UNIT 6

Gravitation

KEPLER'S LAWS OF PLANETARY MOTION

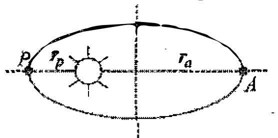
Kepler's three laws of planetary motion are as follows :

- **Kepler's first law (law of orbits) :** All planets move in elliptical orbits with the sun situated at one of the foci of the ellipse.
- **Kepler's second law (law of areas) :** The radius vector drawn from the sun to the planet sweeps out equal areas in equal intervals of time *i.e.* the areal velocity of the planet (or the area swept out by the planet per unit time) around the sun is constant *i.e.* areal velocity $=\frac{d\vec{A}}{dt}=$ a constant, for a planet.
  - Angular momentum $(\vec{L})$ of a planet is related with areal velocity $\left(\frac{d\vec{A}}{dt}\right)$ by the relation

    $$\vec{L}=2m\left(\frac{d\vec{A}}{dt}\right)$$
  - Kepler's second law follows from the law of conservation of angular momentum.
  - The area covered by the radius vector in dt seconds $=\frac{1}{2}r^2\,d\theta$.
  - The areal velocity $=\frac{1}{2}r^2\frac{d\theta}{dt}=\frac{1}{2}r^2\omega=\frac{1}{2}rv.$
  - According to Kepler's second law, the speed of the planet is maximum, when it is closest to the sun and is minimum when the planet is farthest from the sun.
- **Kepler's third law (Law of periods) :** The square of the time period of revolution of a planet around the sun is directly proportional to the cube of semi major axis of the elliptical orbit *i.e.* $T^2 \propto a^3$ where a is the semi major axis of the elliptical orbit of the planet around the sun.

Illustration 1

Let the speed of the planet at the perihelion P be v_p and the sun-planet distance be r_p. Relate (r_p, v_p) to the corresponding quantities at the aphelion (r_a, v_a)



(a) $\frac{r_p}{r_a}=\frac{v_p}{v_a}$ (b) $\frac{r_p^2}{r_a^2}=\frac{v_a}{v_p}$

(c) $\frac{v_p}{v_a}=\frac{r_a}{r_p}$ (d) $\frac{v_p^2}{v_a^2}=\frac{r_a}{r_p}$

Soln. (a) : At the perihelion r_p and v_p are mutually perpendicular so r_a is normal to v_a. The angular momentum is conserved,

$$\therefore\ m\,v_a r_a = m v_p r_p \Rightarrow \frac{v_p}{v_a}=\frac{r_a}{r_p}$$

Illustration 2

The rotation period of an Earth satellite close to the surface of the Earth is 83 minutes. Find the time period of another earth satellite in an orbit at a distance of three earth radii from its surface

Soln. : $T^2 \propto R^3$

$$\therefore\ \left(\frac{T_2}{T_1}\right)^2=\left(\frac{R_2}{R_1}\right)^3 \Rightarrow \left(\frac{T_2}{83\text{ min}}\right)^2=\left(\frac{R+3R}{R}\right)^3$$

$\Rightarrow$ $T_2 = 664$ min.

Illustration 3

A planet of mass m moves around the sun of mass M in an elliptical orbit. The maximum and minimum distances of the planet from the Sun are r_1 and r_2 respectively. Find the time period of the planet.

Soln. : According to Kepler's 3rd law, $T^2 \propto a^3$

where a is the semi-major axis, Here,

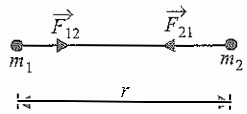
$$a=\left(\frac{r_1+r_2}{2}\right) \Rightarrow T^2 \propto \left(\frac{r_1+r_2}{2}\right)^3$$

or $T \propto (r_1+r_2)^{\frac{3}{2}}$

NEWTON'S LAW OF GRAVITATION

- According to Newton's law of gravitation every body in the universe attracts every other body with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. The direction of this force is along the line joining the two bodies.
- The magnitude of gravitational force acting between two bodies of masses m_1 and m_2 placed distance r apart is

$|\vec{F}_{12}| = |\vec{F}_{21}| = \frac{Gm_1m_2}{r^2}$



Where $\vec{F}_{12} = -\vec{F}_{21}$ and G is the universal gravitational constant.

- It is a universal law.
- Universal gravitational constant G is a scalar quantity. Its value is same throughout the universe and is independent of the nature and size of the bodies as well as the nature of the medium between the bodies.
- The value of G in

  SI system is 6.67×10^{-11} N m^2 kg^{-2} and in

  CGS system is 6.67×10^{-8} dyne cm^2 g^{-2}.
- The dimensional formula for G is $[M^{-1}L^3T^{-2}]$.
- The value of universal gravitational constant G was determined experimentally, first of all, by English scientist **Henry Cavendish** in 1798.
- **Newton's law of gravitation in vector form is**

  $$\vec{F} = -\frac{Gm_1m_2}{r^2}\cdot\hat{r}$$

  –ve sign shows that gravitational force is always attractive.

Characteristics of Gravitational Force

- The characteristics of gravitational force are as follows :
  - It is always attractive in nature.
  - It obeys inverse square law.
  - It is independent of nature of intervening medium.
  - It is independent of the presence or absence of other bodies.
  - It is independent of nature and size of the bodies, till their masses remain the same and the distance between their centres is fixed.
  - It holds good over a wide range of distances. It is found true from interplanetary distances to interatomic distances.
  - It is an action-reaction pair.
  - It is a conservative force as work done by it is path independent.
  - It is a central force. It acts along the line joining the centre of the two bodies.

Principle of Superposition of Gravitation

- It states that the resultant gravitational force $\vec{F}$ acting on a particle due to number of other particles is equal to vector sum of the gravitational forces exerted by individual particles on the given particle.

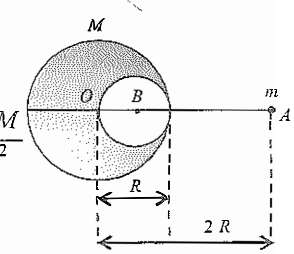
  i.e., $\vec{F} = \vec{F}_{01} + \vec{F}_{02} + \vec{F}_{03} + ... + \vec{F}_{0n} = \sum_{i=1}^{n} \vec{F}_{0i}$

  where $\vec{F}_{01}, \vec{F}_{02}, \vec{F}_{03}, ..., \vec{F}_{0n}$ are the gravitational forces on a particle of mass m_0 due to particles of masses m_1, m_2, ..., m_n respectively.

Illustration 4

A solid sphere of radius (R) gravitationally attracts a particle placed at $(2R)$ from its centre with a force F_1. Now a spherical cavity of radius $(R/2)$ is made in the sphere, as shown in figure and the force becomes F_2. Calculate (F_2/F_1).

Soln. : $F_1 = \frac{GmM}{(2R)^2} = \frac{GmM}{4R^2}$

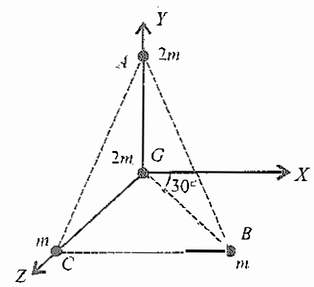


$$F_2 = \frac{GmM}{(2R)^2} - \frac{G(M/8)m}{(3R/2)^2} = \frac{7GmM}{36R^2}$$

$$\therefore \frac{F_2}{F_1} = \frac{7GmM}{36R^2} \times \frac{4R^2}{GmM} = \frac{7}{9}$$

Illustration 5

Masses $2m$, m, m are kept at the vertices of an equilateral triangle of side 1 m, as shown. Another mass 2 m is kept at their centroid. What is the force on the 2 m at the centroid due to other?



Soln. : $\vec{F}_{GA} = \frac{G(2m)(2m)}{1^2}\hat{j}$,

$$\vec{F}_{GB} = \frac{G(m)(2m)}{1^2}(-\hat{i}\cos 30° - \hat{j}\sin 30°)$$

$$\vec{F}_{GC} = \frac{G(m)(2m)}{1^2}(+\hat{i}\cos 30° - \hat{j}\sin 30°)$$

$$\vec{F}_G = \vec{F}_{GA} + \vec{F}_{GB} + \vec{F}_{GC} = 4Gm^2\hat{j} - 2Gm^2(2\sin 30°)\hat{j}$$

$$= 2Gm^2\hat{j}.$$

GRAVITY

- It is defined as the force of attraction exerted by the earth towards its centre on a body lying on or near the surface of the earth.
- It is merely a special case of gravitation and is also called as earth's gravitational pull.
- It is the measure of weight of the body. The weight of the body

  = mass (m) × acceleration due to gravity $(g) = mg$.

  The unit of weight of the body will be the same as those of force.
- It is a vector quantity. It is always directed towards the centre of the earth.

Acceleration due to Gravity

- Acceleration produced in a body due to the force of gravity is called as acceleration due to gravity. It is denoted by symbol 'g' and mathematically it is given by

  $$g = \frac{GM_e}{r^2}$$

  where M_e is the mass of the earth and r is the distance of the body from the centre of the earth.
- If the body is on the surface of the earth *i.e.* $r = R_e$ (radius of the earth), then

  $$g = \frac{GM_e}{R_e^2}$$

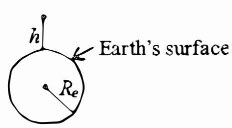
- It is also defined as the acceleration produced in the body while falling freely under the effect of gravity alone.
- It is a vector quantity. Its SI unit is m s^{-2} and its dimensional formula is $[M^0LT^{-2}]$.
- The value of g on the surface of earth is taken to be 9.8 m s^{-2}.
- The value of g varies with altitude, depth, shape and the rotation of earth.
- The acceleration due to gravity (g) is related with gravitational constant (G) by the relation

$$g=\frac{GM_e}{R_e^2}=\frac{G\frac{4}{3}\pi R_e^3\rho}{R_e^2}=\frac{4}{3}\pi GR_e\rho$$

where M_e is the mass of the earth, R_e is the radius of the earth and ρ is the uniform density of the material of the earth.
- The value of acceleration due to gravity is independent of the shape, size, mass etc. of the body but depends upon mass and radius of the earth or planet due to which there is a gravity pull.
- If the radius of a planet decreases by $n\%$, keeping its mass unchanged, the acceleration due to gravity on its surface increases by $2n\%$.
- If the mass of a planet increases by $n\%$, keeping its radius unchanged, the acceleration due to gravity on its surface increases by $n\%$.
- If the density of a planet decreases by $x\%$, keeping its radius unchanged, the acceleration due to gravity decreases by $x\%$.

Variation of Acceleration due to Gravity

- **Due to altitude (h) :** The acceleration due to gravity at height h above the earth's surface is given by



$$g_h=\frac{GM_e}{(R_e+h)^2}=g\left(1+\frac{h}{R_e}\right)^{-2}\quad\left(\because g=\frac{GM_e}{R_e^2}\right)$$

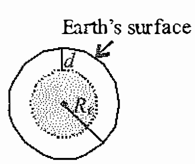
For $h << R_e$

$$\therefore\quad g_h=g\left(1-\frac{2h}{R_e}\right)$$

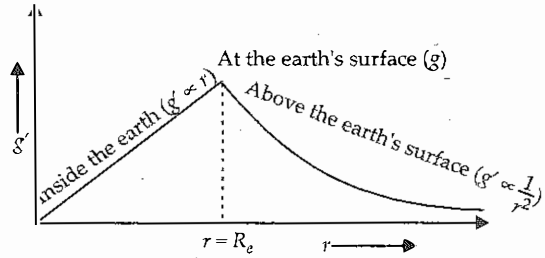
When we move above the earth's surface, the value of acceleration due to gravity goes on decreasing.
- **Due to depth (d) :** The acceleration due to gravity at a depth d below the earth's surface is given by

$$g_d=\frac{GM_e}{R_e^3}(R_e-d)$$

$$=g\left(\frac{R_e-d}{R_e}\right)=g\left(1-\frac{d}{R_e}\right)$$



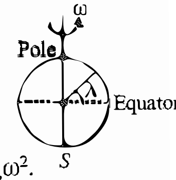
At the centre of the earth, $d=R_e$. $\therefore\ g_d=0$.
- When we move below the earth's surface, the value of acceleration due to gravity also decreases. The value of acceleration due to gravity is maximum at the earth's surface and becomes zero at the centre of the earth.
- The variation of acceleration due to gravity (g') with distance from the centre of the earth (r) is as shown in the figure.



- **Due to rotation of the earth about its axis :** The acceleration due to gravity at latitude λ is given by

$g_\lambda=g-R_e\omega^2\cos^2\lambda$

where ω is the angular speed of rotation of the earth about its axis and its value is 7.3×10^{-5} rad s^{-1}.



  - At the equator, $\lambda=0°$

    $g_\lambda=g_e=g-R_e\omega^2\cos^2 0°=g-R_e\omega^2$.
  - At the poles, $\lambda=90°$

    $g_\lambda=g_p=g-R_e\omega^2\cos^2 90°=g$
- The value of acceleration due to gravity increases from equator to the pole due to rotation of the earth.

$$g_p-g_e=g-(g-R_e\omega^2)=R_e\omega^2$$

- If the earth stops rotating about its axis ($\omega=0$), the value of g will increase everywhere, except at the poles. On the contrary, if there is increase in the angular speed of the earth, then except at the poles the value of g will decrease at all places.
- **Due to shape of the earth :** Earth is not a perfect sphere but it is an ellipsoid. The earth's radius is 21 km larger at the equator than at the poles. Thus, the earth has an equatorial bulge and is flattened at the poles. Both, rotation and equatorial bulge contribute additively to keep the g smaller at the equator than at the poles.

Illustration 6

A body of mass m is placed at a latitude of 45° on earth of radius R and angular speed ω. How does the weight of body change if earth stops rotating about its axis?

Sol: g changes with angular speed ω and λ as

$$g_1=g-R\omega^2\cos^2\lambda.$$

When $\lambda=45°$, $g_1=g-R\omega^2\cos^2 45°$

$$\text{or}\quad g_1=g-\frac{R\omega^2}{2}\qquad\text{...(i)}$$

When $\omega=0$, $g_2=g$ ...(ii)

$$\therefore\quad (g_2-g_1)=g-\left(g-\frac{R\omega^2}{2}\right)=\frac{R\omega^2}{2}$$

$$\therefore\quad \text{Increase in weight of body}=\frac{mR\omega^2}{2}.$$

Illustration 7

The change in the value of g at a height h above the surface of the earth is the same as at a depth d below the surface of earth. When both d and h are much smaller than the radius of earth, then what is the relation between d and h?