Soln.: (b) When Q is doubled then $\bar{Q}$ becomes $2\bar{Q}$. Suppose that the new resultant becomes $\vec{R}'$.

$\therefore\quad \vec{R}' = \vec{P} + 2\vec{Q}$ and $\vec{R}' \perp \vec{P}$

From, $\tan\alpha = \dfrac{B\sin\theta}{A + B\cos\theta} \Rightarrow \tan 90° = \dfrac{2Q\sin\theta}{P + 2Q\cos\theta}$

$\Rightarrow \dfrac{1}{0} = \dfrac{2Q\sin\theta}{P + 2Q\cos\theta}$

$\Rightarrow\quad 2Q\cos\theta = -P \quad \therefore \quad \cos\theta = -\dfrac{P}{2Q}$ ...(i)

Now according to question,

$\vec{R} = \vec{P} + \vec{Q} \quad \therefore \quad R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$

Putting value of $\cos\theta$ from equation number (i)

$$R = \sqrt{P^2 + Q^2 + 2PQ\left(-\frac{P}{2Q}\right)}$$

$\Rightarrow\quad R = \sqrt{P^2 + Q^2 - P^2}$

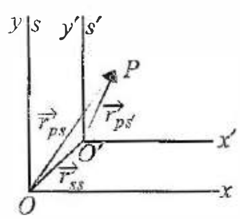
$\Rightarrow\quad R = \sqrt{Q^2} \quad \therefore \quad R = Q$

RELATIVE MOTION

There is no meaning of motion without reference or observer. If reference is not mentioned then we take the ground as a reference of motion. Velocity or displacement of the particle w.r.t. ground is called actual velocity or actual displacement of the body. If we describe the motion of a particle w.r.t. an object which is also moving w.r.t. ground then velocity of particle w.r.t. ground is its actual velocity ($\vec{v}_{act}$) and velocity of the particle w.r.t. moving object is its relative velocity ($\vec{v}_{rel}$) and the velocity of moving object (w.r.t. ground) is the reference velocity ($\vec{v}_{ref}$). In the figure, let s and s' be two reference frames with observers at O and O' respectively position of particle P relative to frame s is $\vec{r}_{ps}$ while position of frame s' relative to frame s is $\vec{r}_{s's}$ at a moment.

According to vector law of addition,

$\vec{r}_{ps} = \vec{r}_{ps'} + \vec{r}_{s's}$



Differentiating the equation w.r.t. time, we get

$$\frac{d\vec{r}_{ps}}{dt} = \frac{d\vec{r}_{ps'}}{dt} + \frac{d\vec{r}_{s's}}{dt}$$

so that $\vec{v}_{ps} = \vec{v}_{ps'} + \vec{v}_{s's}$

Differentiating again, we get

$\vec{a}_{ps} = \vec{a}_{ps'} + \vec{a}_{s's}$

- Relative velocity of a body A with respect to body B, when they are moving in the same direction is given by

  $\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$.

- Relative velocity of a body A with respect to body B when they are moving in the opposite direction is given by $\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$.

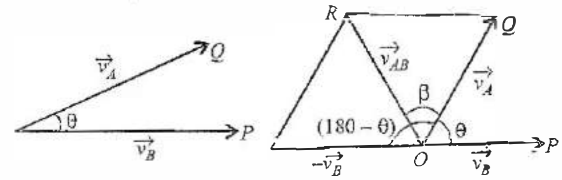
- The magnitude of relative velocity of a body A with respect to body B, when they are inclined at an angle θ is given by

$$v_{AB} = \sqrt{(v_A)^2 + (v_B)^2 + 2(v_A)(v_B)\cos(180° - \theta)}$$

$$= \sqrt{v_A^2 + v_B^2 - 2v_Av_B\cos\theta}$$

$$\tan\beta = \frac{(v_B)\sin(180° - \theta)}{(v_A) + (v_B)\cos(180° - \theta)} = \frac{v_B\sin\theta}{v_A - v_B\cos\theta}$$

where $\vec{v}_A$ and $\vec{v}_B$ are velocities of two bodies A and B, β is an angle which $\vec{v}_{AB}$ makes with the direction of $\vec{v}_A$.



- **Rain man problem**: If rain is falling vertically with a velocity $\vec{v}_r$ and a man is moving horizontally with speed $\vec{v}_m$, the man can protect himself from the rain if he holds his umbrella in the direction of relative velocity of rain w.r.t. man. If θ is the angle which the direction of relative velocity of rain w.r.t. man makes with the vertical,

  then $\tan\theta = \dfrac{v_m}{v_r}$.

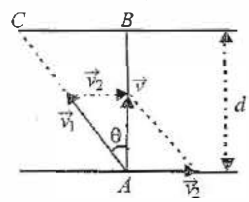
- **Boat-river problem**: Let $\vec{v}_1$ = velocity of boat in still water, $\vec{v}_2$ = velocity of flow of water in river, d = width of river.

  (a) To cross the river in the shortest path: Here it is required that the boat starting from A must reach the opposite point B along the shortest path AB. For the shortest path, the boat should be rowed upstream making an angle θ with AB such that AB gives the direction of resultant velocity.

  So, $\sin\theta = \dfrac{v_2}{v_1}$

  and $v^2 = v_1^2 - v_2^2$

  Also, $t = \dfrac{d}{v} = \dfrac{d}{\sqrt{v_1^2 \;\; v_2^2}}$

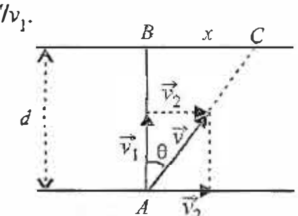


  (b) To cross the river in the shortest time: For the boat to cross the river in the shortest time, the boat should be directed along AB. Let v be the resultant velocity making an angle θ with AB.

  Then $\tan\theta = \dfrac{v_2}{v_1}$ and $v^2 = v_1^2 + v_2^2$

  Time of crossing, $t = d/v_1$.

  Now the boat reaches the point C rather than reaching point B. If $BC = x$, then



  $\tan\theta = \dfrac{v_2}{v_1} = \dfrac{x}{d}$ or $x = d \times \left(\dfrac{v_2}{v_1}\right)$

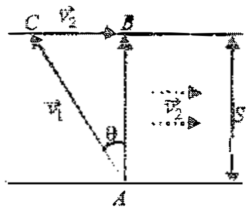
(c) If a man travels downstream in a river, then the time taken by the man to cover a distance d is $t_1 = \frac{d}{v_1 + v_2}$. If a man swims upstream in a river, then the time taken by him to cover a distance d is $t_2 = \frac{d}{v_1 - v_2}$.

So $\frac{t_1}{t_2} = \frac{v_1 - v_2}{v_1 + v_2}$.

Illustration 13

A river is flowing from west to east at a speed of 5 m/min. In what direction should a man on the south bank of the river, capable of swimming at 10 m/min in still water, should swim to cross the river (i) along the shortest path (ii) in shortest time?

Soln. : (i)



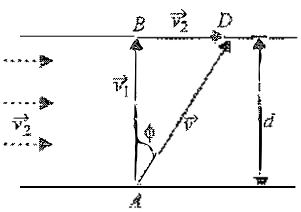
Along shortest path (AB):

The man will cross the river along the shortest path if the resultant of $\vec{v}_2$ and $\vec{v}_1$ is along AB

$\therefore \quad \sin\theta = \frac{v_2}{v_1} = \frac{5}{10} = \frac{1}{2} = \sin 30°$

$\therefore \quad \theta = 30°$ west of north.

(ii)



In shortest time:

$$\text{Time} = \frac{d}{v_1} = \frac{d}{v\cos\phi}$$

Time will be minimum if $\cos\phi$ is maximum *i.e.*, $\phi = 0$.

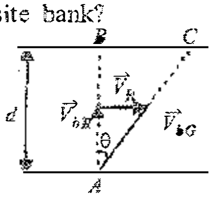
The man should swim along AB *i.e.*, perpendicular to the flow.

Illustration 14

A river 400 m wide is flowing at a rate of 2.0 m s^{-1}. A boat is sailing at a speed of 10 m s^{-1} w.r.t. water, in a direction perpendicular to the river.

(a) Find the time taken by the boat to reach the opposite bank.

(b) How far from the point directly opposite to the starting point does the boat reach the opposite bank?



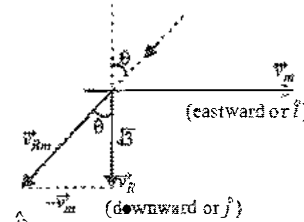
Soln. : (a) $t = \frac{AB}{v_{bR}} = \frac{400}{10}$ sec $= 40$ sec

(b) Drifting of boat along the river $= t \times v_R = 40 \times 2$ m $= 80$ m

Illustration 15

A man is moving due east with a speed 1 km/hr and rain is falling vertically with a speed $\sqrt{3}$ km/hr. At what angle from the vertical the man has to hold his umbrella to keep the rain away. Also find the speed of rain drops w.r.t. man.

Soln. : $\tan\theta = \frac{1}{\sqrt{3}}, \theta = 30°$



$\vec{v}_m = (1 \text{ km hr}^{-1})\hat{i}$

$\vec{v}_R = (\sqrt{3} \text{ km hr}^{-1})(-\hat{j})$

$\vec{v}_{Rm} = \vec{v}_R - \vec{v}_M = (-\sqrt{3}\,\hat{j} - \hat{i})$

$|\vec{v}_{Rm}| = \sqrt{(\sqrt{3})^2 + (1)^2} = 2 \text{ km hr}^{-1}$

Angle of relative velocity with vertical is 30°, so the man has to tilt his umbrella at 30° with vertical towards the east.

Illustration 16

On the ground a man A is moving with velocity $(3\hat{i} - 4\hat{j})$ m s^{-1} and another man B is moving with velocity $(\hat{i} + \hat{j})$ m s^{-1} relative to A. Find the velocity of B w.r.t. ground.

Soln. : $\vec{v}_{BA} = \vec{v}_B - \vec{v}_A$ and $\vec{v}_B = \vec{v}_{BA} + \vec{v}_A$

or $\vec{v}_B = [(\hat{i} + \hat{j}) + (3\hat{i} - 4\hat{j})]$ m $s^{-1} = (4\hat{i} - 3\hat{j})$ m s^{-1}

Illustration 17

A boy is running on the plane road with velocity (v) with a long hollow tube in his hand. The water is falling vertically downwards with velocity u. At what angle to the vertical, he must incline the tube so that the water drops enters in it without touching its side

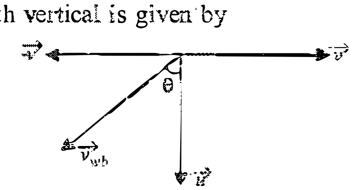
(a) $\tan^{-1}\left(\frac{v}{u}\right)$ (b) $\sin^{-1}\left(\frac{v}{u}\right)$

(c) $\tan^{-1}\left(\frac{u}{v}\right)$ (d) $\cos^{-1}\left(\frac{v}{u}\right)$

Soln. (a) : Velocity of water with respect to boy is given by $\vec{v}_{wb}$ and its inclination with vertical is given by

$\tan\theta = \frac{v}{u}$

$\therefore \quad \theta = \tan^{-1}\left(\frac{v}{u}\right)$



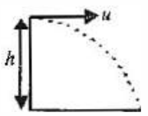
PROJECTILE

- Any body given an initial velocity moves freely in space under the influence of gravity is called a projectile.
- A javelin thrown by an athlete and a ball kicked from the ground level are the examples of projectile motion.
- The path followed by a projectile is called its trajectory. Trajectory of a projectile is a parabola.
- Projectile motion is a two dimensional motion.
- While studying the projectile motion, we have to make two assumptions.

(i) The air resistance has no effect on the projectile motion.

(ii) Trajectories are of short range so that free fall acceleration g remains constant in magnitude and direction.

A Projectile Projected Horizontally from a Height h with Velocity u



(i) Time taken by the projectile to reach the ground is $\sqrt{2h/g}$.

(ii) Time taken by the projectile to reach the ground does not depend upon the velocity of projection *i.e.*, u.

(iii) Horizontal range, $x = ut = u\sqrt{\dfrac{2h}{g}}$.

(iv) Equation of trajectory is $y = \dfrac{g}{2u^2}x^2$.

(v) Resultant velocity of the projectile at any time t is

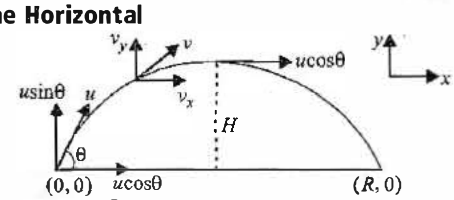
$$v = \sqrt{u^2 + g^2t^2}.$$

(vi) Angle made by the resultant velocity with the horizontal is $\tan\beta = \dfrac{gt}{u}$.

(vii) Velocity of the projectile on striking the ground

$$= \sqrt{u^2 + 2gh}.$$

A Projectile Projected with Velocity u at an Angle θ with the Horizontal



(i) $x = u\cos\theta t$

(ii) $y = u\sin\theta t - \dfrac{1}{2}gt^2$

(iii) Equation of trajectory is

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}.$$

(iv) Velocity of the projectile at any time t is

$$v = \sqrt{(u\cos\theta)^2 + (u\sin\theta - gt)^2}$$

$$= \sqrt{u^2 + g^2t^2 - 2gtu\sin\theta}$$

This velocity make angle β with horizontal.

$$\tan\beta = \frac{u\sin\theta - gt}{u\cos\theta}$$

(v) Horizontal range, $R = \dfrac{u^2\sin 2\theta}{g}$.

(vi) For maximum horizontal range, $\theta = 45°$.

$$R_m = \frac{u^2}{g}$$

(vii) Time of ascent = time of descent $= \dfrac{u\sin\theta}{g}$.

(viii) Time of flight, $T = \dfrac{2u\sin\theta}{g}$.

(ix) Maximum height $H = \dfrac{u^2\sin^2\theta}{2g}$.

(x) Horizontal range of projectile remains same when the angle of projection is (a) θ and 90° – θ (b) (45° + θ) and (45° – θ)

(xi) At the highest point, the projectile possesses velocity only along horizontal direction.

(xii) At the highest point of projectile path, the velocity and acceleration are perpendicular to each other.

(xiii) When horizontal range is n times the maximum height, then $\tan\theta = 4/n$.

(xiv) When the velocity of projection of a projectile thrown at an angle θ with the horizontal is increased n times,

(a) time of ascent becomes n times

(b) time of descent becomes n times

(c) time of flight becomes n times

(d) maximum height is increased by a factor of n^2

(e) horizontal range is increased by a factor of n^2.

Note : When the horizontal range is maximum, the time of flight is

$$T = \frac{2u\sin 45°}{g} = \frac{\sqrt{2}u}{g} \qquad \left(\because \sin 45° = \frac{1}{\sqrt{2}}\right)$$

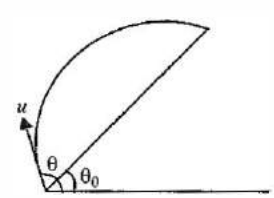
When the horizontal range is maximum, the

$$\text{maximumHeight, } H = \frac{u^2\sin^2 45°}{2g} = \frac{1}{4}\frac{u^2}{g} = \frac{R_m}{4}.$$

- **Effect of air resistance :** The air resistance decreases the maximum height attained and range of the projectile. It also decreases the speed with which the projectile strikes the ground.
- **Effect of variation of g :** Acceleration due to gravity does not remain constant when the range exceeds say 1500 km or so. Then the direction of g changes because g always points towards the centre of earth. Due to this, shape of trajectory changes from parabolic to elliptical.

Motion of Projectile along an Inclined Plane

- When a projectile is projected with velocity u, making an angle θ with the horizontal direction up the inclined plane, whose inclination with the horizontal direction is θ_0, then



(i) Range of projectile along the inclined plane is

$$R = \frac{u^2}{g\cos^2\theta_0}[\sin(2\theta - \theta_0) - \sin\theta_0]$$

(ii) Time of flight on the inclined plane is

$$T = \frac{2u\sin(\theta - \theta_0)}{g\cos\theta_0}$$

(iii) The maximum range on inclined plane is

$$R_{max} = \frac{u^2}{g(1+\sin\theta_0)}$$

(iv) The angle at which the horizontal range on the inclined plane becomes maximum is given as,

$$\theta = \frac{\pi}{4} + \frac{\theta_0}{2}.$$

Illustration 18

A body of mass m is projected horizontally with a velocity v from the top of a tower of height h and it reaches the ground at a distance x from the foot of the tower. If a second body of mass $2m$ is projected horizontally from the top of a tower of height $2h$, it reaches the ground at a distance $2x$ from the foot of the tower. Then what is the horizontal velocity of the second body?

Soln. : For the first body, $h = \frac{1}{2}gt^2$ ... (i)

and $x = vt$ ... (ii)

From equations (i) and (ii), we get

$$h = \frac{1}{2}g\left(\frac{x^2}{v^2}\right) \quad \text{... (iii)}$$

For the second body, let v' be the velocity of projection, then

$$2h = \frac{1}{2}g\left[\frac{(2x)^2}{v'^2}\right] \quad \text{...(iv)}$$

Dividing equations (iii) by equation (iv), we get

$$\frac{1}{2} = \frac{x^2}{v^2} \times \frac{v'^2}{4x^2} \quad \text{or} \quad v' = \sqrt{2}v$$

Illustration 19

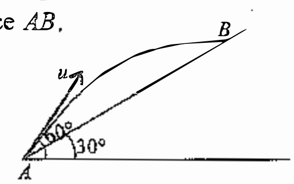
The maximum range of a projectile fired with some initial velocity is found to be 1000 metre, in the absence of wind and air resistance. Find the maximum height h reached by this projectile.

Soln.: $R_{max} = \frac{u^2}{g} = 1000$ m (R is maximum when $\theta = 45°$)

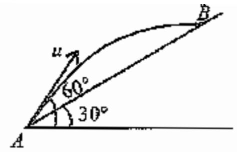
$$H = \frac{u^2 \sin^2\theta}{2g} = \frac{u^2}{2g} \times \sin^2 45° = \frac{u^2}{4g} = \frac{1000}{4} = 250 \text{ m}$$

Illustration 20

Time taken by the projectile to reach from A to B is t, then find the distance AB.



Soln.: Refer the figure below. Horizontal component of velocity at A.



$$u_H = u\cos 60° = \frac{u}{2}$$

$$\therefore \quad AC = u_H \times t = \frac{ut}{2}$$

$$AB = AC\sec 30° = \frac{ut}{2} \times \frac{2}{\sqrt{3}} = \frac{ut}{\sqrt{3}}$$

Illustration 21

A projectile is given an initial velocity of $\hat{i} + 2\hat{j}$. Find the cartesian equation of its path. (g = 10 m s^{-2})

Soln. : $\tan\theta = \frac{u\sin\theta}{u\cos\theta} = \frac{2}{1}$

The desired equation is

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta} = x \times 2 - \frac{10x^2}{2\left(\sqrt{(1)^2+(2)^2}\right)^2\left(\frac{1}{\sqrt{5}}\right)^2}$$

or $y = 2x - 5x^2$

Illustration 22

Two bodies are thrown with the same initial speed at angles α and $(90° - \alpha)$ with the horizontal. What will be the ratio of (i) maximum heights attained by them and (ii) horizontal ranges?

Soln. : Horizontal range, $R = \frac{u^2}{g}\sin 2\theta$

and maximum Height, $H = \frac{u^2\sin^2\theta}{2g}$

case (i) when $\theta = \alpha$; $R_1 = \frac{u^2}{g}\sin 2\alpha$ and $H_1 = \frac{u^2\sin^2\alpha}{2g}$

case (ii) when $\theta = (90° - \alpha)$;

$$R_2 = \frac{u^2\sin 2(90° - \alpha)}{g} = \frac{u^2\sin(180° - 2\alpha)}{g} = \frac{u^2\sin 2\alpha}{g}$$

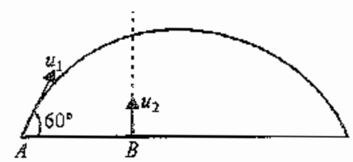
and $H_2 = \frac{u^2\sin^2(90° - \alpha)}{2g} = \frac{u^2\cos^2\alpha}{2g}$

$$\therefore \quad \frac{H_1}{H_2} = \frac{\sin^2\alpha}{\cos^2\alpha} = \tan^2\alpha \text{ and } \frac{R_1}{R_2} = 1$$

Illustration 23

A body is projected with the velocity u_1 from the point A as shown in figure. At the same time another body is projected vertically upwards with the velocity u_2 from the point B. What should be the value of $\frac{u_1}{u_2}$ for both the bodies to collide?

Soln. : The two bodies will collide, if they reach at a point acquiring the same vertical distance in the same time.



$$y = u_1\sin 60° \times t - \frac{1}{2}gt^2 = u_2 t - \frac{1}{2}gt^2$$

$$\text{or} \quad u_1 \times \frac{\sqrt{3}}{2} \times t = u_2 t \quad \text{or} \quad \frac{u_1}{u_2} = \frac{2}{\sqrt{3}}$$