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Mathematics

Indefinite Integrals

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Introduction.

A function $\phi(x)$ is called a primitive or an antiderivative of a function $f(x)$ if $\phi'(x) = f(x)$.

For example, $\frac{x^5}{5}$ is a primitive of x^4 because $\frac{d}{dx}\left(\frac{x^5}{5}\right) = x^4$

Let $\phi(x)$ be a primitive of a function $f(x)$ and let *c* be any constant.

Then $\frac{d}{dx}[\phi(x) + c] = \phi'(x) = f(x)$ $\qquad [\because \ \phi'(x) = f(x)]$

$\Rightarrow \ \phi(x) + c$ is also a primitive of $f(x)$.

Thus, if a function $f(x)$ possesses a primitive, then it possesses infinitely many primitives which are contained in the expression $\phi(x) + c,$ where *c* is a constant.

For example $\frac{x^5}{5}, \frac{x^5}{5} + 2, \frac{x^5}{5} - 1$ etc. are primitives of x^4.

Note: If $F_1(x)$ and $F_2(x)$ are two antiderivatives of a function *f* (*x*) on an interval [*a*, *b*], then the difference between them is a constant.

1. Definition.

Let $f(x)$ be a function. Then the collection of all its primitives is called the indefinite integral of $f(x)$ and is denoted by $\int f(x)\,dx$.

Thus, $\frac{d}{dx}(\phi(x) + c) = f(x) \Rightarrow \int f(x)\,dx = \phi(x) + c$

Where $\phi(x)$ is primitive of $f(x)$ and *c* is an arbitrary constant known as the constant of integration.

Here $\int$ is the integral sign, $f(x)$ is the integrand, *x* is the variable of integration and *dx* is the element of integration.

The process of finding an indefinite integral of a given function is called integration of the function.

It follows from the above discussion that integrating a function $f(x)$ means finding a function $\phi(x)$ such that $\frac{d}{dx}(\phi(x)) = f(x).$

2. Comparison between Differentiation and Integration.

(1) Differentiation and integration both are operations on functions and each gives rise to a function.

(2) Each function is not differentiable or integrable.

(3) The derivative of a function, if it exists, is unique. The integral of a function, if it exists, is not unique.

(4) The derivative of a polynomial function decreases its degree by 1, but the integral of a polynomial function increases its degree by 1.

(5) The derivative has a geometrical meaning, namely, the slope of the tangent to a curve at a point on it. The integral has also a geometrical meaning, namely, the area of some region.

(6) The derivative is used in obtaining some physical quantities like velocity, acceleration etc. of a particle. The integral is used in obtaining some physical quantities like centre of mass, momentum etc.

(7) Differentiation and integration are inverse of each other.

3. Properties of Integrals.

(1) The differentiation of an integral is the integrand itself or the process of differentiation and integration neutralize each other, *i.e.*, $\frac{d}{dx}\left[\int f(x)dx\right] = f(x).$

(2) The integral of the product of a constant and a function is equal to the product of the constant and the integral of the function, *i.e.*, $\int c f(x)dx = c\int f(x)dx$.

(3) Integral of the sum or difference of two functions is equal to the sum or difference of their integrals, *i.e.*, $\int \{f_1(x) \pm f_2(x)\}dx = \int f_1(x)dx \pm \int f_2(x)dx$

In the general form, $\int \{k_1 \,.\, f_1(x) \pm k_2 \,.\, f_2(x) \pm k_3 \,.\, f_3(x) \pm .....\}dx$

$= k_1 \int f_1(x)dx \pm k_2 \int f_2(x)dx \pm k_3 \int f_3(x)dx \pm ....$

4. Fundamental Integration Formulae.

(1)

(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1 \qquad \because \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$

(ii) $\int dx = x + c$

(iii) $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + c,$

(iv) $\int (ax+b)^n dx = \frac{1}{a} \cdot \frac{(ax+b)^{n+1}}{n+1} + c$

(2)

(i) $\int \frac{1}{x} dx = \log|x| + c \qquad \because \frac{d}{dx}(\log|x|) = \frac{1}{x}$

(ii) $\int \frac{1}{ax+b} dx = \frac{1}{a}(\log|ax+b| + c$

(3) $\int e^x dx = e^x + c \qquad \because \frac{d}{dx}(e^x) = e^x$

(4) $\int a^x dx = \frac{a^x}{\log_e a} + c \qquad \because \frac{d}{dx}\left(\frac{a^x}{\log_e a}\right) = a^x$

(5) $\int \sin x \, dx = -\cos x + c \qquad \because \frac{d}{dx}(-\cos x) = \sin x$

(6) $\int \cos x \, dx = \sin x + c \qquad \because \frac{d}{dx}(\sin x) = \cos x$

(7) $\int \sec^2 x \, dx = \tan x + c \qquad \because \frac{d}{dx}(\tan x) = \sec^2 x$

(8) $\int \text{cosec}^2 x \, dx = -\cot x + c \qquad \because \frac{d}{dx}(-\cot x) = \text{cosec}^2 x$

(9) $\int \sec x \tan x \, dx = \sec x + c$ $\quad \because \frac{d}{dx}(\sec x) = \sec x \tan x$

(10) $\int \text{cosec } x \cot x \, dx = -\text{cosec } x + c$ $\quad \because \frac{d}{dx}(-\text{cosec } x) = cosec\, x \cot x$

(11) $\int \tan x \, dx = -\log|\cos x| + c = \log|\sec x| + c$ $\quad \because \frac{d}{dx}(\log \cos x) = -\tan x$

(12) $\int \cot x \, dx = \log|\sin x| + c = -\log|cosec\, x| + c$ $\quad \because \frac{d}{dx}(\log \sin x) = \cot x$

(13) $\int \sec x \, dx = \log|\sec x + \tan x| + c = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$ $\quad \because \frac{d}{dx}\log(\sec x + \tan x) = \sec x$

(14) $\int cosec\, x \, dx = \log|cosec\, x - \cot x| + c = \log \tan \frac{x}{2} + c$ $\quad \because \frac{d}{dx}(\log|cosec\, x - \cot x|) = cosec\, x$

(15) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c = -\cos^{-1} x + c$ $\quad \because \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

(16) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c = -\cos^{-1}\frac{x}{a} + c$ $\quad \because \frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{a^2-x^2}}, \frac{d}{dx}(\cos^{-1}\frac{x}{a}) = \frac{-1}{\sqrt{a^2-x^2}}$

(17) $\int \frac{dx}{1+x^2} = \tan^{-1} x + c = -\cot^{-1} x + c$ $\quad \because \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

(18) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c = \frac{-1}{a}\cot^{-1}\frac{x}{a} + c$ $\quad \because \frac{d}{dx}\left(\tan^{-1}\frac{x}{a}\right) = \frac{a}{a^2+x^2} \quad \frac{d}{dx}\left(\cot^{-1}\frac{x}{a}\right) = \frac{-a}{a^2+x^2}$

(19) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c = -cosec^{-1} x + c$ $\quad \because \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}(cosec^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

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(20) $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c = \frac{-1}{a}\operatorname{cosec}^{-1}\frac{x}{a} + c \quad \because \frac{d}{dx}\left(\sec^{-1}\frac{x}{a}\right) = \frac{a}{x\sqrt{x^2 - a^2}} \quad \frac{d}{dx}(\operatorname{cosec}^{-1}\frac{x}{a}) = \frac{-a}{x\sqrt{x^2 - a^2}}$

Note: In any of the fundamental integration formulae, if x is replaced by $ax + b$, then the same formulae is applicable but we must divide by coefficient of x or derivative of $(ax + b)$ *i.e.*, a. In general, if $\int f(x)dx = \phi(x) + c$,

then $\int f(ax + b)dx = \frac{1}{a}\phi(ax + b) + c$

$\int \sin(ax + b)dx = \frac{-1}{a}\cos(ax + b) + c, \quad \int \sec(ax + b)dx = \frac{1}{a}\log|\sec(ax + b) + \tan(ax + b)| + c$ *etc.*

Some more results:

(i) $\int \frac{1}{x^2 - a^2} = \frac{-1}{a}\coth^{-1}\frac{x}{a} + c = \frac{1}{2a}\log\left|\frac{x - a}{x + a}\right| + c$, when $x > a$

(ii) $\int \frac{1}{a^2 - x^2}dx = \frac{1}{a}\tanh^{-1}\frac{x}{a} + c = \frac{1}{2a}\log\left|\frac{a + x}{a - x}\right| + c$, when $x < a$

(iii) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\{|x + \sqrt{x^2 - a^2}|\} + c = \cosh^{-1}\left(\frac{x}{a}\right) + c$

(iv) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\{|x + \sqrt{x^2 + a^2}|\} + c = \sinh^{-1}\left(\frac{x}{a}\right) + c$

(v) $\int \sqrt{a^2 - x^2}dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right) + c$

(vi) $\int \sqrt{x^2 - a^2}dx = \frac{1}{2}x\sqrt{x^2 - a^2} - \frac{1}{2}a^2\log\{x + \sqrt{x^2 - a^2}\} + c = \frac{1}{2}x\sqrt{x^2 - a^2} - \frac{1}{2}a^2\cosh^{-1}\left(\frac{x}{a}\right) + c$

(vii) $\int \sqrt{x^2 + a^2}dx = \frac{1}{2}x\sqrt{x^2 + a^2} + \frac{1}{2}a^2\log\{x + \sqrt{x^2 + a^2}\} + c = \frac{1}{2}x\sqrt{x^2 + a^2} + \frac{1}{2}a^2\sinh^{-1}\left(\frac{x}{a}\right) + c$

Important Tips

- *The signum function has an antiderivative on any interval which doesn't contain the point x = 0, and does not possess an anti-derivative on any interval which contains the point.*
- *The antiderivative of every odd function is an even function and vice-versa*

5. Integration by Substitution.

(1) **When integrand is a function *i.e.,* $\int f[\phi(x)]\phi'(x)\,dx$:**

Here, we put $\phi(x)=t$, so that $\phi'(x)dx = dt$ and in that case the integrand is reduced to $\int f(t)dt$. In this method, the integrand is broken into two factors so that one factor can be expressed in terms of the function whose differential coefficient is the second factor.

(2) **When integrand is the product of two factors such that one is the derivative of the others *i.e.,* $I=\int f'(x).\,f(x).\,dx$**

In this case we put $f(x)=t$ and convert it into a standard integral.

(3) **Integral of a function of the form $f(ax+b)$:** Here we put $ax+b=t$ and convert it into standard integral. Obviously if $\int f(x)dx = \phi(x)$, then, $\int f(ax+b)dx = \frac{1}{a}\phi(ax+b)+c$

(4) **If integral of a function of the form $\frac{f'(x)}{f(x)}dx = \log[f(x)]+c$**

(5) **If integral of a function of the form, $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}+c \qquad [n \neq -1]$**

(6) **If the integral of a function of the form, $\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)}+c$**

(7) **Standard substitutions**

	Integrand form	**Substitution**
(i)	$\sqrt{a^2 - x^2}$, $\frac{1}{\sqrt{a^2 - x^2}}$, $a^2 - x^2$	$x = a\sin\theta$, $x = a\cos\theta$
(ii)	$\sqrt{x^2 + a^2}$, $\frac{1}{\sqrt{x^2 + a^2}}$, $x^2 + a^2$	$x = a\tan\theta$ or $x = a\sinh\theta$
(iii)	$\sqrt{x^2 - a^2}$, $\frac{1}{\sqrt{x^2 - a^2}}$, $x^2 - a^2$	$x = a\sec\theta$ or $x = a\cosh\theta$
(iv)	$\sqrt{\frac{x}{a+x}}$, $\sqrt{\frac{a+x}{x}}$, $\sqrt{x(a+x)}$, $\frac{1}{\sqrt{x(a+x)}}$	$x = a\tan^2\theta$
(v)	$\sqrt{\frac{x}{a-x}}$, $\sqrt{\frac{a-x}{x}}$, $\sqrt{x(a-x)}$, $\frac{1}{\sqrt{x(a-x)}}$	$x = a\sin^2\theta$
(vi)	$\sqrt{\frac{x}{x-a}}$, $\sqrt{\frac{x-a}{x}}$, $\sqrt{x(x-a)}$, $\frac{1}{\sqrt{x(x-a)}}$	$x = a\sec^2\theta$
(vii)	$\sqrt{\frac{a-x}{a+x}}$, $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
(viii)	$\sqrt{\frac{x-\alpha}{\beta-x}}$, $\sqrt{(x-\alpha)(\beta-x)}$, $(\beta > \alpha)$	$x = \alpha\cos^2\theta + \beta\sin^2\theta$

6. Integration by Parts.

(1) **When integrand involves more than one type of functions**: We may solve such integrals by a rule which is known as integration by parts. We know that,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \Rightarrow d(uv) = udv + vdu \Rightarrow \int d(uv) = \int udv + \int vdu$$

If *u* and *v* are two functions of *x*, then $\int \underset{I}{u}\, \underset{II}{v}\, dx = u\int v\, dx - \int \{\frac{du}{dx}.\int vdx\}dx$ *i.e.*, the integral of the product of two functions = (First function) × (Integral of second function) – Integral of {(Differentiation of first function) × (Integral of second function)}

Integration with the help of above rule is called integration by parts. Before applying this rule proper choice of first and second function is necessary. Normally we use the following methods:

(i) In the product of two functions, one of the function is not directly integrable (*i.e.*, $\log|x|, \sin^{-1}x, \cos^{-1}x, \tan^{-1}x$ ......etc), then we take it as the first function and the remaining function is taken as the second function.

(ii) If there is no other function, then unity is taken as the second function *e.g.* In the integration of $\int \sin^{-1} x\, dx, \int \log x\, dx, 1$ is taken as the second function.

(iii) If both of the function are directly integrable then the first function is chosen in such a way that the derivative of the function thus obtained under integral sign is easily integrable.
Usually, we use the following preference order for the first function. (Inverse, Logarithmic, Algebraic, Trigonometric, exponential). This rule is simply called as "I LATE".

Important Tips

☞ *If* $I_n = \int x^n . e^{ax} dx$, *then* $I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$

☞ *If* $I_n = \int (\log x) dx$, *then* $I_n = x \log x - x$

☞ *If* $I_n = \int \frac{1}{\log x} dx$, *then* $I_n = \log(\log x) + \log x + \frac{(\log x)^2}{2.(2!)} + \frac{(\log x)^3}{3(3!)} + ...$

☞ *If* $I_n = \int (\log x)^n dx$; *then* $I_n = x(\log x)^n - n.I_{n-1}$

☞ *Successive integration by parts can be performed when one of the functions is* x^n *(n is positive integer) which will be successively differentiated and the other is either of the following* $\sin ax, \cos ax, e^{ax}, e^{-ax}, (x+a)^m$ *which will be successively integrated.*

☞ ***Chain rule :*** $\int u.v\, dx = u v_1 - u' v_2 + u'' v_3 - u''' v_4 + ... + (-1)^{n-1} u^{n-1} v_n + (-1)^n \int u^n . v_n\, dx$ *Where* u^n *stands for n*th *differential coefficient of u and* v_n *stands for n*th *integral of v.*

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(2) **Integral is of the form** $\int e^x \{f(x) + f'(x)\} dx$ **:** If the integral is of the form $\int e^x \{f(x) + f'(x)\} dx$, then by breaking this integral into two integrals integrate one integral by parts and keeping other integral as it is, by doing so, we get

(i) $\int e^x [f(x) + f'(x)] dx = e^x f(x) + c$

(ii) $\int e^{mx} [mf(x) + f'(x)] dx = e^{mx} f(x) + c$

(iii) $\int e^{mx}\left[f(x)+\frac{f'(x)}{m}\right]dx = \frac{e^{mx} f(x)}{m}+c$

(3) **Integral is of the form $[x\ f'(x)+f(x)]dx$:** If the integral is of the form $\int [x\ f'(x)+f(x)]dx$ then by breaking this integral into two integrals, integrate one integral by parts and keeping other integral as it is, by doing so, we get, $\int [x\ f'(x)+f(x)]dx = x\ f(x)+c$

(4) **Integrals of the form $\int e^{ax}\sin bx dx$, $\int e^{ax}\cos bx\ dx$:**

Working rule **:** To evaluate $\int e^{ax}\sin bx\ dx$ or $\int e^{ax}\cos bx\ dx$, proceed as follows

(i) Put the given integral equal to *I*.

(ii) Integrate by parts, taking e^{ax} as the first function.

(iii) Again, integrate by parts taking e^{ax} as the first function. This will involve *I*.

(iv) Transpose and collect terms involving *I* and then obtain the value of *I*.

Let $I=\int e^{ax}\sin bx\ dx$. Then $I=\int \underset{\text{I}}{e^{ax}}.\underset{\text{II}}{\sin bx}\ dx = -e^{ax}.\frac{\cos bx}{b} - \int ae^{ax}.\left(\frac{-\cos bx}{b}\right)dx$

$$=\frac{-1}{b}e^{ax}.\cos bx+\frac{a}{b}\int \underset{\text{I}}{e^{ax}}.\underset{\text{II}}{\cos bx}\ dx \quad = \frac{-1}{b}e^{ax}.\cos bx+\frac{a}{b}\left[\frac{e^{ax}.\sin bx}{b}-\int ae^{ax}.\frac{\sin bx}{b}dx\right]$$

$$=\frac{-1}{b}e^{ax}.\cos bx+\frac{a}{b^2}e^{ax}.\sin bx-\frac{a^2}{b^2}\int e^{ax}.\sin bx\ dx \quad = \frac{-1}{b}e^{ax}.\cos bx+\frac{a}{b^2}e^{ax}.\sin bx-\frac{a^2}{b^2}I$$

$$I+I.\frac{a^2}{b^2}=\frac{e^{ax}}{b^2}(a\sin bx-b\cos bx) \Rightarrow I=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+c$$

Thus, $\int e^{ax}\sin bx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)+c \quad = \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin(bx-\tan^{-1}\frac{b}{a})+c$

Similarly $\int e^{ax}.\cos bx\ dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)+c \quad = \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx-\tan^{-1}\frac{b}{a}\right)+c$

Note: $\int e^{ax}.\sin(bx+c)dx = \frac{e^{ax}}{a^2+b^2}[a\sin(bx+c)-b\cos(bx+c)]+k = \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left[(bx+c)-\tan^{-1}\left(\frac{b}{a}\right)\right]+k$

$\int e^{ax}.\cos(bx+c)dx = \frac{e^{ax}}{a^2+b^2}[a\cos(bx+c)+b\sin(bx+c)]+k = \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left[(bx+c)-\tan^{-1}\left(\frac{b}{a}\right)\right]+k$

Important Tips

- $\int x\,e^{ax}\sin(bx+c)dx = \frac{x\,e^{ax}}{a^2+b^2}[a\sin(bx+c)-b\cos(bx+c)]-\frac{e^{ax}}{(a^2+b^2)^2}[(a^2-b^2)\sin bx(bx+c)-2ab\cos bx(bx+c)]+k$
- $\int x.e^{ax}\cos(bx+c)dx = \frac{x.e^{ax}}{a^2+b^2}[a\cos(bx+c)+b\sin(bx+c)]-\frac{e^{ax}}{(a^2+b^2)^2}[(a^2-b^2)\cos(bx+c)+2ab\sin(bx+c)]+k$
- $\int a^x.\sin(bx+c)dx = \frac{a^x}{(\log a)^2+b^2}[(\log a)\sin(bx+c)-b\cos(bx+c)]+k$
- $\int a^x.\cos(bx+c)dx = \frac{a^x}{(\log a)^2+b^2}[(\log a)\cos(bx+c)+b\sin(bx+c)]+k$

7. Evaluation of the various forms of Integrals by use of Standard Results.

(1) Integral of the form $\int \frac{dx}{ax^2+bx+c}$, where ax^2+bx+c can not be resolved into factors.

(2) Integral of the form $\int \frac{px+q}{ax^2+bx+c}dx$.

(3) Integral of the form $\int \frac{dx}{\sqrt{ax^2+bx+c}}$.

(4) Integral of the form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$.

(5) Integral of the form $\int \frac{f(x)}{ax^2+bx+c}dx$, where $f(x)$ is a polynomial of degree 2 or greater than 2.

(6) Integral of the form

(i) $\int \frac{x^2+1}{x^4+kx^2+1}dx$,

(ii) $\int \frac{x^2-1}{x^4+kx^2+1}dx$, Where k is any constant

(7) Integral of the form $\int \sqrt{ax^2+bx+c}\,dx$

(8) Integral of the form $\int (px+q)\sqrt{ax^2+bx+c}\,dx$

(9) Integral of the form $\int \frac{dx}{P\sqrt{Q}}$

(1) **Integrals of the form** $\int \frac{dx}{ax^2+bx+c}$**, where** ax^2+bx+c **cannot be resolved into factors.**

We have, $ax^2+bx+c = a\left(x^2+\frac{b}{a}.x+\frac{c}{a}\right) = a\left[\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)\right] = a\left[\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2-4ac}{4a^2}\right)\right]$

Case (i): When $b^2-4ac>0$

$$\therefore \int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2-\left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2}, \qquad \left[\text{form} \int \frac{dx}{x^2-a^2}\right]$$

$$= \frac{1}{a}.\frac{1}{2.\frac{\sqrt{b^2-4ac}}{2a}}\log\left|\frac{x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}}{x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}}\right|+c = \frac{1}{\sqrt{b^2-4ac}}\log\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|+c$$

Case (ii): When $b^2 - 4ac < 0$

$$\int \frac{dx}{ax^2 + bx + c} = \frac{1}{a}\int \frac{dx}{\left(x + \frac{b}{2a}\right)^2 + \left(\frac{\sqrt{4ac - b^2}}{2a}\right)^2}, \qquad \left[\text{form} \int \frac{dx}{x^2 + a^2}\right]$$

$$= \frac{1}{a} \cdot \frac{1}{\frac{\sqrt{4ac - b^2}}{2a}} \tan^{-1}\left[\frac{x + \frac{b}{2a}}{\frac{\sqrt{4ac - b^2}}{2a}}\right] + c = \frac{2}{\sqrt{4ac - b^2}} \tan^{-1}\left[\frac{2ax + b}{\sqrt{4ac - b^2}}\right] + c$$

Working rule for evaluating $\int \frac{dx}{ax^2 + bx + c}$ **:** To evaluate this form of integrals proceed as follows :

(i) Make the coefficient of x^2 unity by taking '*a*' common from $ax^2 + bx + c$.

(ii) Express the terms containing x^2 and x in the form of a perfect square by adding and subtracting the square of half of the coefficient of *x*.

(iii) Put the linear expression in *x* equal to *t* and express the integrals in terms of *t*.

(iv) The resultant integrand will be either in $\int \frac{dx}{x^2 + a^2}$ or $\int \frac{dx}{x^2 - a^2}$ or $\int \frac{dx}{a^2 - x^2}$ standard form. After using the standard formulae, express the results in terms of *x*.

(2) **Integral of the form** $\int \frac{px + q}{ax^2 + bx + c} dx$ **:** The integration of the function $\frac{px + q}{ax^2 + bx + c}$ is effected by breaking $px + q$ into two parts such that one part is the differential coefficient of the denominator and the other part is a constant.

If *M* and *N* are two constants, then we express $px + q$ as $px + q = M\frac{d}{dx}(ax^2 + bx + c) + N$

$= M.(2ax + b) + N \;\; = (2aM)x + Mb + N$.

Comparing the coefficients of *x* and constant terms on both sides, we have, $p = 2aM \Rightarrow M = \frac{p}{2a}$ and

$q = Mb + N \Rightarrow N = q - Mb = q - \frac{p}{2a}b$.

Thus, *M* and *N* are known. Hence, the given integral is $\int \frac{px + q}{ax^2 + bx + c} dx = \int \frac{\frac{p}{2a}(2ax + b) + \left(q - \frac{p}{2a}b\right)}{ax^2 + bx + c} dx$

$$= \frac{p}{2a}\int \frac{2ax + b}{ax^2 + bx + c} dx + \left(q - \frac{p}{2a}b\right)\int \frac{dx}{ax^2 + bx + c} = \frac{p}{2a}\log|\,ax^2 + bx + c| + \left(q - \frac{p}{2a}b\right)\int \frac{dx}{ax^2 + bx + c} + C$$

The integral on R.H.S. can be evaluated by the method discussed in previous section.

(i) If $b^2 - 4ac < 0$, then $\int \frac{px+q}{ax^2+bx+c}dx = \frac{p}{2a}\log|ax^2+bx+c| + \frac{(2aq-bp)}{a\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}} + k$

(ii) If $b^2 - 4ac > 0$, then

$$\int \frac{px+q}{ax^2+bx+c}dx = \frac{p}{2a}\log|ax^2+bx+c| + \frac{(2aq-bp)}{2a\sqrt{b^2-4ac}}\log\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| + k$$

(3) **Integral of the form** $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ **:** To evaluate this form of integrals proceed as follows :

(i) Make the coefficient of x^2 unity by taking $\sqrt{a}$ common from $\sqrt{ax^2+bx+c}$.

Then, $\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\int \frac{dx}{\sqrt{x^2+\frac{b}{a}x+\frac{c}{a}}}$.

(ii) Put $x^2+\frac{b}{a}x+\frac{c}{a}$, by the method of completing the square in the form, $\sqrt{A^2-X^2}$ or $\sqrt{X^2+A^2}$ or $\sqrt{X^2-A^2}$ where, X is a linear function of x and A is a constant.

(iii) After this, use any of the following standard formulae according to the case under consideration

$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)+c \Rightarrow \int \frac{dx}{\sqrt{x^2+a^2}} = \log|x+\sqrt{x^2+a^2}|+c$ and $\int \frac{dx}{\sqrt{x^2-a^2}} = \log|x+\sqrt{x^2-a^2}|+c$.

Note: If $a<0$, $b^2-4ac>0$, then $\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{-a}}\sin^{-1}\left(\frac{2ax+b}{\sqrt{b^2-4ac}}\right)+k$.

- If $a>0$, $b^2-4ac<0$, then $\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\sinh^{-1}\left[\frac{2ax+b}{\sqrt{4ac-b^2}}\right]+k$

- If $a>0$, $b^2-4ac>0$ $\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\cosh^{-1}\frac{2ax+b}{\sqrt{b^2-4ac}}+k$

(4) **Integral of the form** $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$ **:** To evaluate this form of integrals, first we write,

$$px+q = M\frac{d}{dx}(ax^2+bx+c)+N \Rightarrow \quad px+q = M(2ax+b)+N$$

Where *M* and *N* are constants.

By equating the coefficients of *x* and constant terms on both sides, we get

$$p = 2aM \Rightarrow M = \frac{p}{2a} \text{ and } q = bM+N \Rightarrow N = q-\frac{bp}{2a}$$

In this way, the integral breaks up into two parts given by

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx = \frac{p}{2a}\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}dx + \left(q-\frac{bp}{2a}\right)\int \frac{dx}{\sqrt{ax^2+bx+c}} \qquad = I_1 + I_2, \text{ (say)}$$

Now, $I_1 = \frac{p}{2a}\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}dx$

Putting $ax^2+bx+c \Rightarrow (2ax+b)dx = dt$, we have, $I_1 = \frac{p}{2a}\int t^{-1/2}dt = \frac{p}{2a}.\frac{t^{1/2}}{\frac{1}{2}}+C_1 = \frac{p}{a}\sqrt{ax^2+bx+c}+C_1$

and I_2 is calculated as in the previous section.

Note: $\frac{px+q}{\sqrt{ax^2+bx+c}}dx = \frac{p}{a}\sqrt{ax^2+bx+c} + \frac{2aq-bp}{2a}\int \frac{dx}{\sqrt{ax^2+bx+c}}$.

(5) **Integrals of the form** $\int \frac{f(x)}{ax^2+bx+c}dx$, **where** $f(x)$ **is a polynomial of degree 2 or greater than 2:**

To evaluate the integrals of the above form, divide the numerator by the denominator. Then, the integrals take the form given by $\frac{f(x)}{ax^2+bx+c} = Q(x)+\frac{R(x)}{ax^2+bx+c}dx$

where, $Q(x)$ is a polynomial and $R(x)$ is a linear polynomial in *x*.

Then, we have $\int \frac{f(x)}{ax^2+bx+c}dx = \int Q(x)dx + \int \frac{R(x)}{ax^2+bx+c}dx$

The integrals on R.H.S. can be obtained by the methods discussed earlier.

(6) **Integrals of the form** $\int \frac{x^2+1}{x^4+kx^2+1}dx$ **and** $\int \frac{x^2-1}{x^4+kx^2+1}dx$ **:** To evaluate the integral of the form $I = \int \frac{x^2+1}{x^4+kx^2+1}dx$, proceed as follows

(i) Divide the numerator and denominator by x^2 to get $I = \int \frac{1+\frac{1}{x^2}}{x^2+k+\frac{1}{x^2}}dx.$

(ii) Put $x - \frac{1}{x} = t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$ and $x^2 + \frac{1}{x^2} - 2 = t^2 \Rightarrow x^2 + \frac{1}{x^2} = t^2 + 2.$

Then, the given integral reduces to the form $I = \int \frac{dt}{t^2+2+k}$, which can be integrand as usual.

(iii) To evaluate $I = \int \frac{x^2-1}{x^4+kx^2+1}dx,$ we divide the numerator and denominator by x^2 and get

$$I = \int \frac{1-\frac{1}{x^2}}{x^2+k+\frac{1}{x^2}}dx$$

Then, we put $x + \frac{1}{x} = t \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt$ and $x^2 + \frac{1}{x^2} + 2 = t^2 \Rightarrow x^2 + \frac{1}{x^2} = t^2 - 2.$

Thus, we have $t = \int \frac{dt}{t^2-2+k}$, which can be evaluated as usual.

Important Tips

- ***Algebraic twins:*** $\int \frac{2x^2}{x^4+1}dx = \int \frac{x^2+1}{x^4+1}dx + \int \frac{x^2-1}{x^4+1}dx$

$\int \frac{2}{x^4+1}dx = \int \frac{x^2+1}{x^4+1}dx - \int \frac{x^2-1}{x^4+1}dx$, $\int \frac{2x^2}{x^4+1+kx^2}dx,$ $\int \frac{2}{(x^4+1+kx^2)}dx$

We know the result of $I_1 = \int \frac{x^2+1}{x^4+1}dx$ *and* $I_2 = \int \frac{x^2-1}{x^4+1}dx,$ *so for* $\int \frac{x^2}{x^4+1}dx$ *and for* $\int \frac{dx}{x^4+1}$, *we can use the result of* $\frac{I_1+I_2}{2}$ *and* $\frac{I_1-I_2}{2}$.

- ***Trigonometric twins:*** $\int \sqrt{\tan x}\,dx,$ $\int \sqrt{\cot x}\,dx,$ $\int \frac{dx}{(\sin^4 x+\cos^4 x)},$ $\int \frac{dx}{\sin^6 x+\cos^6 x},$ $\int \frac{\pm \sin x \pm \cos x}{a+b\sin x\cos}dx$

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(7) **Integrals of the forms** $\int \sqrt{ax^2+bx+c}\,dx$ **:** To evaluate this form of integrals, express ax^2+bx+c in the form $a\left[(x+\alpha)^2+\beta^2\right]$ by the method of completing the square and apply the standard result discussed in the above section according to the case as may be.

Note: $\int \sqrt{ax^2+bx+c}\;dx = \frac{(2ax+b)\sqrt{ax^2+bx+c}}{4a} + \frac{4ac-b^2}{8a}\int \frac{dx}{\sqrt{ax^2+bx+c}}$

(8) **Integrals of the form** $\int (px+q)\sqrt{ax^2+bx+c}\,dx$ **:** To evaluate this form of integral, proceed as follows:

(i) First express $(px+q)$ as $px+q = M\frac{d}{dx}(ax^2+bx+c)+N \Rightarrow px+q = M(2ax+b)+N$

Where, *M* and *N* are constant.

(ii) Compare the coefficients of x and constant terms on both sides, will get

$p = 2aM \Rightarrow M = \frac{p}{2a}$ and $q = Mb+N \Rightarrow N = q - Mb = q - \frac{p}{2a}b.$

(iii) Now, write the given integral as

$$\int (px+q)\sqrt{ax^2+bx+c}\,dx = \frac{p}{2a}\int (2ax+b)\sqrt{ax^2+bx+c}\;dx + \left(q-\frac{p}{2a}b\right)\int\sqrt{ax^2+bx+c}\;dx$$

$$= \frac{p}{2a}I_1 + \left(q-\frac{p}{2a}b\right)I_2,\ \text{(say)}.$$

(iv) To evaluate I_1, put $ax^2+bx+c=t$ and to evaluate I_2, follows the method discussed in (7)

(9) **Integrals of the form** $\int \frac{dx}{P\sqrt{Q}}$**, (where *P* and *Q* and linear or quadratic expressions in *x*):** To evaluate such types of integrals, we have following substitutions according to the nature of expressions of *P* and *Q* in *x* :

(i) When *Q* is linear and *P* is linear or quadratic, we put $Q = t^2$.

(ii) When *P* is linear and *Q* is quadratic, we put $P = \frac{1}{t}$.

(iii) When both *P* and *Q* are quadratic, we put $x = \frac{1}{t}$.

8. Integrals of the form $\int \frac{dx}{a + b\cos x}$ and $\int \frac{dx}{a + b\sin x}$.

To evaluate such form of integrals, proceed as follows:

(1) Put $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$.

(2) Replace $1+\tan^2\frac{x}{2}$ in the numerator by $\sec^2\frac{x}{2}$.

(3) Put $\tan\frac{x}{2} = t$ so that $\frac{1}{2}\sec^2\frac{x}{2}dx = dt$.

(4) Now, evaluate the integral obtained which will be of the form $\int \frac{dt}{at^2 + bt + c}$ by the method discussed earlier.

(i) $\int \frac{dx}{a + b\cos x}$

Case I: When $a > b$, then $\int \frac{dx}{a + b\cos x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}} \tan\frac{x}{2}\right) + c$

Case II: When $a < b$, then $\int \frac{dx}{a + b\cos x} = \frac{1}{\sqrt{b^2 - a^2}} \log\left|\frac{\sqrt{b-a}\tan\frac{x}{2} + \sqrt{b+a}}{\sqrt{b-a}\tan\frac{x}{2} - \sqrt{b+a}}\right| + c$

Case III: When $a = b$, then $\int \frac{dx}{a + b\cos x} = \frac{1}{a}\tan\frac{x}{2} + c$.

(ii) $\int \frac{dx}{a + b\sin x}$

Case I: When $a^2 > b^2$ or $a > 0$ and $a > b$, then $\int \frac{dx}{a + b\sin x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left[\frac{a\tan\frac{x}{2} + b}{\sqrt{a^2 - b^2}}\right] + c$

Case II: When $a^2 < b^2$, then $\int \frac{dx}{a + b\sin x} = \frac{1}{\sqrt{b^2 - a^2}} \log\left|\frac{(a\tan\frac{x}{2} + b) - (\sqrt{b^2 - a^2})}{(a\tan\frac{x}{2} + b) + \sqrt{b^2 - a^2}}\right| + c$

Case III: When $a^2 = b^2$

In this case, either $b = a$ or $b = -a$

(a) When $b = a$, then $\int \frac{dx}{a + b\sin x} = \frac{-1}{a}\cot\left(\frac{\pi}{4} + \frac{x}{2}\right) + c = \frac{1}{a}[\tan x - \sec x] + c$

(b) When $b = -a$, then $\int \frac{dx}{a + b\sin x} = \frac{1}{a}\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$.

9. Integrals of the form $\int \frac{dx}{a + b\cos x + c\sin x}, \int \frac{dx}{a\sin x + b\cos x}$.

(1) **Integral of the form** $\int \frac{dx}{a + b\cos x + c\sin x}$ **:** To evaluate such integrals, we put $b = r\cos\alpha$ and $c = r\sin\alpha$.

So that, $r^2 = b^2 + c^2$ and $\alpha = \tan^{-1}\frac{c}{b}$. $\therefore I = \int \frac{dx}{a + r(\cos\alpha\cos x + \sin\alpha\sin x)} = \int \frac{dx}{a + r\cos(x-\alpha)}$

Again, Put $x - \alpha = t \Rightarrow dx = dt$, we have $I = \int \frac{dt}{a + r\cos t}$

Which can be evaluated by the method discussed earlier.

(2) **Integral of the form** $\int \frac{dx}{a\sin x + b\cos x}$ **:** To evaluate this type of integrals we substitute $a = r\cos\theta$, $b = r\sin\theta$ and so $r = \sqrt{a^2 + b^2}$, $\alpha = \tan^{-1}\frac{b}{a}$

So, $\int \frac{dx}{a\sin x + b\cos x} = \frac{1}{r}\int \frac{dx}{\sin(x+\alpha)} = \frac{1}{r}\int \operatorname{cosec}(x+\alpha)dx$

$$= \frac{1}{r}\log\left|\tan\left(\frac{x}{2} + \frac{\alpha}{2}\right)\right| = \frac{1}{\sqrt{a^2+b^2}}\log\left|\tan\left(\frac{x}{2} + \frac{1}{2}\tan^{-1}\frac{b}{a}\right)\right| + c$$

Note: The integral of the above form can be evaluated by using $\cos x = \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}$ and $\sin x = \frac{2\tan x/2}{1 + \tan^2 x/2}$.

Important Tips

- *If* $a > b,\ a^2 > b^2 + c^2$, *then* $\int \frac{dx}{a + b\cos x + c\sin x} = \frac{2}{\sqrt{a^2 - b^2 - c^2}}\tan^{-1}\left[\frac{(a-b)\tan x/2 + c}{\sqrt{a^2 - b^2 - c^2}}\right] + k$
- *If* $a > b,\ a^2 < b^2 + c^2$, *then* $\int \frac{dx}{a + b\cos x + c\sin x} = \frac{1}{\sqrt{b^2 + c^2 - a^2}}\log\left[\frac{(a-b)\tan x/2 + c - \sqrt{b^2 + c^2 - a^2}}{(a-b)\tan x/2 + c + \sqrt{b^2 + c^2 - a^2}}\right] + k$
- *If* $a < b$, $\int \frac{dx}{a + b\cos x + c\sin x} = \frac{-1}{\sqrt{b^2 + c^2 - a^2}}\log\left[\frac{(b-a)\tan x/2 - c - \sqrt{b^2 + c^2 - a^2}}{(b-a)\tan x/2 - c + \sqrt{b^2 + c^2 - a^2}}\right] + k.$

10. Integrals of the form

$$\int \frac{dx}{a+b\cos^2 x}, \int \frac{dx}{a+b\sin^2 x}, \int \frac{dx}{a\sin^2 x+b\cos^2 x}, \int \frac{dx}{(a\sin x+b\cos x)^2}, \int \frac{dx}{a+b\sin^2 x+c\cos^2 x}$$

To evaluate the above forms of integrals proceed as follows:

(1) Divide both the numerator and denominator by $\cos^2 x$.

(2) Replace $\sec^2 x$ in the denominator, if any by $(1+\tan^2 x)$.

(3) Put $\tan x = t \Rightarrow \sec^2 x dx = dt$.

(4) Now, evaluate the integral thus obtained, by the method discussed earlier.

11. Integrals of the form $\int \frac{a \sin x + b \cos x}{c \sin x + d \cos x}$ and $\int \frac{a \sin x + b \cos x + q}{c \sin x + d \cos x + r}$

(1) **Integrals of the form** $\int \frac{a \sin x + b \cos x}{c \sin x + d \cos x} dx$ **:** Such rational functions of **sin *x*** and **cos *x*** may be integrated by expressing the numerator of the integrand as follows:

Numerator = *M* (Diff. of denominator)+*N* (Denominator)

i.e., $a \sin x + b \cos x = M \frac{d}{dx}(c \sin x + d \cos x) + N(c \sin x + d \cos x)$

The arbitary constants *M* and *N* are determined by comparing the coefficients of $\sin x$ and $\cos x$ from two sides of the above identity. Then, the given integral is

$$I = \int \frac{a \sin x + b \cos x}{c \sin x + d \cos x} dx \;\; = \int \frac{M(c \cos x - d \sin x) + N(c \sin x + d \cos x)}{c \sin x + d \cos x} dx \;\; = M \int \frac{c \cos x - d \sin x}{c \sin x + d \cos x} dx + N \int 1 dx$$

$$= M \log | \, c \sin x + d \cos x \, | + Nx + c.$$

(2) **Integrals of the form** $\int \frac{a\sin x + b\cos x + q}{c\sin x + d\cos x + r}dx$ **:** To evaluate this type of integrals, we express the numerator as follows: Numerator $= M(\text{Denominator}) + N(\text{Differentiation of denominator}) + P$

i.e., $(c\sin x + b\cos x + q) = M(c\sin x + d\cos x + r) + N(c\cos x - d\sin x) + P$.

where *M*, *N*, *P* are constants to be determined by comparing the coefficients of $\sin x, \cos x$ and constant term on both sides.

$$\therefore \int \frac{a\sin x + b\cos x + q}{c\sin x + d\cos x + r}dx = \int M\,dx + N\int \frac{\text{Diff. of denominator}}{\text{Denominator}}dx + \int \frac{dx}{c\sin x - d\cos x + r}$$

$$= Mx + N\log|\text{Denominator}| + P\int \frac{dx}{c\sin x + d\cos x + r}.$$

Important Tips

☞ $\int \frac{a\cos x + b\sin x}{c\cos x + d\sin x}dx = \frac{ac+bd}{c^2+d^2}x + \frac{ad-bc}{c^2+d^2}\log|c\cos x + d\sin x| + c$.

12. Integration of Rational Functions by using Partial Fractions.

(1) **Proper rational functions:** Functions of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial and $g(x) \neq 0$, are called rational functions of *x*.

If degree of $f(x)$ is less than degree of $g(x)$, then $\frac{f(x)}{g(x)}$ is called a proper rational function.

(2) **Improper rational function:** If degree of $f(x)$ is greater than or equal to degree of $g(x)$, then $\frac{f(x)}{g(x)}$, is called an improper rational function and every improper rational function can be transformed to a proper rational function by dividing the numerator by the denominator.

For example, $\frac{x^3}{x^2 - 5x + 6}$ is an improper rational function and can be expressed as $(x+5) + \frac{19x - 30}{x^2 - 5x + 6}$, which is the sum of a polynomial $(x+5)$ and a proper function $\frac{19x-30}{x^2-5x+6}$.

(3) **Partial fractions:** Any proper rational function can be broken up into a group of different rational fractions, each having a simple factor of the denominator of the original rational function. Each such fraction is called a partial fraction.

If by some process, we can break a given rational function $\frac{f(x)}{g(x)}$ into different fractions, whose denominators are the factors of $g(x)$, then the process of obtaining them is called the resolution or decomposition of $\frac{f(x)}{g(x)}$ into its partial fractions.

Depending on the nature of the factors of the denominator, the following cases arise.

Case I: When the denominator consists of non-repeated linear factors: To each linear factor $(x-a)$ occurring once in the denominator of a proper fraction, there corresponds a single partial fraction of the form $\frac{A}{x-a}$, where *A* is a constant to be determined.

Case II: When the denominator consists of linear factors, some repeated: To each linear factor $(x-a)$ occurring *r* times in the denominator of a proper rational function, there corresponds a sum of *r* partial fractions of the form.

$$\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\ldots\ldots\ldots.+\frac{A_r}{(x-a)^r}$$

Where $A's$ are constants to be determined. Of course, A_r is not equal to zero.

Case III: When the denominator consists of quadratic factors: To each irreducible non repeated quadratic factor ax^2+bx+c, there corresponds a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$, where *A* and *B* are constants to be determined.

To each irreducible quadratic factor ax^2+bx+c occurring *r* times in the denominator of a proper rational fraction there corresponds a sum of r partial fractions of the form

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots\ldots\ldots\ \ldots\ldots.+\frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

Where, *A*'s and *B*'s are constants to be determined.

(4) **General methods of finding the constants**

(i) In the given proper fraction, first of all factorize the denominator.

(ii) Express the given proper fraction into its partial fractions according to rules given above and multiply both the sides by the denominator of the given fraction.

(iii) Equate the coefficients of like powers of *x* in the resulting identity and solve the equations so obtained simultaneously to find the various constant is short method. Sometimes, we substitute particular values of the variable x in the identity obtained after clearing of fractions to find some or all the constants. For non-repeated linear factors, the values of *x* used as those for which the denominator of the corresponding partial fractions become zero.

Note: If the given fraction is improper, then before finding partial fractions, the given fraction must be expressed as sum of a polynomial and a proper fraction by division.

(5) **Special cases:** Some times a suitable substitution transform the given function to a rational fraction which can be integrated by breaking it into partial fractions.

13. Integration of Trigonometric Functions.

(1) **Integral of the form $\int \sin^m x \cos^n x \, dx$:** (i) To evaluate the integrals of the form $I = \int \sin^m x \cos^n x \, dx$, where *m* and *n* are rational numbers.

(a) Substitute $\sin x = t$, if *n* is odd;

(b) Substitute cos $x = t$, if *m* is odd;

(c) Substitute $\tan x = t$, if $m + n$ is a negative even integer; and

(d) Substitute $\cot x = t$, if $\frac{1}{2}(n-1)$ is an integer.

(e) If *m* and *n* are rational numbers and $\left(\frac{m+n-2}{2}\right)$ is a negative integer, then substitution $\cos x = t$ or $\tan x = t$ is found suitable.

(ii) Integrals of the form $\int R(\sin x, \cos x) dx$, where *R* is a rational function of $\sin x$ and $\cos x$, are transformed into integrals of a rational function by the substitution $\tan \frac{x}{2} = t$, where $-\pi < x < \pi$. This is the so called universal substitution. Sometimes it is more convenient to make the substitution $\cot\frac{x}{2}=t$ for $0<x<2\pi$.

The above substitution enables us to integrate any function of the form $R(\sin x, \cos x)$. However, in practice, it sometimes leads to extremely complex rational function. In some cases, the integral can be simplified by:

(a) Substituting $\sin x = t,$ if the integral is of the form $\int R(\sin x)\cos x\, dx$.

(b) Substituting $\cos x = t,$ if the integral is of the form $\int R(\cos x)\sin x\, dx$.

(c) Substituting tan *x* = *t*, *i.e.*, $dx=\frac{dt}{1+t^2}$, if the integral is dependent only on $\tan x$.

(d) Substituting $\cos x = t$, if $R(-\sin x, \cos x) = -R(\sin x, \cos x)$

(e) Substituting $\sin x = t$, if $R(\sin x, -\cos x) = -R(\sin x, \cos x)$

(f) Substituting $\tan x = t$, if $R(-\sin x, -\cos x) = -R(\sin x, \cos x)$

Important Tips

☞ *To evaluate integrals of the form* $\int \sin mx \cos nx\, dx$, $\int \sin mx . \sin nx\, dx$, $\int \cos mx . \cos nx\, dx$ *and* $\int \cos mx . \sin nx\, dx$, *we use the following trigonometrical identities.*

$$\sin mx . \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x] \Rightarrow \cos mx . \sin nx = \frac{1}{2}[\sin(m+n)x - \sin(m-n)x]$$

$$\sin mx . \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x] \Rightarrow \cos mx . \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$$

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(2) **Reduction formulae for special cases**

(i) $\int \sin^n x\,dx = \frac{-\cos x.\sin^{n-1} x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$

(ii) $\int \cos^n x\,dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$

(iii) $\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$

(iv) $\int \cot^n x\,dx = \frac{-1}{n-1}\cot^{n-1} x - \int \cot^{n-2} x\,dx$

(v) $\int \sec^n x\,dx = \frac{1}{(n-1)}\left[\sec^{n-2} x.\tan x + (n-2)\int \sec^{n-2} x\,dx\right]$

(vi) $\int cosec^n x\,dx = \frac{1}{(n-1)}\left[-\cos ec^{n-2} x.\cot x + (n-2)\int cosec^{n-2} x\,dx\right]$

(vii) $\int \sin^p x \cos^q x\,dx = -\frac{\sin^{q+1} x.\cos^{p-1} x}{p+q} + \frac{p-1}{p+q}\int \sin^{p-2} x.\cos^q x\,dx$

(viii) $\int \sin^p x \cos^q x\,dx = \frac{\sin^{p+1} x.\cos^{q-1} x}{p+q} + \frac{p-1}{p+q}\int \sin^p x.\cos^{q-2} x\,dx$

(ix) $\int \frac{dx}{(x^2+k)^n} = \frac{x}{k(2n-2)(x^2+k)^{n-1}} + \frac{(2n-3)}{k(2n-2)}\int \frac{dx}{(x^2+k)^{n-1}}$

Important Tips

☞ *Reduction formulae for* $I_{(n,m)} = \int \frac{\sin^n x}{\cos^m x}dx$ *is* $I_{(n,m)} = \frac{1}{m-1}.\frac{\sin^{n-1} x}{\cos^{m-1} x} - \frac{(n-1)}{(m-1)}.I_{(n-2,m-2)}$

14. Integration of Hyperbolic Functions

(1) $\int \sinh x\, dx = \cos hx + c$

(2) $\int \cos hx\, dx = \sin hx + c$

(3) $\int \sec h^2 x\, dx = \tan h\, x + c$

(4) $\int \operatorname{cosec} h^2 x\, dx = -\cot hx + c$

(5) $\int \sec hx \tan hx\, dx = -\sec hx + c$

(6) $\int \operatorname{cosec} hx \cot h\, dx = -\operatorname{cosec} hx + c$

15. Integral of the type $f[x, (ax+b)^{m_1/n_1}, (ax+b)^{m_2/n_2} \ldots]$ where *f* is a rational function and m_1, n_1, m_2, n_2 are Integers.

To evaluate such type of integral, we transform it into an integral of rational function by putting $(ax+b) = t^s$, where *s* is the least common multiple (*L.C.M.*) of the numbers n_1, n_2.

Integrals of the form $\int x^m (a + bx^n)^p\, dx$

Case I : If $p \in N$ (Natural number). We expand the integral with the help of binomial theorem and integrate.

16. Integrals using Euler's substitution

Integrals of the form $\int f(x), \sqrt{(ax^2 + bx + c)}\, dx$ are calculated with the aid of one of the three Euler substitution:

(1) $\sqrt{ax^2 + bx + c} = t \pm x\sqrt{a}$, if $a > 0$.

(2) $\sqrt{ax^2 + bx + c} = tx \pm \sqrt{c}$, if $c > 0$.

(3) $\sqrt{ax^2 + bc + c} = (x - \alpha)t$, if $ax^2 + bx + c = a(x - \alpha)(x - \beta)$, *i.e.*, if *x* is real root of $(ax^2 + bx + c)$.

Note: The Euler substitution often lead to rather some calculations, therefore they should be applied only when it is difficult to find another method for calculating the given integral.

17. Some Integrals which cannot be found

Any function continuous on interval (*a, b*) has an antiderivative in that interval. In other words, there exists a function $F(x)$ such that $F'(x) = f(x)$.

However not every antiderivative $F(x)$, even when it exists is expressible in closed form in terms of elementary functions such as polynomials, trigonometric, logarithmic, exponential etc. function. Then we say that such antiderivatives or integrals "cannot be found." Some typical examples are:

(i) $\int \frac{dx}{\log x}$

(ii) $\int e^{x^2} dx$

(iii) $\int \frac{x^2}{1 + x^5} dx$

(iv) $\int \sqrt[3]{1 + x^2}\, dx$

(v) $\int \sqrt{1+x^3}\,dx$

(vi) $\sqrt{1-k^2 \sin^2 x}\,dx$

(vii) $\int e^{-x^2}\,dx$

(viii) $\int \frac{\sin x}{x}\,dx$

(ix) $\int \frac{\cos x}{x}\,dx$

(x) $\int \sqrt{\sin x}\,dx$

(xi) $\int \sin(x^2)\,dx$

(xii) $\int \cos(x^2)\,dx$

(xiii) $\int x \tan x\,dx$ etc.