UNIT

10 Oscillations and Waves

PERIODIC AND OSCILLATORY MOTIONS

- **Periodic motion :** Any motion which repeats itself after regular interval of time is called periodic motion. The time interval after which the motion is repeated is called time period or period of motion.
- **Examples of periodic motion :**
  - Motion of hands of a clock. The period of motion of hour's hand of a clock is 12 hours, of minute's hand of a clock is 1 hour and of second's hand of a clock is 1 minute.
  - The revolution of earth around the sun. Its period of revolution is 1 year
  - Motion of electron around the nucleus.
  - Motion of a ball in a bowl.
  - Motion of a liquid in U-tube.
- A periodic motion can be either rectilinear or closed or open curvilinear.
- In case of periodic motion, force is always directed towards a fixed point which may or may not be on the path of motion.
- **Oscillatory motion :** A periodic motion in which body moves to and fro (or back and forth) along the same path about a fixed point (called mean or equilibrium position) is called oscillatory or vibratory motion. Oscillatory or vibratory motion is a constrained periodic motion between two fixed limits (called extreme positions) on either side of mean position.
- **Examples of oscillatory motion :**
  - Motion of the pendulum of a wall clock.
  - Motion of needle of a sewing machine.
  - Motion of the prongs of a tuning fork.
- Every oscillatory motion is periodic but every periodic motion need not be oscillatory. *e.g*, circular motion is a periodic, but it is not oscillatory motion.

SOME TERMS RELATED TO PERIODIC MOTION

- **Time period :** It is defined as the smallest interval of time after which the motion is repeated. It is denoted by symbol T. Its SI unit is second.
- The orbital period of the planet Mercury is 88 earth days. The Halley's comet appears after every 76 years.
- **Frequency :** It is defined as the number of oscillations per unit time. It is denoted by symbol υ.
- The relation between υ and T is

$$\upsilon = \frac{1}{T}$$

- The SI unit of frequency is hertz(Hz).
  1 hertz = 1 Hz = 1 oscillation per second
  $= 1\ s^{-1}$.
- The beat frequency of heart is 1.25 Hz and its period is 0.8 s.
  Note: The frequency υ is not necessarily an integer.
- **Angular frequency :** $\omega = 2\pi\upsilon = \frac{2\pi}{T}$
  The SI unit of ω is rad s^{-1}.
- **Displacement :** It refers to the change in physical quantity under consideration with time. It represents changes in physical quantities with time such as position, angle, pressure, electric and magnetic fields etc.
- In case of an oscillating simple pendulum, the angle measured from the vertical as a function of time is a displacement variable.
- Displacement variable is measured as a function of time, and it can have both positive and negative values. Displacement can be represented by a mathematical function of time. In case of periodic motion, the displacement can be given by
  $x = f(t) = A\cos\omega t$ or $y = f(t) = A\sin\omega t$
  where A is called amplitude.

PERIODIC FUNCTION

- A function $f(t)$ is said to be periodic, if
  $$f(t) = f(t+T)$$
  where T is the time period of the periodic function.
- Any periodic function can be expressed as a superposition of sine and cosine functions of different time periods with suitable coefficients.
- $\sin\omega t$, $\cos\omega t$ and $\sin\omega t + \cos\omega t$ are the periodic functions with a period $\frac{2\pi}{\omega}$
- $e^{-\omega t}$ and $\log(\omega t)$ are non periodic functions.

SIMPLE HARMONIC MOTION (SHM)

- It is a kind of periodic motion, in which a particle moves to and fro (or up and down) about a mean position under a restoring force, which is always directed towards the mean position and whose magnitude at any instant is directly proportional to the displacement

of the particle from the mean position at that instant.
i.e., $F \propto -x$ or $F = -kx$
where k is known as the force constant. The negative sign shows that restoring force F is always directed towards the mean position.

- The SI unit of k is N m^{-1} and its dimensional formula is $[ML^0T^{-2}]$.
- **Examples of simple harmonic motion :**
  - Motion of bob of a simple pendulum.
  - Motion of a block connected to spring.
- Every periodic motion is not simple harmonic motion. Only that periodic motion governed by the force law $F = -kx$ is simple harmonic.

  Note : Simple harmonic motion is the projection of a uniform circular motion on a diameter of the reference circle.
- **Simple harmonic motion are of two types :**
  - **Linear simple harmonic motion :** When a particle moves to and fro (or up and down) about a mean position along a straight line, then its motion is called linear simple harmonic motion. *e.g.*, motion of a block connected to spring.
  - **Angular simple harmonic motion :** When a system oscillates angularly with respect to a fixed axis, then its motion is called angular simple harmonic motion. *e.g.*, motion of bob of a simple pendulum.
- **Comparison between linear SHM and angular SHM**

	Linear SHM	Angular SHM
1.	Restoring force $F = -kx$ where k is the restoring force constant *i.e.*, restoring force per unit displacement.	Restoring torque $\tau = -C\theta$ where C is the restoring torque constant i.e., restoring torque per unit twist.
2.	Acceleration $a = -\frac{k}{m}x$ where m is the mass of a body.	Angular acceleration $\alpha = -\frac{C}{I}\theta$ where I is the moment of inertia of a body.
3.	The differential equation of linear SHM is $\frac{d^2x}{dt^2} + \omega^2 x = 0$ where $\omega^2 = \frac{k}{m}$	The differential equation of angular SHM is $\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$ where $\omega^2 = \frac{C}{I}$

DISPLACEMENT IN SIMPLE HARMONIC MOTION

- The displacement of a particle in SHM at any instant t from its mean position is given by

$$x = A\cos(\omega t + \phi) \quad \text{...(i)}$$
$$y = A\sin(\omega t + \phi) \quad \text{...(ii)}$$

where A is the amplitude (maximum displacement on either side of mean position) of motion, the argument $(\omega t + \phi)$ is the phase of the motion, ϕ is the initial phase or phase constant (or phase angle) and ω is the angular frequency.

- If s is the span of a particle executing SHM, its amplitude

$$A = \frac{s}{2}$$

VELOCITY IN SIMPLE HARMONIC MOTION

- Velocity of a particle in SHM is given by

$$v = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi)$$
$$v = \omega\sqrt{A^2 - x^2}$$

- In SHM, velocity is maximum at the mean position and minimum at the extreme positions.
- The maximum value of velocity is called velocity amplitude in SHM and is given by
  $v_m = A\omega$.
- The direction of velocity of a particle in SHM is either towards or away from the mean position.
- In SHM, the velocity varies simple harmonically with the same frequency as that of the displacement.
- In SHM, the graph between velocity and displacement is an ellipse as $\frac{v^2}{A^2\omega^2} + \frac{x^2}{A^2} = 1$

  If $\omega = 1$ rad s^{-1}, then the graph will be a circle.

ACCELERATION IN SIMPLE HARMONIC MOTION

- Acceleration of a particle in SHM is given by

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \phi)$$
$$a = -\omega^2 x$$

- In SHM, acceleration is proportional to the displacement and is always directed towards the mean position.
- In SHM, acceleration is maximum at the extreme positions and minimum at the mean position.
- The maximum value of acceleration is called acceleration amplitude in SHM and is given by
  $a_m = \omega^2 A$.
- In SHM, the acceleration varies simple harmonically with the same frequency as that of the displacement.
- In SHM, the graph between acceleration and displacement is a straight line which passes through the origin and has slope $(-\omega^2)$.

PHASE RELATIONSHIP BETWEEN DISPLACEMENT, VELOCITY AND ACCELERATION IN SIMPLE HARMONIC MOTION

- The displacement of a particle executing SHM is given by

$$x = A\cos(\omega t + \phi)$$

Velocity, $v = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi)$

$$= \omega A\cos\left[(\omega t + \phi) + \frac{\pi}{2}\right]$$

Acceleration, $a = \frac{dv}{dt} = -\omega^2 A\cos(\omega t + \phi)$

$= \omega^2 A\cos[(\omega t + \phi) + \pi]$

From above, we get that

Phase of displacement = $(\omega t + \phi)$

Phase of velocity $= \left[(\omega t + \phi) + \frac{\pi}{2}\right]$

Phase of acceleration = $(\omega t + \phi + \pi)$

Thus, we conclude that

- The velocity in SHM is leading the displacement by a phase $\pi/2$ radian.
- The acceleration in SHM is leading the displacement by a phase π radian.
- The acceleration in SHM is leading the velocity by a phase $\pi/2$ radian.

Illustration 1

A particle executes SHM of period 1.2 s and amplitude 8 cm. Find the time it takes to travel 3 cm from the positive extremity of its oscillation.
[Given: cos 51° = 0.625]

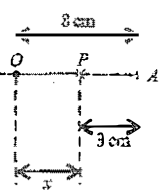
Sol: The equation of SHM, when the particle starts swinging from an extreme position, is

$$x = a\cos\omega t = a\cos\left(\frac{2\pi t}{T}\right)$$

Now x = displacement from mean position

or $x = (8 - 3) = 5$ cm

$\therefore\ 5 = 8\cos\left(\frac{2\pi t}{T}\right)$



or $\cos\left(\frac{2\pi t}{T}\right) = \frac{5}{8} = 0.625$

$= \cos 51° = \cos\left(\frac{51 \times \pi}{180}\right)$

$\therefore\ \frac{2\pi t}{T} = \frac{51 \times \pi}{180}$

or $t = \frac{51 \times T}{2 \times 180} = \frac{51 \times 1.2}{360} = 0.17$ sec

$\therefore$ Time for given travel = 0.17 sec.

Illustration 2

Two particles execute SHM of same amplitude along the same straight line. They cross each other when going in opposite directions, each time their displacement is half of their amplitude. What is the phase difference between them?

Soln.: Let $y_1 = a\sin\omega t$ represent one particle.

$y_2 = a\sin(\omega t + \phi)$ represent other particle.

ϕ = phase difference between the two SHM

When $y_1 = a/2$, we get $\frac{a}{2} = a\sin\omega t$, for one particle.

or $\sin\omega t = \frac{1}{2} = \sin 30°$

or $\omega t = 30°$ ...(i)

For other particle, when $y_2 = \frac{a}{2}$, we get

$\frac{a}{2} = a\sin(\omega t + \phi)$

or $\sin(\omega t + \phi) = \frac{1}{2} = \sin 150°$

$\left[\because \sin 30° = \sin(180° - 30°) = \frac{1}{2}\right]$

or $\omega t + \phi = 150°$ ...(ii)

From (i) and (ii),

$30° + \phi = 150°$ or $\phi = 120°$

$\therefore$ Phase difference = 120°.

Illustration 3

The motion of a particle is given by $x = A\sin\omega t + B\cos\omega t$. The motion of the particle is

(a) not simple harmonic

(b) simple harmonic with amplitude $A + B$

(c) simple harmonic with amplitude $\frac{A + B}{2}$

(d) simple harmonic with amplitude $\sqrt{A^2 + B^2}$

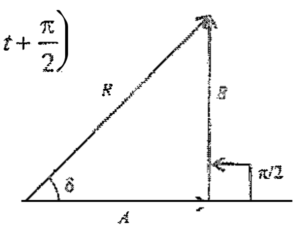
Soln. (d) : $x = A\sin\omega t + B\cos\omega t$

$x = A\sin\omega t + B\sin\left(\omega t + \frac{\pi}{2}\right)$

$\Rightarrow x = R\sin(\omega t + \delta)$

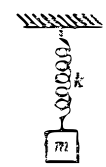
where $R = \sqrt{A^2 + B^2}$

and $\delta = \tan^{-1}\left(\frac{B}{A}\right)$



SPRING PENDULUM

- When a mass m is suspended from a massless spring of spring constant k, then its time period is given by



$$T = 2\pi\sqrt{\frac{m}{k}}$$

- If M be the mass of the spring and mass m is suspended from it, then the time period is given by

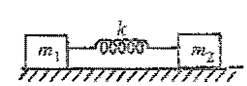
$$T = 2\pi\sqrt{\frac{m + (M/3)}{k}}$$

- If a spring of spring constant k is divided into N equal parts and one such part is attached to a mass m, then the time period is given by

$$T = 2\pi\sqrt{\frac{m}{Nk}}$$

- If two masses m_1 and m_2 are connected by a spring of spring constant k, then the time period is given by

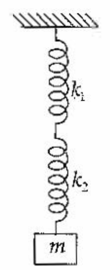
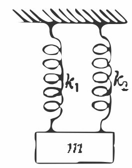
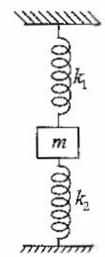
$$T = 2\pi\sqrt{\frac{\mu}{k}}$$



$\left(\text{where } \mu = \text{reduced mass} = \frac{m_1 m_2}{m_1 + m_2}\right)$

COMPOSITE SPRING PENDULUM

- If a spring pendulum is made by using two springs of spring constants k_1 and k_2 respectively and mass m, the following three situations are possible.

Series	Parallel	
		
The effective spring constant is given by $\frac{1}{k_s}=\frac{1}{k_1}+\frac{1}{k_2}$ or $k_s=\frac{k_1k_2}{k_1+k_2}$	The effective spring constant is given by $k_p=k_1+k_2$	The effective spring constant is given by $k_p=k_1+k_2$
The time period is given by $T=2\pi\sqrt{\frac{m}{k_s}}$ $=2\pi\sqrt{\frac{m(k_1+k_2)}{k_1k_2}}$	The time period is given by $T=2\pi\sqrt{\frac{m}{k_p}}$ $=2\pi\sqrt{\frac{m}{(k_1+k_2)}}$	The time period is given by $T=2\pi\sqrt{\frac{m}{k_p}}$ $=2\pi\sqrt{\frac{m}{(k_1+k_2)}}$

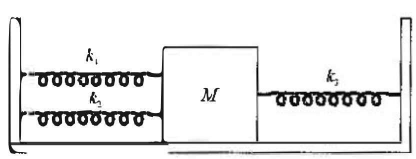
Note : When a spring of spring constant k, is cut into two pieces of lengths l_1 and l_2, then

Spring constant of length $l_1 = k_1 = k\left(1+\frac{l_2}{l_1}\right)$

Spring constant of length $l_2 = k_2 = k\left(1+\frac{l_1}{l_2}\right)$

Illustration 4

At equilibrium, the springs are released, Mass M is oscillating under the influence of three springs k_1, k_2 and k_3 as shown. Its frequency (υ) of oscillation is $\frac{1}{2\pi}\sqrt{\frac{k_{eq}}{M}}$, where k_{eq} is such that

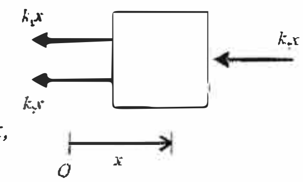


Frictionless floor

(a) $k_{eq} = k_1 + k_2 + k_3$ (b) $k_{eq} = \frac{k_1+k_2+k_3}{3}$

(c) $\frac{1}{k_{eq}} = \frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}$ (d) $k_{eq} = (k_1 + k_2 - k_3)$

Soln. (a) : When the mass M is displaced slightly to the right, the resultant forces acting on the mass M is as shown.



$F_{net} = -(k_1 + k_2 + k_3)\cdot x,$

Thus, $k_{eq} = k_1 + k_2 + k_3$

$\Rightarrow$ (a) is correct.

Illustration 5

Two bodies A and B of equal mass are suspended from two separate massless springs of constant k_1 and k_2 respectively. If the bodies oscillate vertically such that their maximum velocities are equal, the ratio of the amplitude of A to that of B is

(a) $\sqrt{\frac{k_1}{k_2}}$ (b) $\sqrt{\frac{k_2}{k_1}}$

(c) $\frac{k_1}{k_2}$ (d) $\frac{k_2}{k_1}$

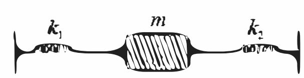
Soln. (b) : $v_{max} = \omega A$, so, $\omega_A A_A = \omega_B A_B \Rightarrow \frac{A_A}{A_B} = \frac{\omega_B}{\omega_A}$

$$= \sqrt{\frac{k_2}{m}} \div \sqrt{\frac{k_1}{m}} = \sqrt{\frac{k_2}{k_1}}.$$

$\Rightarrow$ (b) is correct.

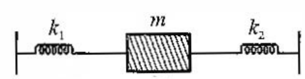
Illustration 6

Two springs, of force constants k_1 and k_2 are connected to a mass m as shown. The frequency of oscillation of the mass is υ. If both k_1 and k_2 are made four times their original values, the frequency of oscillation becomes



(a) 2υ (b) $\upsilon/2$
(c) $\upsilon/4$ (d) 4υ

Soln. (a) : In the given figure two springs are connected in parallel. Therefore the effective spring constant is given by



$k_{eff} = k_1 + k_2$

Frequency of oscillation,

$$\upsilon = \frac{1}{2\pi}\sqrt{\frac{k_{eff}}{m}} = \frac{1}{2\pi}\sqrt{\frac{k_1+k_2}{m}} \quad \text{... (i)}$$

As k_1 and k_2 are increased four times

New frequency,

$$\upsilon' = \frac{1}{2\pi}\sqrt{\frac{4(k_1+k_2)}{m}} = 2\upsilon \quad \text{(using (i).}$$

Illustration 7

A mass M is suspended from a spring of negligible mass. The spring is pulled a little and then released so that the mass executes SHM of time period T. If the mass is increased by m, the time period becomes $5T/3$. Then the ratio of m/M is

(a) 3/5 (b) 25/9
(c) 16/9 (d) 5/3.

Soln. (c) : Initially, $T = 2\pi\sqrt{M/k}$

Finally, $\frac{5T}{3} = 2\pi\sqrt{\frac{M+m}{k}}$

$\therefore \quad \frac{5}{3} \times 2\pi\sqrt{\frac{M}{k}} = 2\pi\sqrt{\frac{M+m}{k}}$

or $\quad \frac{25}{9}\frac{M}{k} = \frac{M+m}{k}$

or $\quad 9m + 9M = 25M$ or $\frac{m}{M} = \frac{16}{9}$.

ENERGY IN SIMPLE HARMONIC MOTION

- The kinetic energy of a particle in SHM is given by

  Kinetic energy $K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi)$

  $= \frac{1}{2}m\omega^2(A^2 - x^2)$
- Kinetic energy of a particle executing SHM is periodic with period $T/2$. It is zero at extreme positions and maximum at mean position.
- The potential energy of a particle in SHM is given by

  Potential energy $U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t + \phi)$

  $= \frac{1}{2}m\omega^2 A^2 \cos^2(\omega t + \phi)$
- Potential energy of a particle executing SHM is periodic with period $T/2$. It is zero at the mean position and maximum at the extreme positions.
- Total energy of a particle in SHM is given by

  $E = K + U$

  $= \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) + \frac{1}{2}m\omega^2 A^2\cos^2(\omega t + \phi)$

  $= \frac{1}{2}m\omega^2 A^2$
- In SHM, total energy remains constant at all instants and at all displacements. It depends upon the mass, amplitude and frequency of vibration of the particle.
- In SHM, at the mean position total energy is in the form of its kinetic energy and at the extreme positions, total energy is in the form of its potential energy.
- The average value of kinetic energy or potential energy of a particle in SHM in a time of one complete oscillation,

  $\langle K \rangle = \langle U \rangle = \frac{1}{4}m\omega^2 A^2$.

  But the average value of total energy of a particle in SHM in one complete oscillation,

  $\langle E \rangle = \frac{1}{2}m\omega^2 A^2$.

  If frequency of oscillation in SHM is υ, then frequency of oscillation of KE = frequency of oscillation of PE = 2υ.
- In SHM, the frequency of oscillation of total energy is zero.

Illustration 8

If E_P and E_K represent the potential energy and kinetic energy of a body undergoing S.H.M, then (E is the total energy of the body), at a position where the displacement is half the amplitude,

(a) $E_P = \frac{E}{2}, E_K = \frac{E}{2}$ (b) $E_P = \frac{3E}{4}, E_K = \frac{E}{4}$
(c) $E_P = \frac{E}{4}, E_K = \frac{3E}{4}$ (d) $E_P = \frac{E}{3}, E_K = \frac{2E}{3}$

Soln. (c) : $E_K = \frac{1}{2}m\omega^2(A^2 - x^2)$ At $x = \frac{A}{2}$

$E_K = \frac{1}{2}m\omega^2\left(A^2 - \frac{A^2}{4}\right) = \frac{3}{4}\left(\frac{1}{2}m\omega^2 A^2\right) = \frac{3}{4}E$

Also, $E_P = E - E_K = E - \frac{3E}{4} = \frac{E}{4}$

$\Rightarrow$ (c) is correct.

Illustration 9

The total mechanical energy of a spring-mass system in simple harmonic motion is $E = \frac{1}{2}m\omega^2 A^2$. Suppose the oscillating particle is replaced by another particle of double the mass while the amplitude A remains the same. The new mechanical energy will

(a) become $2E$ (b) become $E/2$
(c) become $\sqrt{2}E$ (d) remain E

Soln (d) : $E = \frac{1}{2}m\omega^2 A^2 = \frac{1}{2}m\left(\sqrt{\frac{k}{m}}\right)^2 A^2$

$E = \frac{1}{2}kA^2$

Total energy depends on k of spring and amplitude A. It is independent of mass.

Illustration 10

Starting from the origin, a body oscillates simple harmonically with a period of 2 s. After what time will its kinetic energy be 75% of the total energy?

(a) $\frac{1}{12}$ s (b) $\frac{1}{6}$ s
(c) $\frac{1}{4}$ s (d) $\frac{1}{3}$ s.

Soln. (b) : During simple harmonic motion,

Kinetic energy

$= \frac{1}{2}mv^2 = \frac{1}{2}m(a\omega\cos\omega t)^2$

Total energy $E = \frac{1}{2}ma^2\omega^2$

$\because$ (Kinetic energy) $= \frac{75}{100}(E)$

or $\frac{1}{2}ma^2\omega^2\cos^2\omega t = \frac{75}{100}\times\frac{1}{2}ma^2\omega^2$

or $\cos^2\omega t = \frac{3}{4} \Rightarrow \cos\omega t = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$

$\therefore$ $\omega t = \frac{\pi}{6}$

or $t = \frac{\pi}{6\omega} = \frac{\pi}{6(2\pi/T)} = \frac{2\pi}{6\times 2\pi} = \frac{1}{6}$ s

SIMPLE PENDULUM

- The time period of a simple pendulum is given by

$$T = 2\pi\sqrt{\frac{L}{g}}.$$

where L is the length of the pendulum and g is the acceleration due to gravity.

- Time period of a simple pendulum is independent of mass, shape and material of bob and it is also independent of the amplitude of oscillation provided it is small.
- Time period of a simple pendulum depends on L as $T \propto \sqrt{L}$, so the graph between T and L will be a parabola while between T^2 and L will be a straight line.
- Time period of a simple pendulum depends on acceleration due to gravity as $T \propto \left(1/\sqrt{g}\right)$. With increase in g, T will decrease or vice versa.
- If the length of a simple pendulum is comparable with the radius of earth (R_e), then time period T is given by

$$T = 2\pi\sqrt{\frac{1}{g\left(\frac{1}{L}+\frac{1}{R_e}\right)}}$$

where R_e is the radius of the earth.

Special cases :

- If $L \ll R_e$, then

$$T = 2\pi\sqrt{\frac{R_e}{g}} = 2\pi\sqrt{\frac{6.4\times 10^6}{9.8}} = 84.6 \text{ min}$$

- If $L = R_e$, then

$$T = 2\pi\sqrt{\frac{R_e}{2g}} = 2\pi\sqrt{\frac{6.4\times 10^6}{2\times 9.8}} = 60 \text{ min}$$

- The simple pendulum having time period of 2 s is called **second pendulum.**
- If a simple pendulum is suspended in a lift and lift is accelerating downwards with an acceleration a, then its time period is given by

$$T = 2\pi\sqrt{\frac{L}{g-a}}$$

- If a simple pendulum is suspended in a lift and lift is accelerating upwards with an acceleration a, then its time period is given by

$$T = 2\pi\sqrt{\frac{L}{g+a}}$$

- If a simple pendulum is suspended in a lift and the lift is moving upwards or downwards with constant velocity v, then its time period is given by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

- If a simple pendulum is suspended in a lift and lift is freely falling with acceleration g, then its time period is given by

$$T = 2\pi\sqrt{\frac{L}{g-g}} = \infty$$

- If a simple pendulum is suspended in a carriage which is accelerating horizontally with an acceleration a, then its time period is given by

$$T = 2\pi\sqrt{\frac{L}{(\sqrt{g^2+a^2})}}$$

- If a simple pendulum is suspended from the roof of a trolley which is moving down an inclined plane of inclination θ, then the time period is given by

$$T = 2\pi\sqrt{\frac{L}{g\cos\theta}}$$

- If a simple pendulum whose bob is of density ρ oscillates in a non-viscous liquid of density $\sigma(\sigma < \rho)$, then its time period is given by

$$T = 2\pi\sqrt{\frac{L}{\left(1-\frac{\sigma}{\rho}\right)g}}$$

- If the simple pendulum has charge q and is oscillating in a uniform electric field E, which is

Opposite to g	In the direction of g	Perpendicular to g
Electrostatic force qE is opposite to the force of gravity mg, then time period is given by $T = 2\pi\sqrt{\frac{L}{g-\frac{qE}{m}}}$	Electrostatic force qE is in the direction of force of gravity mg, then time period is given by $T = 2\pi\sqrt{\frac{L}{g+\frac{qE}{m}}}$	Electrostatic force qE is perpendicular to force gravity mg, then time period is given by $T = 2\pi\sqrt{\frac{L}{\left(g^2+\left(\frac{qE}{m}\right)^2\right)^{1/2}}}$

EXPRESSIONS FOR TIME PERIOD OF SOME SIMPLE HARMONIC MOTIONS

- **Conical pendulum :** The time period of a conical pendulum is given by

$$T = 2\pi\sqrt{\frac{L\cos\theta}{g}}$$

where L is the length of the string and θ is the angle which the string makes with the vertical.

- **Torsional pendulum :** The time period of a torsional pendulum is given by

$$T = 2\pi\sqrt{\frac{I}{C}}$$

where I is the moment of inertia of the disc about the suspension wire as axis of rotation and C is the restoring torque per unit twist.

$$C = \frac{\pi\eta r^4}{2L}$$

where r is the radius, L is the length and η is the modulus of rigidity of a wire respectively.

- **Liquid in U-tube :** The time period of oscillation of a liquid in U-tube is given by

$$T = 2\pi\sqrt{\frac{L}{2g}} = 2\pi\sqrt{\frac{h}{g}}$$

where L = length of liquid column in a U-tube,
h = height of liquid column in each limb of U-tube
Also $h = L/2$

- **Floating cylinder in a liquid :** The time period of oscillation of floating cylinder in a liquid is given by

$$T = 2\pi\sqrt{\frac{m}{A\sigma g}}$$

where m is the mass of a cylinder, A is the area of cross section of a cylinder, σ is the density of a liquid

$$\text{or} \quad T = 2\pi\sqrt{\frac{h\rho}{\sigma g}} = 2\pi\sqrt{\frac{h'}{g}}$$

where h is the height of cylinder of density ρ and σ is the density of a liquid in which cylinder is floating, h' is the height of the cylinder inside the liquid.

- Time period of LC oscillations of a circuit containing capacitance C and inductance L is given by

$$T = 2\pi\sqrt{LC}$$

- If a wire of length L, area of cross-section A, Young's modulus Y is stretched by suspending a mass m, then the mass can oscillate with time period

$$T = 2\pi\sqrt{\frac{mL}{YA}}$$

- If a gas is enclosed in a cylinder of volume V fitted with piston of cross section area A and mass M and the piston is slightly depressed and released, the piston can oscillate with a time period

$$T = 2\pi\sqrt{\frac{MV}{BA^2}}$$

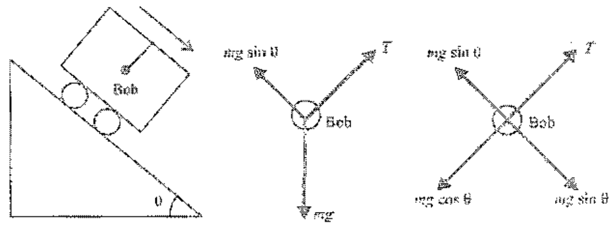
where B is the bulk modulus of elasticity of the gas.

Illustration 11

A simple pendulum is set up on a trolley which slides down a frictionless inclined plane making an angle θ with horizontal. The time period of pendulum is

(a) $2\pi\sqrt{\frac{l}{g}}$
(b) $2\pi\sqrt{\frac{l\cos\theta}{g}}$
(c) $2\pi\sqrt{\frac{l}{g\cos\theta}}$
(d) $2\pi\sqrt{\frac{l}{g(1-\cos\theta)}}$

Soln. (c) :



At equilibrium, inside the trolley,
Tension = $mg\cos\theta$
Hence effective $g' = g\cos\theta$

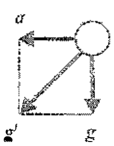
$$T = 2\pi\sqrt{\frac{l}{g'}} = 2\pi\sqrt{\frac{l}{g\cos\theta}}$$

Illustration 12

A simple pendulum is suspended from the roof of a trolley which moves in a horizontal direction with an acceleration a. Then the time period is given by $T = 2\pi\sqrt{\frac{l}{g'}}$ where g' is equal to

(a) g
(b) $g - a$
(c) $g + a$
(d) $\sqrt{g^2 + a^2}$

Soln. (d) :



$$g' = \sqrt{g^2 + a^2}$$

Illustration 13

A simple pendulum of length l_1 has a time period of 4 s and another simple pendulum of length l_2 has a time period 3 s. Then the time period of another pendulum of length $(l_1 - l_2)$ is

(a) $\sqrt{3}$ s
(b) 1 s
(c) $\sqrt{\frac{3}{4}}$ s
(d) $\sqrt{7}$ s

Soln. (d) : $4\text{ s} = 2\pi\sqrt{\frac{l_1}{g}}$ or $l_1 = \frac{4g}{\pi^2}$

$$3\text{ s} = 2\pi\sqrt{\frac{l_2}{g}} \text{ or } l_2 = \frac{9g}{4\pi^2}$$

$$(l_1 - l_2) = \frac{g}{\pi^2}\left(4 - \frac{9}{4}\right) = \frac{7g}{4\pi^2}$$

$$T = 2\pi\sqrt{\frac{(l_1 - l_2)}{g}} = 2\pi\sqrt{\frac{7g}{4\pi^2 \cdot g}} = \sqrt{7}\text{ s.}$$

UNDAMPED AND DAMPED OSCILLATIONS

- **Undamped oscillations :** When a system oscillates with a constant amplitude which does not change with time, its oscillations are called undamped oscillations.
- The energy of the system executing undamped oscillations remains constant and is independent of time.
- The dissipative forces (*i.e.*, frictional or viscous forces) are not present in the system executing undamped oscillations.
- **Damped oscillations :** When a system oscillates with a decreasing amplitude with time, its oscillations are called damped oscillations.
- The energy of the system executing damped oscillations will go on decreasing with time but the oscillations of the system remain periodic.
- The dissipative forces or damping forces are active in the oscillating system which are generally the frictional or viscous forces.
- The damping force is given by $F_d = -bv$, where, v is the velocity of the oscillator and b is damping constant. Negative sign shows that damping force acts opposite to the velocity at every moment.
- The SI unit of b is kg s^{-1}.
- The differential equation of damped harmonic oscillator is given by

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

- The displacement of the damped oscillator at any instant t is given by

$$x = Ae^{-bt/2m}\cos(\omega' t + \phi)$$

where ω' is the angular frequency of the damped oscillator is given by

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

- If the damping constant b is small, then $\omega' \approx \omega$, where ω is the angular frequency of the undamped oscillator.
- The mechanical energy E of the damped oscillator at any instant t is given by

$$E = \frac{1}{2}kA^2e^{-bt/m}$$

- **Maintained oscillations :** Due to the damping forces, the amplitude of oscillator will go on decreasing with time. If we can feed the energy to the damped oscillatory system at the same rate at which it is dissipating the energy, then the amplitude of such oscillations would become constant. Such oscillations are called maintained oscillations.

FORCED OSCILLATIONS AND RESONANCE

- **Free oscillations :** When a system oscillates with its own natural frequency, without the help of any external periodic force, its oscillations are called free oscillations.
- **Forced or driven oscillations :** When a system oscillates with the help of an external periodic force, other than its own natural angular frequency, its oscillations are called forced or driven oscillations.
- The differential equation of forced damped harmonic oscillator is given by

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0\cos\omega_d t$$

where ω_d is the angular frequency of the external force.
- The displacement of the forced damped harmonic oscillator at any instant t is given by

$$x = A\cos(\omega_d t + \phi)$$

where $A = \dfrac{F_0}{\{m^2(\omega^2 - \omega_d^2)^2 + \omega_d^2 b^2\}^{1/2}}$

$$= \frac{F_0}{m\left[(\omega^2 - \omega_d^2)^2 + \left(\frac{\omega_d b}{m}\right)^2\right]^{1/2}}$$

and $\tan\phi = \dfrac{-v_0}{\omega_d x_0}$

where ω is the natural angular frequency of the oscillator, x_0 and v_0 are the displacement and velocity of the oscillator at time $t = 0$, when the periodic force is applied.
- **Resonance :** It is the phenomenon in which a system is made to oscillate by external force whose frequency is equal to the natural frequency of the system. At resonance, the amplitude of the system is maximum. It is a special case of forced oscillation.

  Condition for resonance is $\omega = \omega_d$.

WAVE MOTION

- It is a kind of disturbance which travels through a material medium (having properties of elasticity and inertia) on account of repeated periodic vibrations of the particles of the medium about their mean position, the disturbance being handed on from one particle to the adjoining particle and so on, without any net transport of the medium. It transports both, energy and momentum without the transport of matter.

Types of Waves

- Waves can be classified into following three categories :
  - Mechanical waves
  - Electromagnetic waves
  - Matter waves
- **Mechanical waves :** These waves require a medium for their propagation. *e.g.*, water waves, sound waves, seismic waves, etc. Sometimes mechanical waves are also called elastic waves.
- **Electromagnetic waves :** These waves do not require any medium for their propagation. *e.g.*, light waves, X-rays, micro waves, etc. All electromagnetic waves travel through vacuum at the same speed c ($c = 3 \times 10^8$ m s^{-1})
- **Matter waves :** These waves are associated with moving particles of matter like electrons, protons, neutrons, atoms and molecules. These waves arise in quantum mechanical description of nature. These waves are also known as **de Broglie waves**.