Knowledge... Everywhere

Physics

Motion in Two Dimension

(Circular Motion)

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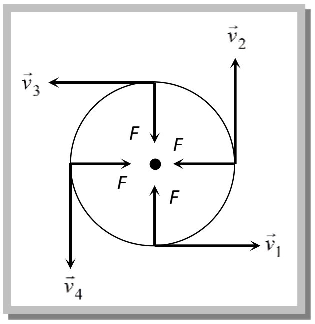
Circular motion is another example of motion in two dimensions. To create circular motion in a body it must be given some initial velocity and a force must then act on the body which is always directed at right angles to instantaneous velocity.



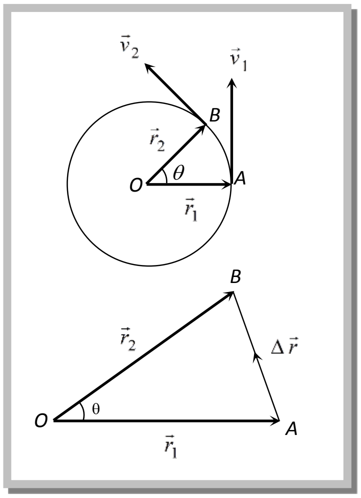
Since this force is always at right angles to the displacement due to the initial velocity therefore no work is done by the force on the particle. Hence, its kinetic energy and thus speed is unaffected. But due to simultaneous action of the force and the velocity the particle follows resultant path, which in this case is a circle. Circular motion can be classified into two types – Uniform circular motion and non-uniform circular motion.

CIRCULAR MOTION

1. Variables of Circular Motion.

(1) **Displacement and distance:** When particle moves in a circular path describing an angle θ during time *t* (as shown in the figure) from the position *A* to the position *B*, we see that the magnitude of the position vector $\vec{r}$ (that is equal to the radius of the circle) remains constant. *i.e.,* $|\vec{r}_1| = |\vec{r}_2| = r$ and the direction of the position vector changes from time to time.

(i) Displacement**:** The change of position vector or the displacement $\Delta\vec{r}$ of the particle from position *A* to the position *B* is given by referring the figure.



$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$$

$$\Rightarrow \Delta r = |\Delta\vec{r}| = |\vec{r}_2 - \vec{r}_1| \qquad \Delta r = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta}$$

Putting $r_1 = r_2 = r$ we obtain

$$\Delta r = \sqrt{r^2 + r^2 - 2r.r\cos\theta}$$

$$\Rightarrow \Delta r = \sqrt{2r^2(1-\cos\theta)} = \sqrt{2r^2\left(2\sin^2\frac{\theta}{2}\right)}$$

$$\Delta r = 2r\sin\frac{\theta}{2}$$

(ii) Distance**:** The distanced covered by the particle during the time t is given as

d = length of the arc AB = $r\,\theta$

(iii) Ratio of distance and displacement**:** $\dfrac{d}{\Delta r} = \dfrac{r\theta}{2r\sin\theta/2} = \dfrac{\theta}{2}\text{cosec}\,(\theta/2)$

(2) **Angular displacement (θ):** The angle turned by a body moving on a circle from some reference line is called angular displacement.

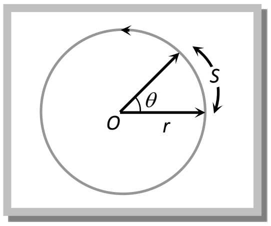
(i) Dimension = $[M^0L^0T^0]$ (as $\theta = arc/\text{radius}$) .

(ii) Units = Radian or Degree. It is sometimes also specified in terms of fraction or multiple of revolution.

(iii) $2\pi\,\text{rad} = 360^o = 1\,\text{Revolution}$

(iv) Angular displacement is a axial vector quantity.

Its direction depends upon the sense of rotation of the object and can be given by Right Hand Rule; which states that if the curvature of the fingers of right hand represents the sense of rotation of the object, then the thumb, held perpendicular to the curvature of the fingers, represents the direction of angular displacement vector.



(v) Relation between linear displacement and angular displacement $\vec{s} = \vec{\theta} \times \vec{r}$

or $s = r\theta$

(3) **Angular velocity (ω):** Angular velocity of an object in circular motion is defined as the time rate of change of its angular displacement.

(i) Angular velocity $\omega = \dfrac{\text{angle traced}}{\text{time taken}} = \underset{\Delta t \to 0}{Lt}\dfrac{\Delta\theta}{\Delta t} = \dfrac{d\theta}{dt}$

$\therefore \qquad \omega = \dfrac{d\theta}{dt}$

(ii) Dimension: $[M^0L^0T^{-1}]$

(iii) Units: Radians per second ($rad.s^{-1}$) or Degree per second.

(iv) Angular velocity is an axial vector.

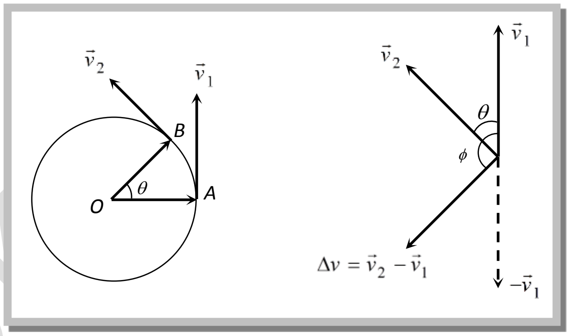
(v) Relation between angular velocity and linear velocity $\vec{v} = \vec{\omega} \times \vec{r}$

Its direction is the same as that of $\Delta\theta$. For anticlockwise rotation of the point object on the circular path, the direction of ω, according to Right hand rule is along the axis of circular path directed upwards. For clockwise rotation of the point object on the circular path, the direction of ω is along the axis of circular path directed downwards.

Note: It is important to note that nothing actually moves in the direction of the angular velocity vector $\vec{\omega}$. The direction of $\vec{\omega}$ simply represents that the rotational motion is taking place in a plane perpendicular to it.

(vi) For uniform circular motion ω remains constant whereas for non-uniform motion ω varies with respect to time.

(4) **Change in velocity:** We want to know the magnitude and direction of the change in velocity of the particle which is performing uniform circular motion as it moves from *A* to *B* during time *t* as shown in figure. The change in velocity vector is given as



$$\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$$

or $$|\Delta\vec{v}| = |\vec{v}_2 - \vec{v}_1| \Rightarrow \Delta v = \sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos\theta}$$

For uniform circular motion $v_1 = v_2 = v$

So $$\Delta v = \sqrt{2v^2(1-\cos\theta)} = 2v\sin\frac{\theta}{2}$$

The direction of $\Delta\vec{v}$ is shown in figure that can be given as

$$\phi = \frac{180^o - \theta}{2} = (90^o - \theta/2)$$

Note: Relation between linear velocity and angular velocity.

In vector form $\vec{v} = \vec{\omega} \times \vec{r}$

(5) **Time period (*T*):** In circular motion, the time period is defined as the time taken by the object to complete one revolution on its circular path.

(i) Units: second.

(ii) Dimension: $[M^0 L^0 T]$

(iii) Time period of second's hand of watch = 60 *second*.

(iv) Time period of minute's hand of watch = 60 *minute*

(v) Time period of hour's hand of watch = 12 *hour*

(6) **Frequency (*n*):** In circular motion, the frequency is defined as the number of revolutions completed by the object on its circular path in a unit time.

(i) Units: s^{-1} or hertz (*Hz*).

(ii) Dimension: $[M^0 L^0 T^{-1}]$

Note: Relation between time period and frequency: If *n* is the frequency of revolution of an object in circular motion, then the object completes *n* revolutions in 1 second. Therefore, the object will complete one revolution in 1/*n* second.

$$\therefore T = 1/n$$

Relation between angular velocity, frequency and time period: Consider a point object describing a uniform circular motion with frequency *n* and time period *T*. When the object completes one revolution, the angle traced at its axis of circular motion is 2π radians. It means, when time *t* = *T*, $\theta = 2\pi$ radians. Hence, angular velocity $\omega = \frac{\theta}{t} = \frac{2\pi}{T} = 2\pi n$ $(\because T = 1/n)$

$$\omega = \frac{2\pi}{T} = 2\pi n$$

If two particles are moving on same circle or different coplanar concentric circles in same direction with different uniform angular speeds ω_A and ω_B respectively, the angular velocity of *B* relative to *A* will be

$$\omega_{rel} = \omega_B - \omega_A$$

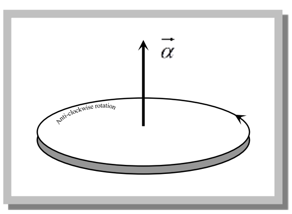
So the time taken by one to complete one revolution around *O* with respect to the other (i.e., time in which *B* complete one revolution around *O* with respect to the other (*i.e.*, time in which *B* completes one more or less revolution around *O* than *A*)

$$T = \frac{2\pi}{\omega_{rel}} = \frac{2\pi}{\omega_2 - \omega_1} = \frac{T_1 T_2}{T_1 - T_2} \qquad \left[\text{as } T = \frac{2\pi}{\omega}\right]$$

Special case: If $\omega_B = \omega_A, \omega_{rel} = 0$ and so $T = \infty$., particles will maintain their position relative to each other. This is what actually happens in case of geostationary satellite ($\omega_1 = \omega_2$ = constant)

(7) **Angular acceleration (α):** Angular acceleration of an object in circular motion is defined as the time rate of change of its angular velocity.

(i) If $\Delta\omega$ be the change in angular velocity of the object in time interval *t* and $t + \Delta t$, while moving on a circular path, then angular acceleration of the object will be



$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

(ii) Units: *rad. s*$^{-2}$

(iii) Dimension: $[M^0 L^0 T^{-2}]$

(iv) Relation between linear acceleration and angular acceleration $\vec{a} = \vec{\alpha} \times \vec{r}$

(v) For uniform circular motion since ω is constant so $\alpha = \frac{d\omega}{dt} = 0$

(vi) For non-uniform circular motion $\alpha \neq 0$

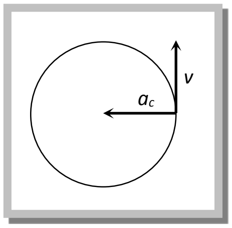
Note: Relation between linear (tangential) acceleration and angular acceleration $\vec{a} = \vec{\alpha} \times \vec{r}$

a. For uniform circular motion angular acceleration is zero, so tangential acceleration also is equal to zero.
b. For non-uniform circular motion $a \neq 0$ (because $\alpha \neq 0$).

2. Centripetal Acceleration.

(1) Acceleration acting on the object undergoing uniform circular motion is called centripetal acceleration.

(2) It always acts on the object along the radius towards the center of the circular path.



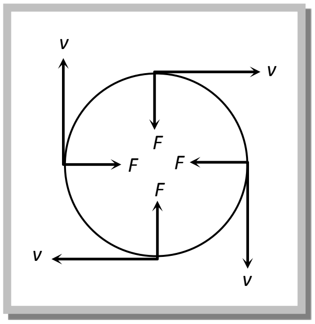
(3) Magnitude of centripetal acceleration $a = \frac{v^2}{r} = \omega^2 r = 4\pi n^2 r = \frac{4\pi^2}{T^2} r$

(4) Direction of centripetal acceleration: It is always the same as that of $\Delta\vec{\upsilon}$. When Δt decreases, $\Delta\theta$ also decreases. Due to which $\Delta\vec{\upsilon}$ becomes more and more perpendicular to $\vec{\upsilon}$. When $\Delta t \to 0$, $\Delta\vec{\upsilon}$ becomes perpendicular to the velocity vector. As the velocity vector of the particle at an instant acts along the tangent to the circular path, therefore $\Delta\vec{\upsilon}$ and hence the centripetal acceleration vector acts along the radius of the circular path at that point and is directed towards the center of the circular path.

3. Centripetal Force.

According to Newton's first law of motion, whenever a body moves in a straight line with uniform velocity, no force is required to maintain this velocity. But when a body moves along a circular path with uniform speed, its direction changes continuously *i.e.* velocity keeps on changing on account of a change in direction. According to Newton's second law of motion, a change in the direction of motion of the body can take place only if some external force acts on the body.

Due to inertia, at every point of the circular path; the body tends to move along the tangent to the circular path at that point (in figure). Since everybody has directional inertia, a velocity cannot change by itself and as such we have to apply a force. But this force should be such that it changes the direction of velocity and not its magnitude. This is possible only if the force acts perpendicular to the direction of velocity. Because the velocity is along the tangent, this force must be along the radius (because the radius of a circle at any point is perpendicular to the tangent at that point). Further, as this force is to move the body in a circular path, it must acts towards the center. This center-seeking force is called the centripetal force.



Hence, centripetal force is that force which is required to move a body in a circular path with uniform speed. The force acts on the body along the radius and towards center.

(1) Formulae for centripetal force: $F = \frac{mv^2}{r} = m\omega^2 r = m\,4\pi^2 n^2 r = \frac{m4\pi^2 r}{T^2}$

(2) Centripetal force in different situation

Situation	Centripetal Force
A particle tied to a string and whirled in a horizontal circle	Tension in the string
Vehicle taking a turn on a level road	Frictional force exerted by the road on the tyres
A vehicle on a speed breaker	Weight of the body or a component of weight
Revolution of earth around the sun	Gravitational force exerted by the sun

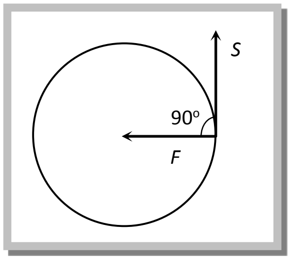
Electron revolving around the nucleus in an atom	Coulomb attraction exerted by the protons in the nucleus
A charged particle describing a circular path in a magnetic field	Magnetic force exerted by the agent that sets up the magnetic field

4. Centrifugal Force.

It is an imaginary force due to incorporated effects of inertia. When a body is rotating in a circular path and the centripetal force vanishes, the body would leave the circular path. To an observer *A* who is not sharing the motion along the circular path, the body appears to fly off tangential at the point of release. To another observer *B*, who is sharing the motion along the circular path (*i.e.*, the observer *B* is also rotating with the body with the same velocity), the body appears to be stationary before it is released. When the body is released, it appears to *B*, as if it has been thrown off along the radius away from the center by some force. In reality no force is actually seen to act on the body. In absence of any real force the body tends to continue its motion in a straight line due to its inertia. The observer *A* easily relates this events to be due to inertia but since the inertia of both the observer *B* and the body is same, the observer *B* cannot relate the above happening to inertia. When the centripetal force ceases to act on the body, the body leaves its circular path and continues to moves in its straight-line motion but to observer *B* it appears that a real force has actually acted on the body and is responsible for throwing the body radially out-words. This imaginary force is given a name to explain the effects on inertia to the observer who is sharing the circular motion of the body. This inertial force is called centrifugal force. Thus centrifugal force is a fictitious force which has significance only in a rotating frame of reference.

5. Work done by Centripetal Force.

The work done by centripetal force is always zero as it is perpendicular to velocity and hence instantaneous displacement.



Work done = Increment in kinetic energy of revolving body

Work done = 0

Also $W = \vec{F}.\vec{S} = F \cdot S\cos\theta$

$= F \cdot S \cos 90^\circ = 0$

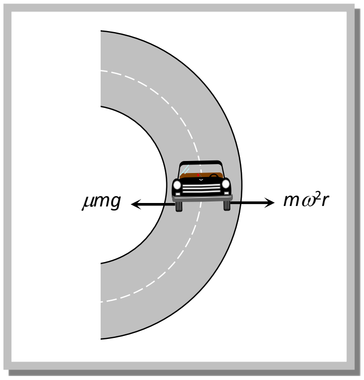
Example: (i) When an electron revolve around the nucleus in hydrogen atom in a particular orbit, it neither absorb nor emit any energy means its energy remains constant.

(ii) When a satellite established once in a orbit around the earth and it starts revolving with particular speed, then no fuel is required for its circular motion.

6. Skidding of Vehicle on a Level Road.

When a vehicle turns on a circular path it requires centripetal force.

If friction provides this centripetal force then vehicle can move in circular path safely if



Friction force ≥ required centripetal force

$$\mu mg \geq \frac{mv^2}{r}$$

$$\therefore \quad v_{safe} \leq \sqrt{\mu rg}$$

This is the maximum speed by which vehicle can turn in a circular path of radius *r*, where coefficient of friction between the road and tyre is μ.

7. Skidding of Object on a Rotating Platform.

On a rotating platform, to avoid the skidding of an object (mass m) placed at a distance r from axis of rotation, the centripetal force should be provided by force of friction.

Centripetal force = Force of friction

$$m\omega^2 r = \mu mg$$

$$\therefore \quad \omega_{\max} = \sqrt{(\mu g / r)},$$

Hence maximum angular velocity of rotation of the platform is $\sqrt{(\mu g / r)}$, so that object will not skid on it.

8. Bending of a Cyclist.

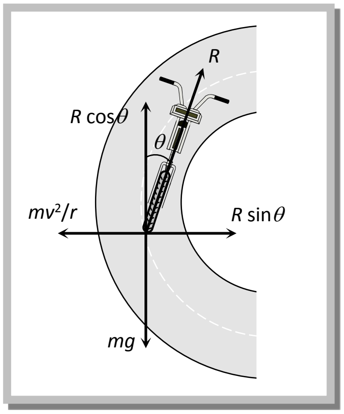
A cyclist provides himself the necessary centripetal force by leaning inward on a horizontal track, while going round a curve. Consider a cyclist of weight mg taking a turn of radius r with velocity v. In order to provide the necessary centripetal force, the cyclist leans through angle θ inwards as shown in figure.

The cyclist is under the action of the following forces:

The weight mg acting vertically downward at the center of gravity of cycle and the cyclist.

The reaction R of the ground on cyclist. It will act along a line-making angle θ with the vertical.

The vertical component $R\cos\theta$ of the normal reaction R will balance the weight of the cyclist, while the horizontal component $R\sin\theta$ will provide the necessary centripetal force to the cyclist.



$$R\sin\theta = \frac{mv^2}{r} \quad .....(i)$$

and $$R\cos\theta = mg \quad .....(ii)$$

Dividing equation (i) by (ii), we have

$$\frac{R\sin\theta}{R\cos\theta} = \frac{mv^2/r}{mg}$$

or $$\tan\theta = \frac{v^2}{rg} \quad ....(iii)$$

Therefore, the cyclist should bend through an angle $\theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$

It follows that the angle through which cyclist should bend will be greater, if

(i) The radius of the curve is small *i.e.* the curve is sharper

(ii) The velocity of the cyclist is large.

Note: For the same reasons, an ice skater or an airplane has to bend inwards, while taking a turn.

9. Banking of a Road.

For getting a centripetal force cyclist bend towards the center of circular path but it is not possible in case of four wheelers.

Therefore, outer bed of the road is raised so that a vehicle moving on it gets automatically inclined towards the center.

In the figure (A) shown reaction R is resolved into two components, the component $R\cos\theta$ balances weight of vehicle

$\therefore \quad R\cos\theta = mg$ ......(i)

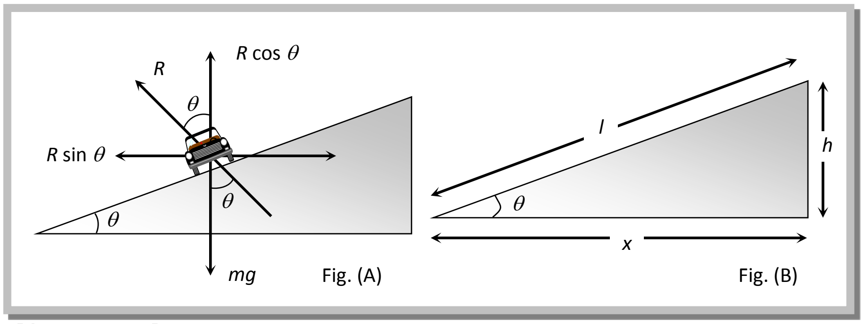
and the horizontal component $R\sin\theta$ provides necessary centripetal force as it is directed towards center of desired circle

Thus $\quad R\sin\theta = \frac{mv^2}{r}$ ......(ii)

Dividing (ii) by (i), we have

$\tan\theta = \frac{v^2}{rg}$ ...... (iii)

or $\tan\theta = \frac{\omega^2 r}{g} = \frac{v\omega}{rg}$ ...... (iv) [As $v = r\omega$]



If l = width of the road, h = height of the outer edge from the ground level then from the figure (B)

$\tan\theta = \frac{h}{x} = \frac{h}{l}$ ....(v) [since θ is very small]

From equation (iii), (iv) and (v) $\tan\theta = \frac{v^2}{rg} = \frac{\omega^2 r}{g} = \frac{v\omega}{rg} = \frac{h}{l}$

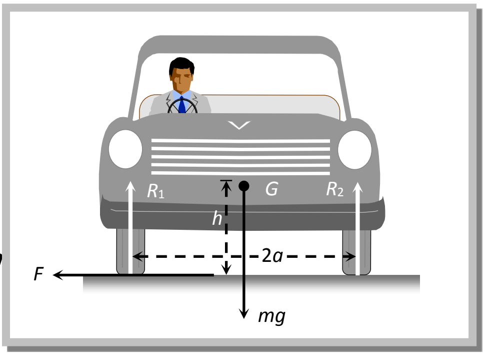
Note:

a. If friction is also present between the tyres and road then $\frac{v^2}{rg} = \frac{\mu + \tan\theta}{1 - \mu\tan\theta}$

b. Maximum safe speed on a banked frictional road $v = \sqrt{\frac{rg(\mu + \tan\theta)}{1 - \mu\tan\theta}}$

10. Overturning of Vehicle.

When a car moves in a circular path with speed more than maximum speed then it overturns and it's inner wheel leaves the ground first



Weight of the car = mg

Speed of the car = v

Radius of the circular path = r

Distance between the center of wheels of the car = $2a$

Height of the center of gravity (G) of the car from the road level = h

Reaction on the inner wheel of the car by the ground = R_1

Reaction on the outer wheel of the car by the ground = R_2

When a car move in a circular path, horizontal force F provides the required centripetal force

$$i.e.,\ F = \frac{mv^2}{R} \quad ....(i)$$

For rotational equilibrium, by taking the moment of forces R_1, R_2 and F about G

$$Fh + R_1 a = R_2 a \quad ....(ii)$$

As there is no vertical motion so $R_1 + R_2 = mg$ ....(iii)

By solving (i), (ii) and (iii)

$$R_1 = \frac{1}{2}M\left[g - \frac{v^2 h}{ra}\right] \quad ....(iv)$$

and $R_2 = \frac{1}{2}M\left[g + \frac{v^2h}{ra}\right]$ ....(v)

It is clear from equation (iv) that if v increases value of R_1 decreases and for $R_1 = 0$

$$\frac{v^2h}{ra} = g \quad \text{or} \quad v = \sqrt{\frac{gra}{h}}$$

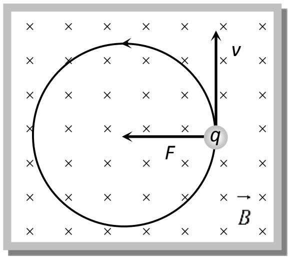
i.e. the maximum speed of a car without overturning on a flat road is given by $v = \sqrt{\frac{gra}{h}}$

11. Motion of Charged Particle in Magnetic Field.

When a charged particle having mass m, charge q enters perpendicularly in a magnetic field B, with velocity v then it describes a circular path of radius r.

Because magnetic force (qvB) works in the perpendicular direction of v and it provides required centripetal force

Magnetic force = Centripetal force



$$qvB = \frac{mv^2}{r}$$

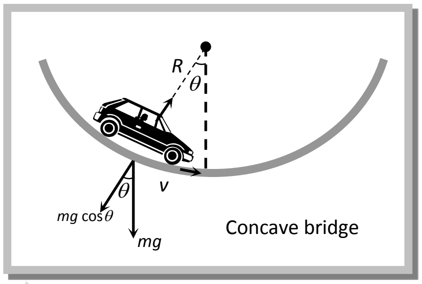
$\therefore$ radius of the circular path $r = \frac{mv}{qB}$

12. Reaction of Road on Car.

(1) When car moves on a concave bridge then

Centripetal force = $R - mg\cos\theta = \frac{mv^2}{r}$

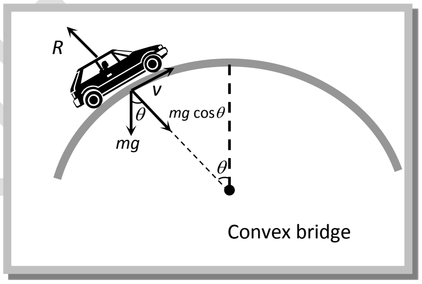
and reaction $R = mg\cos\theta + \frac{mv^2}{r}$



(2) When car moves on a convex bridge

Centripetal force = $mg\cos\theta - R = \frac{mv^2}{r}$

and reaction $R = mg\cos\theta - \frac{mv^2}{r}$



13. Non-Uniform Circular Motion.

If the speed of the particle in a horizontal circular motion changes with respect to time, then its motion is said to be non-uniform circular motion.

Consider a particle describing a circular path of radius *r* with center at *O*. Let at an instant the particle be at *P* and $\vec{\upsilon}$ be its linear velocity and $\vec{\omega}$ be its angular velocity.

Then, $\vec{\upsilon} = \vec{\omega} \times \vec{r}$ .....(i)

Differentiating both sides of *w.r.t.* time *t* we have

$\frac{d\vec{\upsilon}}{dt} = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$ .....(ii)

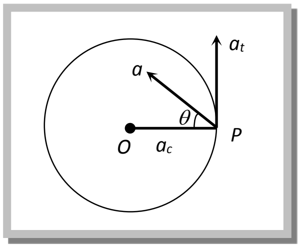
Here, $\frac{d\vec{v}}{dt} = \vec{a}$, (Resultant acceleration)

$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{\upsilon}$

$\frac{d\vec{\omega}}{dt} = \vec{\alpha}$ (Angular acceleration)

$\vec{a} = \vec{a}_t + \vec{a}_c$ .....(iii)

$\frac{d\vec{r}}{dt} = \vec{\upsilon}$ (Linear velocity)



Thus the resultant acceleration of the particle at *P* has two component accelerations

(1) **Tangential acceleration:** $\vec{a_t} = \vec{\alpha} \times \vec{r}$

It acts along the tangent to the circular path at *P* in the plane of circular path.

According to right hand rule since $\vec{\alpha}$ and $\vec{r}$ are perpendicular to each other, therefore, the magnitude of tangential acceleration is given by

$$|\vec{a}_t| = |\vec{\alpha} \times \vec{r}| = \alpha r \sin 90^o = \alpha r.$$

(2) **Centripetal (Radial) acceleration:** $\vec{a_c} = \vec{\omega} \times \vec{v}$

It is also called centripetal acceleration of the particle at *P*.

It acts along the radius of the particle at *P*.

According to right hand rule since $\vec{\omega}$ and $\vec{\upsilon}$ are perpendicular to each other, therefore, the magnitude of centripetal acceleration is given by

$$|\vec{a}_c| = |\vec{\omega} \times \vec{\upsilon}| = \omega \upsilon \sin 90^o = \omega \upsilon = \omega(\omega r) = \omega^2 r = \upsilon^2 / r$$

(3) **Tangential and centripetal acceleration in different motions**

Centripetal acceleration	Tangential acceleration	Net acceleration	Type of motion
$a_c = 0$	$a_t = 0$	$a = 0$	Uniform translatory motion
$a_c = 0$	$a_t \neq 0$	$a = a_t$	Accelerated translatory motion
$a_c \neq 0$	$a_t = 0$	$a = a_c$	Uniform circular motion
$a_c \neq 0$	$a_t \neq 0$	$a = \sqrt{a_c^2 + a_t^2}$	Non-uniform circular motion

Note: Here a_t governs the magnitude of $\vec{v}$ while $\vec{a}_c$ its direction of motion.

(4) **Force:** In non-uniform circular motion the particle simultaneously possesses two forces

Centripetal force: $F_c = ma_c = \dfrac{mv^2}{r} = mr\omega^2$

Tangential force: $F_t = ma_t$

Net force: $F_{net} = ma = m\sqrt{a_c^2 + a_t^2}$

Note: In non-uniform circular motion work done by centripetal force will be zero since $\vec{F}_c \perp \vec{v}$

a. In non-uniform circular motion work done by tangential of force will not be zero since $F_t \neq 0$
b. Rate of work done by net force in non-uniform circular = rate of work done by tangential force

i.e. $P = \frac{dW}{dt} = \vec{F}_t.\vec{v}$

14. Equations of Circular Motion.

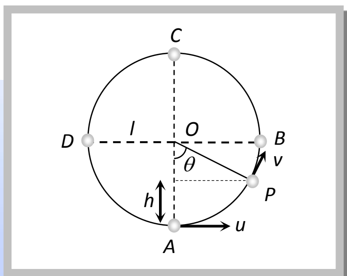
For accelerated motion	For retarded motion
$\omega_2 = \omega_1 + \alpha t$	$\omega_2 = \omega_1 - \alpha t$
$\theta = \omega_1 t + \frac{1}{2}\alpha t^2$	$\theta = \omega_1 t - \frac{1}{2}\alpha t^2$
$\omega_2^2 = \omega_1^2 + 2\alpha\theta$	$\omega_2^2 = \omega_1^2 - 2\alpha\theta$
$\theta_n = \omega_1 + \frac{\alpha}{2}(2n-1)$	$\theta_n = \omega_1 - \frac{\alpha}{2}(2n-1)$

Where
ω_1 = Initial angular velocity of particle
ω_2 = Final angular velocity of particle
α = Angular acceleration of particle
θ = Angle covered by the particle in time t
θ_n = Angle covered by the particle in n^{th} second

15. Motion in Vertical Circle.

This is an example of non-uniform circular motion. In this motion body is under the influence of gravity of earth. When body moves from lowest point to highest point. Its speed decrease and becomes minimum at highest point. Total mechanical energy of the body remains conserved and *KE* converts into *PE* and vice versa.

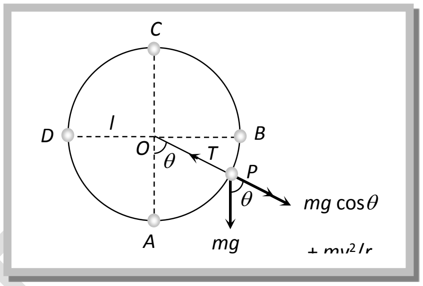
(1) **Velocity at any point on vertical loop:** If *u* is the initial velocity imparted to body at lowest point then. Velocity of body at height *h* is given by



$$v = \sqrt{u^2 - 2gh} = \sqrt{u^2 - 2gl(1-\cos\theta)} \qquad [\text{As } h = l - l\cos\theta = l(1-\cos\theta)]$$

Where l in the length of the string

(2) **Tension at any point on vertical loop**: Tension at general point P, According to Newton's second law of motion.



Net force towards center = centripetal force

$$T - mg\cos\theta = \frac{mv^2}{l} \qquad \text{Or} \qquad T = mg\cos\theta + \frac{mv^2}{l}$$

$$T = \frac{m}{l}[u^2 - gl(2 - 3\cos\theta)] \qquad [\text{As } v = \sqrt{u^2 - 2gl(1-\cos\theta)}\,]$$

(3) **Velocity and tension in a vertical loop at different positions**

Position	Angle	Velocity	Tension
A	0°	u	$\frac{mu^2}{l} + mg$
B	90°	$\sqrt{u^2 - 2gl}$	$\frac{mu^2}{l} - 2mg$
C	180°	$\sqrt{u^2 - 4gl}$	$\frac{mu^2}{l} - 5mg$
D	270°	$\sqrt{u^2 - 2gl}$	$\frac{mu^2}{l} - 2mg$

It is clear from the table that: $T_A > T_B > T_C$ and $T_B = T_D$

$T_A - T_B = 3mg,$

$T_A - T_C = 6mg$

and $T_B - T_C = 3mg$

(4) **Various conditions for vertical motion:**

Velocity at lowest point	Condition
$u_A > \sqrt{5gl}$	Tension in the string will not be zero at any of the point and body will continue the circular motion.
$u_A = \sqrt{5gl}$,	Tension at highest point C will be zero and body will just complete the circle.
$\sqrt{2gl} < u_A < \sqrt{5gl}$,	Particle will not follow circular motion. Tension in string become zero somewhere between points *B* and *C* whereas velocity remain positive. Particle leaves circular path and follow parabolic trajectory.
$u_A = \sqrt{2gl}$	Both velocity and tension in the string becomes zero between *A* and *B* and particle will oscillate along semi-circular path.
$u_A < \sqrt{2gl}$	velocity of particle becomes zero between *A* and *B* but tension will not be zero and the particle will oscillate about the point *A*.

Note: *K.E.* of a body moving in horizontal circle is same throughout the path but the *K.E.* of the body moving in vertical circle is different at different places.

If body of mass *m* is tied to a string of length *l* and is projected with a horizontal velocity *u* then:

Height at which the velocity vanishes is $h = \frac{u^2}{2g}$

Height at which the tension vanishes is $h = \frac{u^2 + gl}{3g}$

(5) **Critical condition for vertical looping:** If the tension at *C* is zero, then body will just complete revolution in the vertical circle. This state of body is known as critical state. The speed of body in critical state is called as critical speed.

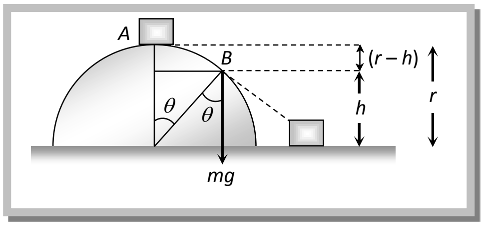
From the above table $T_C = \frac{mu^2}{l} - 5mg = 0 \Rightarrow u = \sqrt{5gl}$

It means to complete the vertical circle the body must be projected with minimum velocity of $\sqrt{5gl}$ at the lowest point.

(6) **Various quantities for a critical condition in a vertical loop at different positions :**

Quantity	Point *A*	Point *B*	Point *C*	Point *D*	Point *P*
Linear velocity (*v*)	$\sqrt{5gl}$	$\sqrt{3gl}$	$\sqrt{gl}$	$\sqrt{3gl}$	$\sqrt{gl(3+2\cos\theta)}$
Angular velocity (ω)	$\sqrt{\frac{5g}{l}}$	$\sqrt{\frac{3g}{l}}$	$\sqrt{\frac{g}{l}}$	$\sqrt{\frac{3g}{l}}$	$\sqrt{\frac{g}{l}(3+2\cos\theta)}$
Tension in String (*T*)	6 *mg*	3 *mg*	0	3 *mg*	$3mg\,(1+\cos\theta)$
Kinetic Energy (*KE*)	$\frac{5}{2}mgl$	$\frac{3}{2}mgl$	$\frac{1}{2}mgl$	$\frac{3}{2}mgl$	$\frac{mgl}{2}(3+2\cos\theta)$
Potential Energy (*PE*)	0	*mgl*	2 *mgl*	*mgl*	$mgl\,(1-\cos\theta)$
Total Energy (*TE*)	$\frac{5}{2}mgl$	$\frac{5}{2}mgl$	$\frac{5}{2}mgl$	$\frac{5}{2}mgl$	$\frac{5}{2}mgl$

(7) **Motion of a block on frictionless hemisphere**: A small block of mass *m* slides down from the top of a frictionless hemisphere of radius *r*. The component of the force of gravity ($mg\cos\theta$) provides required centripetal force but at point *B* it's circular motion ceases and the block lose contact with the surface of the sphere.



For point *B*, by equating the forces, $mg\cos\theta = \frac{mv^2}{r}$ .....(i)

For point *A* and *B*, by law of conservation of energy

Total energy at point *A* = Total energy at point *B*

K.E.$_{(A)}$ + P.E.$_{(A)}$ = K.E.$_{(B)}$ + P.E.$_{(B)}$

$$0 + mgr = \frac{1}{2}mv^2 + mgh \;\Rightarrow\; v = \sqrt{2g(r-h)} \quad .....(ii)$$

and from the given figure $h = r\cos\theta$ .....(iii)

By substituting the value of *v* and *h* from eq^n (ii) and (iii) in eq^n (i)

$$mg\left(\frac{h}{r}\right) = \frac{m}{r}\left(\sqrt{2g(r-h)}\right)^2$$

$$\Rightarrow \qquad h = 2(r-h) \;\Rightarrow\; h = \frac{2}{3}r$$

i.e. the block lose contact at the height of $\frac{2}{3}r$ from the ground.

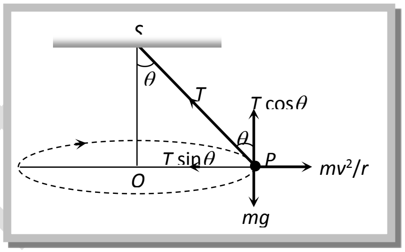
and angle from the vertical can be given by $\cos\theta = \frac{h}{r} = \frac{2}{3}$ $\quad\therefore\; \theta = \cos^{-1}\frac{2}{3}$.

16. Conical Pendulum.

This is the example of uniform circular motion in horizontal plane.

A bob of mass m attached to a light and in-extensible string rotates in a horizontal circle of radius r with constant angular speed ω about the vertical. The string makes angle θ with vertical and appears tracing the surface of a cone. So this arrangement is called conical pendulum.

The force acting on the bob are tension and weight of the bob.

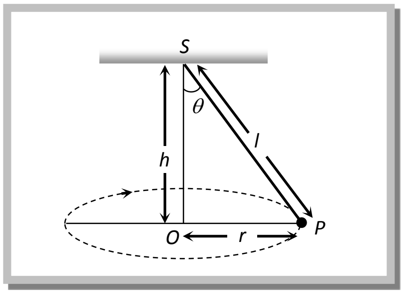


From the figure $T\sin\theta = \dfrac{mv^2}{r}$ ....(i)

and $T\cos\theta = mg$ ....(ii)

(1) Tension in the string: $T = mg\sqrt{1+\left(\dfrac{v^2}{rg}\right)^2}$

$$T = \frac{mg}{\cos\theta} = \frac{mgl}{\sqrt{l^2-r^2}} \qquad \left[\text{As } \cos\theta = \frac{h}{l} = \frac{\sqrt{l^2-r^2}}{l}\right]$$



(2) Angle of string from the vertical: $\tan\theta = \dfrac{v^2}{rg}$

(3) Linear velocity of the bob: $v = \sqrt{gr\tan\theta}$

(4) Angular velocity of the bob: $\omega = \sqrt{\dfrac{g}{r}\tan\theta} = \sqrt{\dfrac{g}{h}} = \sqrt{\dfrac{g}{l\cos\theta}}$

(5) Time period of revolution: $T_P = 2\pi\sqrt{\dfrac{l\cos\theta}{g}} = 2\pi\sqrt{\dfrac{h}{g}} = 2\pi\sqrt{\dfrac{\sqrt{l^2-r^2}}{g}} = 2\pi\sqrt{\dfrac{r}{g\tan\theta}}$