Knowledge... Everywhere

Mathematics

Continuity

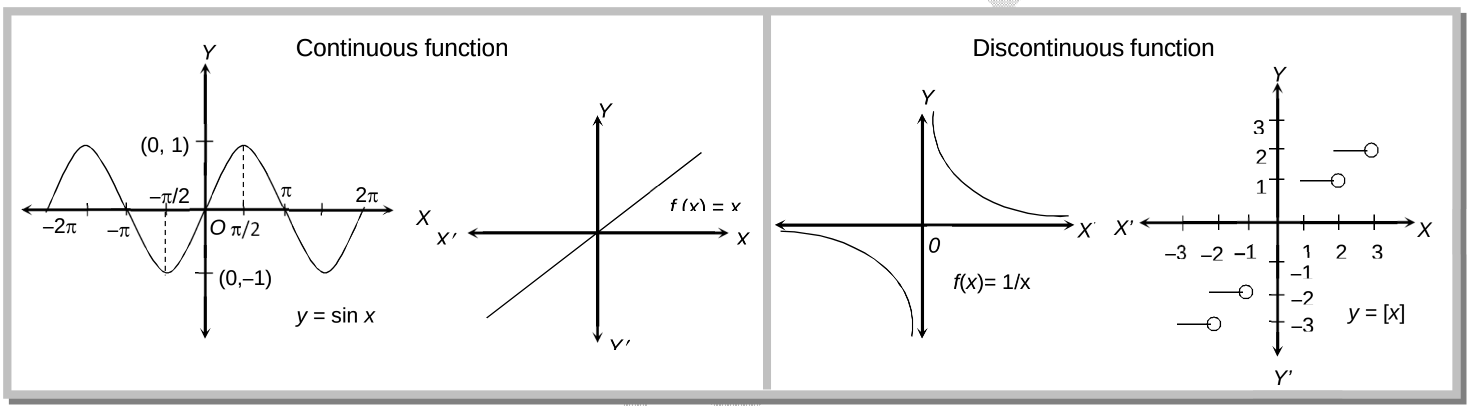
Table of Content

1. Introduction.

The word 'Continuous' means without any break or gap. If the graph of a function has no break, or gap or jump, then it is said to be continuous.

A function which is not continuous is called a discontinuous function.

While studying graphs of functions, we see that graphs of functions $\sin x$, x, $\cos x$, e^x etc. are continuous but greatest integer function [x] has break at every integral point, so it is not continuous. Similarly $\tan x, \cot x, \sec x, \frac{1}{x}$ etc. are also discontinuous function.



2. Continuity of a Function at a Point.

A function $f(x)$ is said to be continuous at a point $x = a$ of its domain iff $\lim_{x \to a} f(x) = f(a)$. *i.e.* a function $f(x)$ is continuous at $x = a$ if and only if it satisfies the following three conditions:

(1) $f(a)$ exists. ('*a*' lies in the domain of *f*)

(2) $\lim_{x \to a} f(x)$ exist *i.e.* $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$ or R.H.L. = L.H.L.

(3) $\lim_{x \to a} f(x) = f(a)$ (limit equals the value of function).

Cauchy's definition of continuity: A function f is said to be continuous at a point *a* of its domain *D* if for every $\varepsilon > 0$ there exists $\delta > 0$ (dependent on ε) such that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

Comparing this definition with the definition of limit we find that $f(x)$ is continuous at $x = a$ if $\lim_{x\to a} f(x)$ exists and is equal to $f(a)$ *i.e.*, if $\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a+} f(x)$.

Heine's definition of continuity: A function f is said to be continuous at a point *a* of its domain *D*, converging to *a*, the sequence $< a_n >$ of the points in *D* converging to *a*, the sequence < $f(a_n) >$ converges to $f(a) i.e.$ $\lim a_n = a \Rightarrow \lim f(a_n) = f(a)$. This definition is mainly used to prove the discontinuity to a function.

Note: Continuity of a function at a point, we find its limit and value at that point, if these two exist and are equal, then function is continuous at that point.

Formal definition of continuity: The function $f(x)$ is said to be continuous at $x = a$, in its domain if for any arbitrary chosen positive number $\in > 0$, we can find a corresponding number δ depending on $\in$ such that $|f(x) - f(a)| < \in \; \forall x$ for which $0 < |x - a| < \delta$.

3. Continuity from Left and Right.

Function $f(x)$ is said to be

(1) Left continuous at *x* = *a* if $\lim_{x\to a-0} f(x) = f(a)$

(2) Right continuous at $x = a$ if $\lim_{x\to a+0} f(x) = f(a)$.

Thus a function $f(x)$ is continuous at a point $x = a$ if it is left continuous as well as right continuous at $x = a$.

4. Continuity of a Function in Open and Closed Interval.

Open interval: A function $f(x)$ is said to be continuous in an open interval (*a, b*) iff it is continuous at every point in that interval.

Note**:** This definition implies the non-breakable behavior of the function $f(x)$ in the interval (*a*, *b*).

Closed interval: A function $f(x)$ is said to be continuous in a closed interval [*a, b*] iff,

(1) *f* is continuous in (*a*, *b*)

(2) *f is* continuous from the right at '*a*' *i.e.* $\lim_{x \to a^+} f(x) = f(a)$

(3) *f is* continuous from the left at '*b*' *i.e.* $\lim_{x \to b^-} f(x) = f(b)$.

5. Continuous Function.

(1) **A list of continuous functions:**

Function *f*(*x*)	**Interval in which *f*(*x*) is continuous**
(i) Constant *K*	$(-\infty, \infty)$
(ii) x^n, (*n* is a positive integer)	$(-\infty, \infty)$
(iii) x^{-n} (*n* is a positive integer)	$(-\infty, \infty) - \{0\}$
(iv) $\|x - a\|$	$(-\infty, \infty)$
(v) $p(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_n$	$(-\infty, \infty)$
(vi) $\frac{p(x)}{q(x)}$, where *p*(*x*) and *q*(*x*) are polynomial in *x*	$(-\infty, \infty) - \{x : q(x) = 0\}$
(vii) $\sin x$	$(-\infty, \infty)$
(viii) $\cos x$	$(-\infty, \infty)$
(ix) $\tan x$	$(-\infty, \infty) - \{(2n + 1)\pi/2 : n \in I\}$
(x) $\cot x$	$(-\infty, \infty) - \{n\pi : n \in I\}$

(xi)	$\sec x$	$(-\infty, \infty) - \{(2n+1)\pi/2 : n \in I\}$
(xii)	$\text{cosec } x$	$(-\infty, \infty) - \{n\pi : n \in I\}$
(xiii)	e^x	$(-\infty, \infty)$
(xiv)	$\log_e x$	$(0, \infty)$

(2) **Properties of continuous functions:** Let $f(x)$ and $g(x)$ be two continuous functions at $x = a$. Then

(i) $cf(x)$ is continuous at $x = a$, where c is any constant

(ii) $f(x) \pm g(x)$ is continuous at $x = a$.

(iii) $f(x) . g(x)$ is continuous at $x = a$.

(iv) $f(x) / g(x)$ is continuous at $x = a$, provided $g(a) \neq 0$.

Important Tips

- *A function $f(x)$ is said to be continuous if it is continuous at each point of its domain.*
- *A function $f(x)$ is said to be everywhere continuous if it is continuous on the entire real line R i.e. $(-\infty, \infty)$. Eg. Polynomial function e^x, $\sin x, \cos x$, constant, x^n, $|x-a|$ etc.*
- *Integral function of a continuous function is a continuous function.*
- *If g(x) is continuous at x = a and f(x) is continuous at x = g (a) then (fog) (x) is continuous at $x = a$.*
- *If f(x) is continuous in a closed interval [a, b] then it is bounded on this interval.*
- *If f(x) is a continuous function defined on [a, b] such that f (a) and f (b) are of opposite signs, then there is atleast one value of x for which f(x) vanishes. i.e. if f (a)>0, f (b) < 0 $\Rightarrow \exists c \in (a, b)$ such that f(c) = 0.*
- *If f(x) is continuous on [a, b] and maps [a, b] into [a, b] then for some x $\in$ [a, b] we have f(x) = x.*

(3) **Continuity of composite function:** If the function $u = f(x)$ is continuous at the point $x = a$, and the function $y = g(u)$ is continuous at the point $u = f(a)$, then the composite function $y = (gof)(x) = g(f(x))$ is continuous at the point $x = a$.

6. Discontinuous Function.

(1) **Discontinuous function:** A function '*f*' which is not continuous at a point $x = a$ in its domain is said to be discontinuous there at. The point '*a*' is called a point of discontinuity of the function.

The discontinuity may arise due to any of the following situations.

(i) $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$ or both may not exist

(ii) $\lim_{x \to a^+} f(x)$ as well as $\lim_{x \to a^-} f(x)$ may exist, but are unequal.

(iii) $\lim_{x \to a^+} f(x)$ as well as $\lim_{x \to a^-} f(x)$ both may exist, but either of the two or both may not be equal to $f(a)$.

Important Tips

- *A function f is said to have removable discontinuity at x = a if* $\lim_{x + a^+} f(x) = \lim_{x + a^-} f(x)$ *but their common value is not equal to f (a).*
Such a discontinuity can be removed by assigning a suitable value to the function f at x = a.
- *If* $\lim_{x \to a} f(x)$ *does not exist, then we cannot remove this discontinuity. So this become a non-removable discontinuity or essential discontinuity.*
- *If f is continuous at x = c and g is discontinuous at x = c, then*
(a) f +g and f – g are discontinuous (b) f.g may be continuous
- *If f and g are discontinuous at x = c, then f + g, f – g and fg may still be continuous.*
- *Point functions (domain and range consists one value only) is not a continuous function.*