UNIT 5

Motion of System of Particles and Rigid Body

RIGID BODY

- Ideally, a rigid body is one for which the distances between different particles of the body do not change, even though there are forces acting on them.
- In pure translatory motion, every particle of the body moves with the same velocity at any instant of time.
- **Axis of rotation :** The line along which the body is fixed is known as axis of rotation. Examples of rotation about an axis is a ceiling fan and a merry-go-round etc.
- In pure rotation of a rigid body about a fixed axis, every particle of the rigid body moves in a circle which lies in a plane perpendicular to the axis and has its centre on the axis. Every point in the rotating rigid body has the same angular velocity at any instant of time.

CENTRE OF MASS

- The centre of mass of a body is a point where the entire mass of the body can be supposed to be concentrated. In fact, nature of motion executed by the body shall remain unaffected if all the forces acting on the body were applied directly at this point.
- For a **system of two particles** of masses m_1 and m_2 having their position vectors as $\vec{r}_1$ and $\vec{r}_2$ respectively, with respect to origin of the coordinate system, the position vector $\vec{R}_{CM}$ of the centre of mass is given by

$$\vec{R}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

If $m_1 = m_2 = m$ (say), then $\vec{R}_{CM} = \frac{\vec{r}_1 + \vec{r}_2}{2}$

Thus, the centre of mass of two equal masses lies exactly at the centre of the line joining the two masses.

- For a **system of N-particles** of masses $m_1, m_2, m_3 ... m_N$ having their position vectors as $\vec{r}_1, \vec{r}_2, \vec{r}_3, ..., \vec{r}_N$ respectively, with respect to the origin of the coordinate system, the position vector $\vec{R}_{CM}$ of the centre of mass is given by

$$\vec{R}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + ... + m_N\vec{r}_N}{m_1 + m_2 + ... + m_N} = \frac{\sum_{i=1}^{N} m_i\vec{r}_i}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i\vec{r}_i}{M}$$

- The coordinates of centre of mass are given by

$$X_{CM} = \frac{\sum_{i=1}^{N} m_i x_i}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i x_i}{M}$$

$$Y_{CM} = \frac{\sum_{i=1}^{N} m_i y_i}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i y_i}{M}$$

$$Z_{CM} = \frac{\sum_{i=1}^{N} m_i z_i}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i z_i}{M}$$

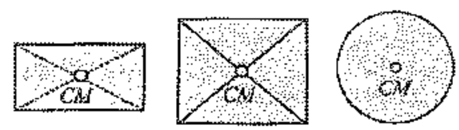
- For a **continuous distribution of mass,** the coordinates of centre of mass are given by

$$X_{CM} = \frac{1}{M}\int x\,dm;\ Y_{CM} = \frac{1}{M}\int y\,dm;\ Z_{CM} = \frac{1}{M}\int z\,dm$$

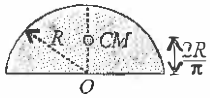
- The position of the centre of mass of a system is independent of the choice of coordinate system.
- The position of the centre of mass depends on the shape of the body and the distribution of its mass. Hence it may lie within or outside the material of the body.
- In symmetrical bodies in which the distribution of mass is homogeneous, the centre of mass coincides with the centre of symmetry or geometrical centre.
- The centre of mass changes its position only under the translatory motion but remains unchanged in rotatory motion.

Centre of mass of some well known rigid bodies are given below :

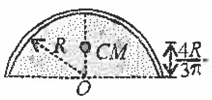
- Centre of mass of a uniform rectangular, square or circular plate lies at its centre as shown in the figure.



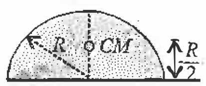
- Centre of mass of a uniform semicircular ring of radius R lies at a distance of $h = \frac{2R}{\pi}$ from its centre, on the axis of symmetry as shown in the figure



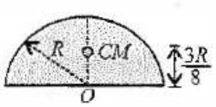
- Centre of mass of a uniform semicircular disc of radius R lies at a distance of $h = \frac{4R}{3\pi}$ from the centre on the axis of symmetry as shown in figure.



- Centre of mass of a hemispherical shell of radius R lies at a distance of $h = \frac{R}{2}$ from its centre on the axis of symmetry as shown in figure.



- Centre of mass of a solid hemisphere of radius R lies at a distance of $h = \frac{3R}{8}$ from its centre on the axis of symmetry as shown in the figure



- **Velocity of centre of mass**

$$\vec{v}_{CM} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + ... + m_N\vec{v}_N}{m_1 + m_2 + ... + m_N} = \frac{\sum_{i=1}^{N} m_i\vec{v}_i}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i\vec{v}_i}{M}$$

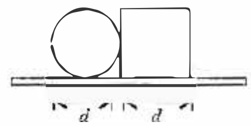
- **Acceleration of centre of mass**

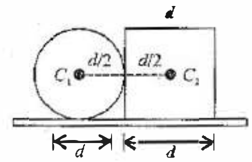
$$\vec{a}_{CM} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + ... + m_N\vec{a}_N}{m_1 + m_2 + ... + m_N} = \frac{\sum_{i=1}^{N} m_i\vec{a}_i}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i\vec{a}_i}{M}$$

- If total external force acting on the system is zero, then the total linear momentum of the system is conserved. Also, when the total external force acting on the system is zero, the velocity of centre of mass remains constant.

Illustration 1

A circular plate of diameter d is kept in contact with a square plate of edge d as shown in figure. The density of the material and the thickness are same everywhere. Show that the centre of mass of the composite system lies inside the square plate.





Soln. The centre of masses of the circular plate and the square are at their centres namely C_1 and C_2 respectively. Now the masses of the plates are proportional to their areas (ρ = density, t = thickness).

$$m_1 = \rho \cdot t \cdot \left(\frac{\pi d^2}{4}\right); \quad m_2 = \rho \cdot t \cdot d^2$$

With C_1 as the origin

$$x_{CM} = \frac{m_1(0) + m_2(d)}{m_1 + m_2} = \frac{\rho \cdot t \cdot d^2 \cdot d}{\rho t\left(\frac{\pi d^2}{4} + d^2\right)} = \frac{d}{\left(\frac{\pi}{4} + 1\right)}$$

But $1 + \frac{\pi}{4} = 1 + \frac{3.14}{4} < 2 \Rightarrow \frac{1}{1 + \frac{\pi}{4}} > \frac{1}{2}$

or $\frac{d}{1 + \frac{\pi}{4}} > \frac{d}{2}$

Thus the centre of mass of the system lies inside the square plate.

Illustration 2

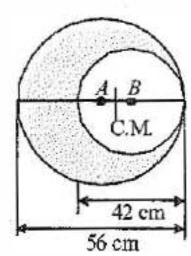
Two particles A and B, initially at rest move towards each other under mutual force of attraction. At the instant when the velocity of A is V and the velocity of B is $2V$, show that the velocity of the centre of mass of the system is zero.

Soln. : Net force acting on the system is zero. Hence the centre of mass continues to be at rest.

$\therefore$ Velocity of centre of mass = zero.

Illustration 3

A circular plate of uniform thickness has a diameter of 56 cm. A circular portion of diameter 42 cm is removed from one edge of the plate as shown in the figure. Find the position of the centre of mass of the remaining portion.



Soln. : Let r_1 be the distance of the centre of mass of remaining portion from centre of the bigger circle. Remaining area

$$A_1 = \frac{\pi[(56)^2 - (42)^2]}{4}$$

$$\therefore \quad A_1 r_1 = A_2 r_2 \quad \text{or } r_1 = \left(\frac{A_2}{A_1}\right) r_2$$

or $r_1 = \dfrac{\pi(42)^2 \times 4}{\pi[(56)^2 - (42)^2] \times 4} \times \dfrac{7}{1}$

or $r_1 = \dfrac{42 \times 42 \times 7}{98 \times 14} = 9$ cm

$\therefore$ $r_1 = 9$ cm = Required distance of C.M.of remaining part.

Illustration 4

Three identical spheres, each of mass M, are placed at the corners of a right angled triangle with mutually perpendicular sides equal to 2 m. Taking their point of intersection as the origin, find the position vector of centre of mass.

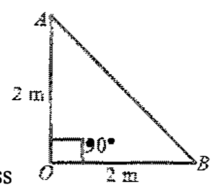
Soln.: $x = \dfrac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$

$\therefore$ $x = \dfrac{(M \times 0) + (M \times 0) + (M \times 2)}{M + M + M} = \dfrac{2M}{3M} = \dfrac{2}{3}$

$\therefore$ x-coordinate $= 2/3\ \hat{i}$

Again $y = \dfrac{(M \times 2) + (M \times 0) + (M \times 0)}{M + M + M}$

or $y = \dfrac{2M}{3M} = \dfrac{2}{3}$



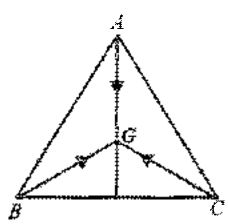
$\therefore$ y-coordinate $= 2/3\ \hat{j}$

$\therefore$ Position vector of centre of mass

$= \dfrac{2}{3}\hat{i} + \dfrac{2}{3}\hat{j} = \dfrac{2}{3}(\hat{i} + \hat{j})$

Illustration 5

Three particles A, B and C of equal mass move with equal speed V along the medians of an equilateral triangle as shown in figure.



They collide at the centroid G of the triangle. After the collision, A comes to rest, B retraces its path with the speed V. What is the velocity of C?

Soln. : Before collision, net momentum of the system is zero. The three vectors meeting at a point G, are in equilibrium.

Hence the momentum after collision should also be zero. A comes to rest. Its momentum is zero.

$\therefore$ Momentum of B + momentum of $C = 0$

$\therefore$ $MV + MV' = 0$ or $V' = -V$

$\therefore$ velocity of $C = -V$

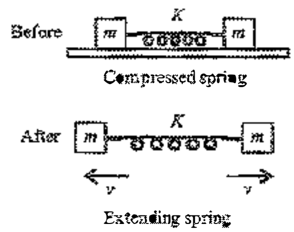
The velocity of C is in the opposite direction of velocity B.

Illustration 6

Show that internal forces can change the kinetic energy but not linear momentum.

Soln. : $\vec{F}_{ext} = M\vec{a}_{CM} \Rightarrow \vec{F}_{ext} = \dfrac{Md\vec{v}_{CM}}{dt}$

$\vec{F}_{ext} = \dfrac{d}{dt}\left(M\vec{v}_{CM}\right)$



Only external forces can change the momentum of a system. Internal forces can give rise to kinetic energy. A compressed spring attached to two blocks, converts its potential energy
into kinetic energy. This K.E. is due to internal forces.

ANGULAR VELOCITY AND ANGULAR ACCELERATION

- **Angular velocity :** It is defined as the time rate of change of angular displacement and is given by

  $\omega = \dfrac{d\theta}{dt}$

- Angular velocity is directed along the axis of rotation. Angular velocity is a vector quantity. Its SI unit is rad/s and its dimensional formula is $[M^0L^0T^{-1}]$.

- **Relationship between linear velocity and angular velocity**

  The linear velocity of a particle of a rigid body rotating about a fixed axis is given by

  $\vec{v} = \vec{\omega} \times \vec{r}$

  where $\vec{r}$ is the position vector of the particle with respect to an origin along the fixed axis.

- As in pure translational motion, all particles of the body have the same linear velocity at any instant of time. Similarly, in pure rotational motion, all particles of the body have the same angular velocity at any instant of time.

- For rotation about a fixed axis, the direction of the angular velocity $\vec{\omega}$ does not change with time. Its magnitude, however may change from instant to instant. In general rotation, both the magnitude and direction of ω may change from instant to instant.

- **Angular acceleration :** It is defined as the time rate of change of angular velocity and it is given by

  $\vec{\alpha} = \dfrac{d\vec{\omega}}{dt}$

- Angular acceleration is a vector quantity. Its SI unit is rad s^{-2} and its dimensional formula is $[M^0L^0T^{-2}]$

- **Equations of rotational motion**
  - $\omega = \omega_0 + \alpha t$
  - $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
  - $\omega^2 - \omega_0^2 = 2\alpha\theta$

  where the symbols have their usual meaning.

  These equations are valid for uniform angular acceleration.

Illustration 7

What is the value of linear velocity $\vec{v}$ if $\vec{\omega} = 3\hat{i} - 4\hat{j} + \hat{k}$ and $\vec{r} = 5\hat{i} - 6\hat{j} + 6\hat{k}$?

Soln. : $\vec{v} = \vec{\omega} \times \vec{r}$
or

$$\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -4 & 1 \\ 5 & -6 & 6 \end{vmatrix} = \hat{i}(-24+6) - \hat{j}(18-5) + \hat{k}(-18+20)$$

$$\vec{v} = -18\hat{i} - 13\hat{j} + 2\hat{k}.$$

Illustration 8

In an electric clock, the extremity of the hour hand moves one twentieth as fast as that of the minute hand. What is the length of the hour hand if the minute hand is 10 cm long?

Soln.: The angular speed of the hour hand,

$$\omega_H = \frac{2\pi}{3600 \times 12} \text{ rad/sec}$$

The angular speed of the minute hand,

$$\omega_M = \frac{2\pi}{3600} \text{ rad/sec.}$$

Let r_H be the length of the hour hand.

$\therefore$ The linear speed of the extremity of the hour hand is given by $v = r\omega$

$$\therefore \quad v_H = r_H \omega_H \quad \text{...(i)}$$

Similarly, for the minute hand,

$$v_M = r_M \omega_M \quad \text{...(ii)}$$

Now, by problem,

$$v_H = \frac{1}{20} v_M \quad \text{or} \quad r_H \omega_H = \frac{1}{20} r_M \omega_M$$

$$\therefore \quad r_H \times \frac{2\pi}{3600 \times 12} = \frac{1}{20} \times 10 \times \frac{2\pi}{3600} \text{ or } r_H = 6 \text{ cm}$$

MOMENT OF INERTIA

- Moment of inertia of a rigid body about a given axis of rotation is defined as the sum of the products of masses of all the particles of the body and square of their respective perpendicular distances from the axis of rotation.

  It is denoted by symbol I and is given by

  $$I = \sum_{i=1}^{N} m_i r_i^2$$

- Moment of inertia is a scalar quantity. Its SI unit is kg m^2 and its dimensional formula is $[ML^2T^0]$. It depends upon
  - position of the axis of rotation
  - orientation of the axis of rotation
  - shape of the body
  - size of the body
  - distribution of mass of the body about the axis of rotation.
- **Radius of gyration :** It is defined as the distance from the axis of rotation at which, if whole mass of the body were concentrated, the moment of inertia of the body would be same as with the actual distribution of the mass of body. It is denoted by symbol K.
- Radius of gyration of a body about an axis of rotation may also be defined as the root mean square distance of the particles from the axis of rotation.

  $$i.e., \; K = \sqrt{\frac{r_1^2 + r_2^2 + \ldots + r_N^2}{N}}$$

- The moment of inertia of a body about a given axis is equal to the product of mass of the body and square of its radius of gyration about that axis.

  i.e., $I = MK^2$.
- The SI unit of radius of gyration is metre and its dimensional formula is $[M^0LT^0]$
- **Theorem of perpendicular axes :** The moment of inertia of a planar lamina about an axis perpendicular to its plane is equal to the sum of its moments of inertia about two perpendicular axes concurrent with perpendicular axis and lying in the plane of the body.

  $$I_z = I_x + I_y$$

  where x and y are two perpendicular axes in the plane and z axis is perpendicular to its plane.
- **Theorem of parallel axes :** The moment of inertia of a body about any axis is equal to the sum of the moment of inertia of the body about a parallel axis passing through its centre of mass and the product of its mass and the square of the distance between the two parallel axes.

  $$I = I_{CM} + Md^2$$

  where I_{CM} is the moment of inertia of the body about an axis passing through the centre of mass and d is the perpendicular distance between two parallel axes.