Mathematics

Differentiability

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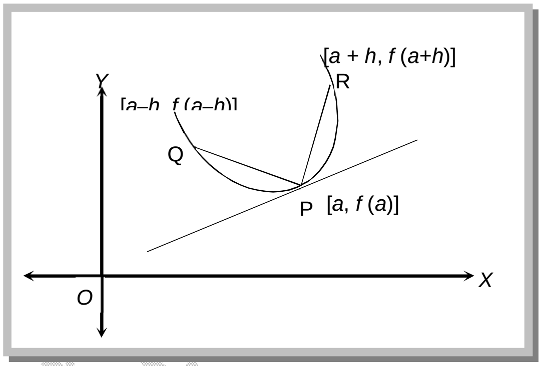
1. Differentiability of a Function at a Point.

(1) **Meaning of differentiability at a point:** Consider the function $f(x)$ defined on an open interval (b, c) let $P(a, f(a))$ be a point on the curve $y = f(x)$ and let Q $(a-h, f(a-h))$ and $R(a+h, f(a+h))$ be two neighboring points on the left hand side and right hand side respectively of the point P.



Then slope of chord $PQ = \dfrac{f(a-h)-f(a)}{(a-h)-a} = \dfrac{f(a-h)-f(a)}{-h}$

And, slope of chord $PR = \dfrac{f(a+h)-f(a)}{a+h-a} = \dfrac{f(a+h)-f(a)}{h}$.

$\because$ As $h \to 0$, point Q and R both tends to P from left hand and right hand respectively. Consequently, chords PQ and PR becomes tangent at point P.

Thus, $\lim\limits_{h\to 0} \dfrac{f(a-h)-f(a)}{-h} = \lim\limits_{h\to 0}$ (slope of chord PQ) = $\lim\limits_{Q\to P}$ (slope of chord PQ)

Slope of the tangent at point P, which is limiting position of the chords drawn on the left hand side of point P and $\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ = $\lim\limits_{h\to 0}$ (slope of chord PR) = $\lim\limits_{R\to P}$ (slope of chord PR).

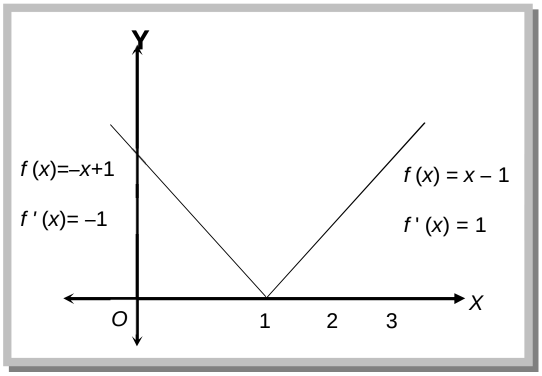
$\Rightarrow$ Slope of the tangent at point P, which is the limiting position of the chords drawn on the right hand side of point P.

Now, $f(x)$ is differentiable at $x = a \Leftrightarrow \lim\limits_{h\to 0} \dfrac{f(a-h)-f(a)}{-h} = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$

$\Leftrightarrow$ There is a unique tangent at point P.

Thus, $f(x)$ is differentiable at point P, iff there exists a unique tangent at point P. In other words, $f(x)$ is differentiable at a point P iff the curve does not have P as a corner point. *i.e.*, "the function is not differentiable at those points on which function has jumps (or holes) and sharp edges."

Let us consider the function $f(x) = |x-1|$, which can be graphically shown,



Which show $f(x)$ is not differentiable at $x = 1$. Since, $f(x)$ has sharp edge at $x = 1$.

Mathematically: The right hand derivative at $x = 1$ is 1 and left-hand derivative at $x = 1$ is –1. Thus, $f(x)$ is not differentiable at $x = 1$.

(2) **Right hand derivative:** Right hand derivative of $f(x)$ at $x = a$, denoted by $f'(a+0)$ or $f'(a+)$, is the $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$.

(3) **Left hand derivative:** Left hand derivative of $f(x)$ at $x = a,$ denoted by $f'(a-0)$ or $f'(a-)$, is the $\lim_{h \to 0} \frac{f(a-h)-f(a)}{-h}$.

(4) A function $f(x)$ is said to be differentiable (finitely) at $x = a$ if $f'(a+0) = f'(a-0)$ = finite

i.e., $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h \to 0} \frac{f(a-h)-f(a)}{-h}$ = finite and the common limit is called the derivative of $f(x)$ at $x = a$, denoted by $f'(a)$. Clearly, $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ $\{x \to a$ from the left as well as from the right$\}$.

2. Differentiability in an Open Interval.

A function $f(x)$ defined in an open interval (a, b) is said to be differentiable or derivable in open interval (a, b) if it is differentiable at each point of (a, b).

Differentiability in a closed interval: A function $f:[a,b] \to R$ is said to be differentiable in [a, b] if

(1) $f'(x)$ exists for every x such that $a < x < b$ *i.e.* f is differentiable in (a, b).

(2) Right hand derivative of f at $x = a$ exists.

(3) Left hand derivative of f at $x = b$ exists.

Everywhere differentiable function: If a function is differentiable at each $x \in R$, then it is said to be everywhere differentiable. *e.g.*, A constant function, a polynomial function, $\sin x, \cos x$ etc. are everywhere differentiable.

Some standard results on differentiability

(1) Every polynomial function is differentiable at each $x \in R$.

(2) The exponential function $a^x, a > 0$ is differentiable at each $x \in R$.

(3) Every constant function is differentiable at each $x \in R$.

(4) The logarithmic function is differentiable at each point in its domain.

(5) Trigonometric and inverse trigonometric functions are differentiable in their domains.

(6) The sum, difference, product and quotient of two differentiable functions is differentiable.

(7) The composition of differentiable function is a differentiable function.

Important Tips

- *If f is derivable in the open interval (a, b) and also at the end points 'a' and 'b', then f is said to be derivable in the closed interval [a, b].*
- *A function f is said to be a differentiable function if it is differentiable at every point of its domain.*
- *If a function is differentiable at a point, then it is continuous also at that point.*

i.e. Differentiability $\Rightarrow$ Continuity, but the converse need not be true.

- *If a function 'f' is not differentiable but is continuous at x = a, it geometrically implies a sharp corner or kink at x = a.*
- *If f(x) is differentiable at x = a and g(x) is not differentiable at x =a, then the product function f(x).g(x) can still be differentiable at x = a.*
- *If f(x) and g(x) both are not differentiable at x = a then the product function f(x).g(x) can still be differentiable at x = a.*
- *If f(x) is differentiable at x = a and g(x) is not differentiable at x = a then the sum function f(x) + g(x) is also not differentiable at x = a*
- *If f(x) and g(x) both are not differentiable at x = a, then the sum function may be a differentiable function.*