- **Equations of rotational motion**
  - $\omega = \omega_0 + \alpha t$
  - $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
  - $\omega^2 - \omega_0^2 = 2\alpha\theta$

  where the symbols have their usual meaning.

  These equations are valid for uniform angular acceleration.

Illustration 7

What is the value of linear velocity $\vec{v}$ if $\vec{\omega} = 3\hat{i} - 4\hat{j} + \hat{k}$ and $\vec{r} = 5\hat{i} - 6\hat{j} + 6\hat{k}$?

Soln. : $\vec{v} = \vec{\omega} \times \vec{r}$
or

$$\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -4 & 1 \\ 5 & -6 & 6 \end{vmatrix} = \hat{i}(-24+6) - \hat{j}(18-5) + \hat{k}(-18+20)$$

$$\vec{v} = -18\hat{i} - 13\hat{j} + 2\hat{k}.$$

Illustration 8

In an electric clock, the extremity of the hour hand moves one twentieth as fast as that of the minute hand. What is the length of the hour hand if the minute hand is 10 cm long?

Soln.: The angular speed of the hour hand,

$$\omega_H = \frac{2\pi}{3600 \times 12} \text{ rad/sec}$$

The angular speed of the minute hand,

$$\omega_M = \frac{2\pi}{3600} \text{ rad/sec.}$$

Let r_H be the length of the hour hand.

$\therefore$ The linear speed of the extremity of the hour hand is given by $v = r\omega$

$$\therefore \quad v_H = r_H \omega_H \qquad \text{...(i)}$$

Similarly, for the minute hand,

$$v_M = r_M \omega_M \qquad \text{...(ii)}$$

Now, by problem,

$$v_H = \frac{1}{20} v_M \text{ or } r_H \omega_H = \frac{1}{20} r_M \omega_M$$

$$\therefore \quad r_H \times \frac{2\pi}{3600 \times 12} = \frac{1}{20} \times 10 \times \frac{2\pi}{3600} \text{ or } r_H = 6 \text{ cm}$$

MOMENT OF INERTIA

- Moment of inertia of a rigid body about a given axis of rotation is defined as the sum of the products of masses of all the particles of the body and square of their respective perpendicular distances from the axis of rotation.

  It is denoted by symbol I and is given by

  $$I = \sum_{i=1}^{N} m_i r_i^2$$

- Moment of inertia is a scalar quantity. Its SI unit is kg m^2 and its dimensional formula is $[ML^2T^0]$. It depends upon
  - position of the axis of rotation
  - orientation of the axis of rotation
  - shape of the body
  - size of the body
  - distribution of mass of the body about the axis of rotation.
- **Radius of gyration** : It is defined as the distance from the axis of rotation at which, if whole mass of the body were concentrated, the moment of inertia of the body would be same as with the actual distribution of the mass of body. It is denoted by symbol K.
- Radius of gyration of a body about an axis of rotation may also be defined as the root mean square distance of the particles from the axis of rotation.

  $$i.e., \; K = \sqrt{\frac{r_1^2 + r_2^2 + \ldots + r_N^2}{N}}$$

- The moment of inertia of a body about a given axis is equal to the product of mass of the body and square of its radius of gyration about that axis.

  i.e., $I = MK^2$.
- The SI unit of radius of gyration is metre and its dimensional formula is $[M^0LT^0]$
- **Theorem of perpendicular axes :** The moment of inertia of a planar lamina about an axis perpendicular to its plane is equal to the sum of its moments of inertia about two perpendicular axes concurrent with perpendicular axis and lying in the plane of the body.

  $$I_z = I_x + I_y$$

  where x and y are two perpendicular axes in the plane and z axis is perpendicular to its plane.
- **Theorem of parallel axes :** The moment of inertia of a body about any axis is equal to the sum of the moment of inertia of the body about a parallel axis passing through its centre of mass and the product of its mass and the square of the distance between the two parallel axes.

  $$I = I_{CM} + Md^2$$

  where I_{CM} is the moment of inertia of the body about an axis passing through the centre of mass and d is the perpendicular distance between two parallel axes.

Moment of inertia and radius of gyration of some regular bodies about specific axis is given below.

S. No.	Body	Axis of rotation	Moment of inertia (I)	Radius of gyration (K)
1	Uniform circular ring of mass M and radius R	(i) about an axis passing through its centre and perpendicular to its plane	MR^2	R
		(ii) about its diameter	$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$
		(iii) about a tangent in its own plane	$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$
		(iv) about a tangent perpendicular to its plane	$2MR^2$	$R\sqrt{2}$
2	Uniform circular disc of mass M and radius R	(i) about an axis passing through its centre and perpendicular to its plane	$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$
		(ii) about its diameter	$\frac{1}{4}MR^2$	$\frac{R}{2}$
		(iii) about a tangent in its own plane	$\frac{5}{4}MR^2$	$\sqrt{5}\frac{R}{2}$
		(iv) about a tangent perpendicular to its plane	$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$
3	Solid sphere of radius R and mass M	(i) about its diameter	$\frac{2}{5}MR^2$	$\sqrt{\frac{2}{5}}R$
		(ii) about a tangential axis	$\frac{7}{5}MR^2$	$\sqrt{\frac{7}{5}}R$
4	Hollow sphere of radius R and mass M	(i) about its diameter	$\frac{2}{3}MR^2$	$\sqrt{\frac{2}{3}}R$
		(ii) about a tangential axis	$\frac{5}{3}MR^2$	$\sqrt{\frac{5}{3}}R$
5	Solid cylinder of length l, radius R and mass M	(i) about its own axis	$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$
		(ii) about an axis passing through its centre and perpendicular to its own axis	$M\left[\frac{l^2}{12}+\frac{R^2}{4}\right]$	$\sqrt{\frac{l^2}{12}+\frac{R^2}{4}}$
		(iii) about the diameter of one of the faces of cylinder	$M\left[\frac{l^2}{3}+\frac{R^2}{4}\right]$	$\sqrt{\frac{l^2}{3}+\frac{R^2}{4}}$
6	Hollow cylinder of mass M, length l and radius R	(i) about its own axis	MR^2	R
		(ii) about an axis passing through its centre and perpendicular to its own axis	$M\left(\frac{R^2}{2}+\frac{l^2}{12}\right)$	$\sqrt{\frac{R^2}{2}+\frac{l^2}{12}}$
7	Thin rod of length L	(i) about an axis passing through its centre and perpendicular to the rod	$\frac{ML^2}{12}$	$\frac{L}{\sqrt{12}}$
		(ii) about an axis passing through one end and perpendicular to the rod	$\frac{ML^2}{3}$	$\frac{L}{\sqrt{3}}$
8	Rectangular lamina of length l and breadth b	(i) about an axis passing through its centre and perpendicular to its plane	$M\left[\frac{l^2+b^2}{12}\right]$	$\sqrt{\frac{l^2+b^2}{12}}$
9	Uniform cone of radius R and height h	(ii) about an axis through its centre and joining its vertex to centre of base	$\frac{3}{10}MR^2$	$R\sqrt{\frac{3}{10}}$
10	Parallelepiped of length l, breadth b and height h, mass M	(iii) about its central axis	$M\left(\frac{l^2+b^2}{12}\right)$	$\sqrt{\frac{l^2+b^2}{12}}$

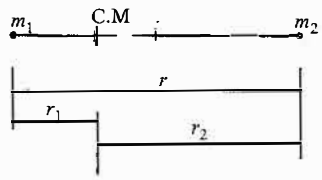
Illustration 9

Calculate the moment of inertia of a diatomic molecule about an axis passing through its centre of mass and perpendicular to the line joining the two atoms.

Given: The two masses are m_1 and m_2 separated by a distance r.

Soln. : $m_1r_1 = m_2(r_2) = m_2(r - r_1)$

$\therefore \quad m_1r_1 = m_2r - m_2r_1$



or $\quad r_1 = \dfrac{m_2r}{(m_1 + m_2)}$

Now $r_2 = r - r_1 = r - \dfrac{m_2r}{(m_1 + m_2)}$

$r_2 = \dfrac{m_1r}{(m_1 + m_2)} \quad \therefore \; I = m_1r_1^2 + m_2r_2^2$

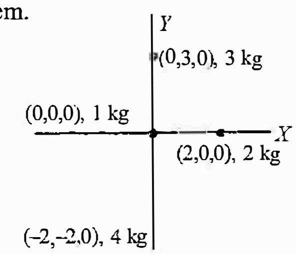
or $\quad I = \dfrac{m_1 \times (m_2r)^2}{(m_1 + m_2)^2} + \dfrac{m_2 \times (m_1r)^2}{(m_1 + m_2)^2}$

or $\quad I = \dfrac{m_1m_2r^2(m_2 + m_1)}{(m_1 + m_2)^2} \quad$ or $\quad I = \dfrac{m_1m_2r^2}{(m_1 + m_2)}$

or $\quad I = \mu r^2$, where $\mu = \dfrac{m_1m_2}{(m_1 + m_2)}$

= reduced mass of system.

Illustration 10



Point masses 1, 2, 3 and 4 kg are lying at the point (0, 0, 0), (2, 0, 0), (0, 3, 0) and (−2, −2, 0) respectively. Find the moment of inertia of this system about X-axis.

Soln. : It is evident from the question that the Z-coordinates of all the four particles are zero. Hence they lie in X - Y plane.

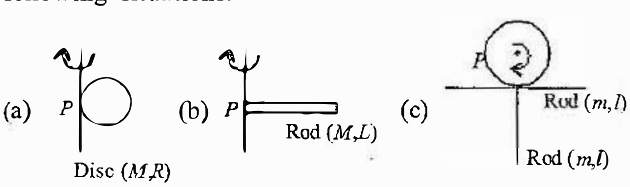
$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + m_4r_4^2$

$= (1 \times 0) + (2 \times 0) + (3 \times 3^2) + (4 \times 2^2)$

$= 27 + 16 = 43$ kgm^2.

Illustration 11

Find the moment of inertia about the axis shown in the following situations.

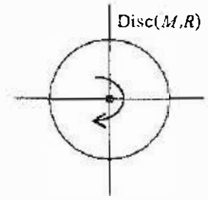


Soln.: (a) $I_z = \dfrac{MR^2}{2}$

Using perpendicular axes theorem

$I_x + I_y = I_z$

But $I_x = I_y$ from symmetry,

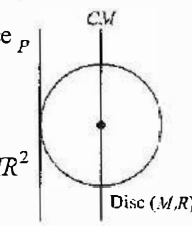


$\Rightarrow \quad 2I_y = I_z = \dfrac{MR^2}{2}$

$I_y = \dfrac{MR^2}{4}$

Now,

The I_y is through CM of the disc, hence

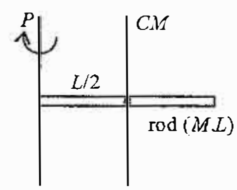


$I_P = I_{cm} + Md^2$

(using parallel axes theorem)

$I_P = \dfrac{MR^2}{4} + M(R)^2 \Rightarrow I_P = \dfrac{5}{4}MR^2$

(b) For the rod, $I_{cm} = \dfrac{ML^2}{12}$



$I_P = I_{cm} + Md^2$

(using parallel axes theorem)

$I_P = \dfrac{ML^2}{12} + M\left(\dfrac{L}{2}\right)^2$

$= \dfrac{ML^2}{12} + \dfrac{ML^2}{4} = \dfrac{(1+3)ML^2}{12}$

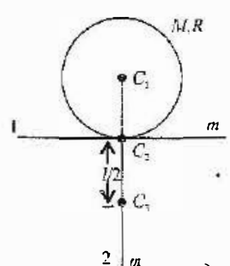
$\Rightarrow \quad I_P = \dfrac{ML^2}{3}$

(c) About C_1

$I_{ring} = MR^2$

$I_{rod\ 1} = I_{C_2} + M(R)^2$

(As C_2 is the centre of mass of rod 1 and the distance between the C_1 and C_2 is $d = R$).]



$\Rightarrow I_{Rod\ 1} = \left(\dfrac{ml^2}{12} + mR^2\right)$

Similarly, $I_{Rod\ 2} = I_{C_2} + m\left(R + \dfrac{l}{2}\right)^2$

(As the centre of mass of rod 2 is C_3 and it is $d = \left(R + \dfrac{l}{2}\right)$ away from C_1).

$\Rightarrow \quad I_{rod\ 2} = \dfrac{ml^2}{12} + m\left(R + \dfrac{l}{2}\right)^2$

Thus, $I_{system} = I_{ring} + I_{rod\ 1} + I_{rod\ 2}$

$= (MR^2) + \left(\dfrac{ml^2}{12} + mR^2\right) + \left(\dfrac{ml^2}{12} + m\left(R + \dfrac{l}{2}\right)^2\right)$

TORQUE

- The torque or moment of force is a measure of the turning effect of force about the axis of rotation. It is denoted by the symbol $\vec{\tau}$.

  Torque $\vec{\tau} = \vec{r} \times \vec{F}$

  In magnitude, $\tau = rF \sin\theta$

  where θ is the angle between r and F.