ is called the **Periodic function** and least positive value of T is called the **Period** of the function.

$\because$ $\sinh x = \sinh(2\pi i + x)$

$\cosh x = \cosh(2\pi i + x)$

and $\tanh x = \tanh(\pi i + x)$

Therefore the period of these functions are respectively $2\pi i, 2\pi i$ and πi. Also period of cosech x, sech x and $\coth x$ are respectively $2\pi i$, $2\pi i$ and πi.

Note: Remember that if the period of $f(x)$ is T, then period of $f(nx)$ will be $\left(\frac{T}{n}\right)$.

Hyperbolic function are neither periodic functions nor their curves are periodic but they show the algebraic properties of periodic functions and having imaginary period.

9. Inverse Hyperbolic Functions.

If $\sinh y = x,$ then y is called the inverse hyperbolic sine of x and it is written as $y = \sinh^{-1} x$. Similarly $cosech^{-1} x, \cosh^{-1} x, \tanh^{-1} x$ etc. can be defined.

(1) **Domain and range of Inverse hyperbolic function**

Function	**Domain**	**Range**
$\sinh^{-1} x$	R	R
$\cosh^{-1} x$	[1, ∞)	R
$\tanh^{-1} x$	$(-1, 1)$	R
$\coth^{-1} x$	R – [–1, 1]	R_0
$\text{sech}^{-1} x$	(0, 1]	R
$\text{cosech}^{-1} x$	R_0	R_0

(2) **Relation between inverse hyperbolic function and inverse circular function**

Method: Let $\sinh^{-1} x = y$

$\Rightarrow \quad x = \sinh y = -i\sin(iy) \Rightarrow ix = \sin(iy) \Rightarrow iy = \sin^{-1}(ix)$

$\Rightarrow y = -i\sin^{-1}(ix) \Rightarrow \sinh^{-1} x = -i\sin^{-1}(ix)$

Therefore we get the following relations

(i) $\sinh^{-1} x = -i\sin^{-1}(ix)$

(ii) $\cosh^{-1} x = -i\cos^{-1} x$

(iii) $\tanh^{-1} x = -i\tan^{-1}(ix)$

(iv) $\sec h^{-1}x = -i\sec^{-1} x$

(v) $\text{cosech}^{-1}x = i\,\text{cosec}^{-1}(ix)$

(3) **To express any one inverse hyperbolic function in terms of the other inverse hyperbolic functions**

To express $\sinh^{-1} x$ in terms of the others

(i) Let $\sinh^{-1} x = y \Rightarrow x = \sinh y \Rightarrow \text{cosech}\, y = \dfrac{1}{x} \Rightarrow y = cosec^{-1}\left(\dfrac{1}{x}\right)$

(ii) $\because \cosh y = \sqrt{1+\sinh^2 y} = \sqrt{1+x^2}$

$\because y = \cosh^{-1}\sqrt{1+x^2} \Rightarrow \sinh^{-1} x = \cosh^{-1}\sqrt{1+x^2}$

(iii) $\because \tanh y = \dfrac{\sinh y}{\cosh y} = \dfrac{\sinh y}{\sqrt{1+\sinh^2 y}} = \dfrac{x}{\sqrt{1+x^2}}$

$\therefore y = \tanh^{-1}\dfrac{x}{\sqrt{1+x^2}} \Rightarrow \sinh^{-1} x = \tanh^{-1}\dfrac{x}{\sqrt{1+x^2}}$

(iv) $\because \coth y = \dfrac{\sqrt{1+\sinh^2 y}}{\sinh y} = \dfrac{\sqrt{1+x^2}}{x}$

$\therefore y = \coth^{-1}\dfrac{\sqrt{1+x^2}}{x} \Rightarrow \sinh^{-1} x = \coth^{-1}\dfrac{\sqrt{1+x^2}}{x}$

(v) $\because \sec hy = \dfrac{1}{\cosh y} = \dfrac{1}{\sqrt{1+\sinh^2 y}} = \dfrac{1}{\sqrt{1+x^2}}$

$$y = \operatorname{sec h}^{-1}\frac{1}{\sqrt{1+x^2}} \Rightarrow \sinh^{-1}x = \operatorname{sec h}^{-1}\frac{1}{\sqrt{1+x^2}}$$

(vi) Also, $\sinh^{-1}x = \operatorname{cosech}^{-1}\left(\frac{1}{x}\right)$

From the above, it is clear that

$$\coth^{-1}x = \tanh^{-1}\left(\frac{1}{x}\right)$$

$$\operatorname{sec h}^{-1}x = \cosh^{-1}\left(\frac{1}{x}\right)$$

$$\operatorname{cosech}^{-1} = \sinh^{-1}\left(\frac{1}{x}\right)$$

Note: If x is real then all the above six inverse functions are single valued.

(4) **Relation between inverse hyperbolic functions and logarithmic functions**

Method:

Let $\sinh^{-1}x = y$

$$\Rightarrow x = \sinh y = \frac{e^y - e^{-y}}{2} \Rightarrow e^{2y} - 2xe^y - 1 = 0 \Rightarrow e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

But $e^y > 0, \forall y$ and $x < \sqrt{x^2+1}$

$$\therefore e^y = x + \sqrt{x^2+1} \Rightarrow y = \log(x + \sqrt{x^2+1})$$

$$\therefore \sinh^{-1}x = \log(x+\sqrt{x^2+1})$$

By the above method we can obtain the following relations between inverse hyperbolic functions and principal values of logarithmic functions.

(i) $\sinh^{-1}x = \log(x+\sqrt{x^2+1}) \qquad (-\infty < x < \infty)$

(ii) $\cosh^{-1}x = \log(x+\sqrt{x^2-1}) \qquad (x \geq 1)$

(iii) $\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) \qquad |x|<1$

(iv) $\coth^{-1} x = \frac{1}{2}\log\left(\frac{x+1}{x-1}\right) \qquad |x|>1$

(v) $\operatorname{sech}^{-1} x = \log\left(\frac{1+\sqrt{1-x^2}}{x}\right) \qquad 0<x\le 1$

(vi) $\operatorname{cosech}^{-1} x = \log\left(\frac{1+\sqrt{1+x^2}}{x}\right) \qquad (x \ne 0)$

Note: Formulae for values of $\operatorname{cosech}^{-1} x$, $\operatorname{sech}^{-1} x$ and $\coth^{-1} x$ may be obtained by replacing x by $\frac{1}{x}$ in the values of $\sinh^{-1} x, \cosh^{-1} x$ and $\tanh^{-1} x$ respectively.

10. Separation of Inverse Trigonometric and Inverse Hyperbolic Functions.

If $\sin(\alpha + i\beta) = x + iy$ then $(\alpha + i\beta)$, is called the inverse sine of $(x+iy)$. We can write it as,

$\sin^{-1}(x+iy) = \alpha + i\beta$

Here the following results for inverse functions may be easily established.

(1)

$$\cos^{-1}(x+iy) = \frac{1}{2}\cos^{-1}\left[(x^2+y^2) - \sqrt{(1-x^2+y^2)^2 + 4x^2y^2}\right] + \frac{i}{2}\cosh^{-1}\left[(x^2+y^2) + \sqrt{(1-x^2+y^2)^2 + 4x^2y^2}\right]$$

(2) $\sin^{-1}(x+iy) = \frac{\pi}{2} - \cos^{-1}(x+iy)$

$$= \frac{\pi}{2} - \frac{1}{2}\cos^{-1}\left[(x^2+y^2) - \sqrt{(1-x^2+y^2)^2 + 4x^2y^2}\right] - \frac{i}{2}\cosh^{-1}\left[(x^2+y^2) + \sqrt{(1-x^2+y^2)^2 + 4x^2y^2}\right]$$

(3) $\tan^{-1}(x+iy) = \frac{1}{2}\tan^{-1}\left(\frac{2x}{1-x^2-y^2}\right) + \frac{i}{2}\tanh^{-1}\left(\frac{2y}{1+x^2+y^2}\right) = \frac{1}{2}\tan^{-1}\left(\frac{2x}{1-x^2-y^2}\right) + \frac{i}{4}\log\left[\frac{x^2+(1+y)^2}{x^2+(1-y)^2}\right]$

(4) $\sin^{-1}(\cos\theta + i\sin\theta) = \cos^{-1}(\sqrt{\sin\theta}) + i\sinh^{-1}(\sqrt{\sin\theta})$ or $\cos^{-1}(\sqrt{\sin\theta}) + i\log(\sqrt{\sin\theta} + \sqrt{1+\sin\theta})$

(5) $\cos^{-1}(\cos\theta + i\sin\theta) = \sin^{-1}(\sqrt{\sin\theta}) - i\sinh^{-1}(\sqrt{\sin\theta})$ or $\sin^{-1}(\sqrt{\sin\theta}) - i\log(\sqrt{\sin\theta} + \sqrt{1+\sin\theta})$

(6) $\tan^{-1}(\cos\theta + i\sin\theta) = \frac{\pi}{4} + \frac{i}{4}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$, $(\cos\theta) > 0$

and $\tan^{-1}(\cos\theta + i\sin\theta) = \left(-\frac{\pi}{4}\right) + \frac{1}{4}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$, $(\cos\theta) < 0$

Since each inverse hyperbolic function can be expressed in terms of logarithmic function, therefore for separation into real and imaginary parts of inverse hyperbolic function of complex quantities use the appropriate method.

Note: Both inverse circular and inverse hyperbolic functions are many valued.