**Soln.:** At height $h$, $g_h = g\left(1 - \frac{2h}{R}\right)$ where $h << R$

or $\quad g - g_h = \frac{2hg}{R} \quad$ or $\quad \Delta g_h = \frac{2hg}{R} \quad$ ...(i)

At depth $d$, $g_d = g\left(1 - \frac{d}{R}\right)$ where $d << R$

or $\quad g - g_d = \frac{dg}{R}$

or $\quad \Delta g_d = \frac{dg}{R} \quad$ ...(ii)

From (i) and (ii), when $\Delta g_h = \Delta g_d$

$$\frac{2hg}{R} = \frac{dg}{R} \quad \text{or} \quad d = 2h.$$

## GRAVITATIONAL FIELD

- The space around a material body in which its gravitational pull can be experienced is called its gravitational field.

### Intensity of Gravitational Field

- The intensity of the gravitational field of a body at a point in the field is defined as the force experienced by a body of unit mass placed at that point provided the presence of unit mass does not disturb the original gravitational field. It is denoted by symbol $E$.
- The intensity of gravitational field at a point due to a body of mass $M$, at a distance $r$ from the centre of the body is

$$E = -\frac{GM}{r^2}$$

where –ve sign shows that the gravitational intensity is of attractive force.

- Intensity of gravitational field is a vector quantity. Its dimensional formula is $[M^0LT^{-2}]$.
- Unit of intensity of gravitational field in SI system is N $kg^{-1}$ and in CGS system is dyne $g^{-1}$.

## GRAVITATIONAL POTENTIAL

- The gravitational potential at a point in the gravitational field of a body is defined as the amount of work done in bringing a unit mass from infinity to that point. It is denoted by symbol $V$.
- The gravitational potential at a point in the gravitational field due to a body of mass $M$ at a distance $r$ from the centre of the body is given by

$$V = -\frac{GM}{r}$$

- Gravitational potential is a scalar quantity. Its dimensional formula is $[M^0L^2T^{-2}]$.
- Unit of gravitational potential in SI system is J $kg^{-1}$ and in CGS system is erg $g^{-1}$.
- Gravitational potential ($V$) is related with gravitational field intensity ($E$) by a relation

$$E = -\frac{dV}{dr}$$

### Gravitational Field Intensity and Gravitational Potential due to a Spherical Shell

- The gravitational field intensity and gravitational potential due to a spherical shell of radius $R$ and mass $M$ at a point distant $r$ from the centre of the shell is given as follows:
  - At a point outside the shell *i.e.* $r > R$

$$E_{outside} = -\frac{GM}{r^2}, \; V_{outside} = -\frac{GM}{r}$$

  - At a point on the surface of the shell *i.e.* $r = R$

$$E_{surface} = -\frac{GM}{R^2}, \; V_{surface} = -\frac{GM}{R}$$

  - At a point inside the shell *i.e.* $r < R$

$$E_{inside} = 0, \; V_{inside} = -\frac{GM}{R}$$

### Gravitational Field Intensity and Gravitational Potential due to a Solid Sphere

- The gravitational field intensity and gravitational potential due to a solid sphere of radius $R$ and mass $M$ at a distance $r$ from the centre of the sphere is given as follows:
  - At a point outside the sphere *i.e.* $r > R$

$$E_{outside} = -\frac{GM}{r^2}, \; V_{outside} = -\frac{GM}{r}$$

  - At a point on the surface of the sphere *i.e.* $r = R$

$$E_{surface} = -\frac{GM}{R^2}, \; V_{surface} = -\frac{GM}{R}$$

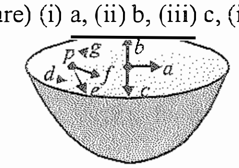
  - At a point inside the sphere *i.e.* $r < R$

$$E_{inside} = -\frac{GMr}{R^3}, \; V_{inside} = -\frac{GM(3R^2 - r^2)}{2R^3}$$

**Note :** $V_{centre} = \frac{3}{2} V_{surface}$

**Illustration 8**

The gravitational intensity at the centre of a hemispherical shell of uniform mass density has the direction indicated by the arrow (see figure) (i) a, (ii) b, (iii) c, (iv) 0.



**Soln.:** The gravitational potential is constant at all points inside a spherical shell. Therefore, the gravitational potential gradient at all points inside the spherical shell is zero [*i.e.* as $V$ is constant, $dV/dr = 0$]. Since gravitational intensity is equal to negative of the gravitational potential gradient, hence the gravitational intensity is zero at all points inside a hollow spherical shell.

This indicates that the gravitational forces acting on a particle at any point inside a spherical shell, will be symmetrically placed.

Therefore if we remove the upper hemi-spherical shell, the net gravitational force acting on the particle at the centre $Q$ or at some other point $P$ will be acting downwards which will also be the direction of gravitational intensity. It is so because, the gravitational intensity at a point is the gravitational force per unit mass at that point. Hence the gravitational intensity at the centre $Q$ will be along $c$, *i.e.*, option (iii) is correct.

**Illustration 9**

How is gravitational potential on the surface of earth, at a given point, related with acceleration due to gravity $g$? Take radius of earth = $R$.

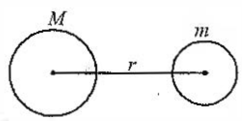
**Soln.:** Gravitational potential = $V$

$$\therefore V = -\frac{GM}{R} = -\frac{R \cdot GM}{R^2} = -Rg.$$

### Gravitational Potential Energy

- The gravitational potential energy of a body at a point in a gravitational field of another body is defined as the amount of work done in bringing the given body from infinity to that point.
- Gravitational potential energy = Gravitational potential × mass of the body
- The gravitational potential energy of mass $m$ in the gravitational field of mass $M$ at a distance $r$ from it is

$$U = -\frac{GMm}{r}$$




where, $r$ is the distance between $M$ and $m$.

- The gravitational potential energy of a mass $m$ at a distance $r$ (> $R_e$) from the centre of the earth is

$$U = mV = -\frac{GM_e m}{r}$$

- Gravitational potential energy of a mass at infinite distance from the earth is zero.
- Gravitational potential energy is a scalar quantity. Its dimensional formula is $[ML^2T^{-2}]$ and SI unit is J.
- Gravitational potential energy of a body of mass $m$ at height $h$ above the earth's surface is given by

$$U_h = \frac{-GM_e m}{(R_e + h)}$$

- Gravitational potential energy of a body of mass $m$ on the earth's surface is given by

$$U_s = \frac{-GM_e m}{R_e}$$

- The change in potential energy when a body of mass $m$ is moved vertically upwards through a height $h$ from the earth's surface is given by

$$\Delta U = U_h - U_s = GM_e m\left[\frac{1}{R_e} - \frac{1}{R_e + h}\right]$$

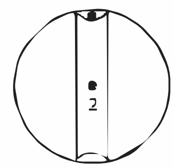
$$= \frac{GM_e mh}{R_e^2\left(1+\frac{h}{R_e}\right)} = \frac{mgh}{\left(1+\frac{h}{R_e}\right)} \quad \left(\because g = \frac{GM_e}{R_e^2}\right)$$

For $h << R_e$, $\Delta U = mgh$.

**Illustration 10**

A diametrical tunnel is dug across the earth. A ball is dropped into the tunnel from one side. What is the velocity of the ball when it reaches the centre of the Earth? (Given : gravitational potenital at the centre of earth $= 3GM/2$)

**Soln.:** As only the conservative gravitational force is acting,




$$U_1 + K_1 = U_2 + K_2$$

$$U_1 = \frac{-GMm}{R}, K_1 = 0$$

$$U_2 = \frac{-3GMm}{2R}$$

$$K_2 = \frac{1}{2}mv^2 \Rightarrow \frac{-GMm}{R} + 0 = \frac{-3}{2}\frac{GMm}{R} + \frac{1}{2}mv^2$$

$$\Rightarrow \frac{GMm}{2R} = \frac{1}{2}mv^2 \quad \text{or } v^2 = \frac{GM}{R}$$

$$\text{Now, } mg = \frac{GMm}{R^2} \text{ or } g = \frac{GM}{R^2} \Rightarrow v^2 = \left(\frac{GM}{R^2}\right)R = gR$$

$$\Rightarrow v = \sqrt{gR}$$

**Illustration 11**

If the acceleration due to gravity at the surface of the earth is $g$, then what will be the work done in slowly lifting a body of mass $m$ from the earth's surface to a height $R$ equal to the radius of the earth?

**Soln. :** Let $W_{ex}$ be the required work done in this case.

$$W_{ex} = \Delta U = U_f - U_o$$

$$U_f = -\frac{GMm}{(2R)}; \quad U_o = -\frac{GMm}{R}$$

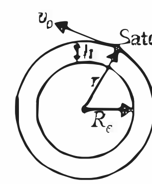
$$\Delta U = -\frac{GMm}{2R} + \frac{GMm}{R} = \frac{GMm}{2R}$$

$$\text{Now, } mg = \frac{GMm}{R^2} \text{ or } g = \frac{GM}{R^2} \Rightarrow \Delta U = \frac{mgR}{2}$$

$$\Rightarrow W_{ex} = \frac{mgR}{2}$$

### SATELLITE

- Satellite is natural or artificial body describing orbit around a planet under its gravitational attraction. Moon is a natural satellite while INSAT-1B is an artificial satellite of the earth.
- **Orbital speed of a satellite :** It is defined as the minimum speed required to put the satellite into a given orbit around earth.




- Orbital speed of the satellite, when it is revolving around the earth at a height $h$ is given by

$$v_o = \sqrt{\frac{GM_e}{r}} = \sqrt{\frac{GM_e}{R_e + h}} = R_e\sqrt{\frac{g}{R_e + h}} \quad \left(\text{As } g = \frac{GM_e}{R_e^2}\right)$$

When the satellite is orbiting close to the earth's surface, *i.e.*, $h << R_e$, then

$$v_o = R_e\sqrt{\frac{g}{R_e}} = \sqrt{gR_e}$$

$$v_o = \sqrt{9.8\times 6.4\times 10^6} = 7.92\times 10^3\ \text{ms}^{-1} \approx 8\ \text{km s}^{-1}$$

- The orbital speed of the satellite is independent of the mass of the satellite. The orbital speed of the satellite depends upon the mass and radius of the earth/planet around which the revolution of satellite is taking place. The direction of orbital speed of the satellite at an instant is along the tangent to the orbital path of satellite at that instant.
- **Time period of a satellite :** It is the time taken by satellite to complete one revolution around the earth and it is given by

$$T = \frac{2\pi r}{v_o} = 2\pi\sqrt{\frac{r^3}{GM_e}} = 2\pi\sqrt{\frac{(R_e+h)^3}{GM_e}} = \frac{2\pi}{R_e}\sqrt{\frac{(R_e+h)^3}{g}}$$

For a satellite orbiting close to the earth's surface *i.e.* $h << R_e$

$$T = 2\pi\sqrt{\frac{R_e}{g}} = 84.6\ \text{min}.$$

- The period of revolution of the satellite depends upon its height above earth's surface. Larger is the height of the satellite, the greater will be its time period of revolution.
- **Height of satellite above the earth's surface**

$$h = \left(\frac{T^2R_e^2g}{4\pi^2}\right)^{1/3} - R_e$$

- **Kinetic energy of a satellite**

$$K = \frac{1}{2}mv_o^2 = \frac{1}{2}\frac{GM_em}{r} = \frac{1}{2}\frac{GM_em}{(R_e+h)}$$

- **Potential energy of a satellite**

$$U = -\frac{GM_em}{r} = -\frac{GM_em}{R_e+h}$$

- **Total energy (mechanical) of a satellite**

$$E = K + U = -\frac{GM_em}{2r} = -\frac{GM_em}{2(R_e+h)}$$

For satellite orbiting very close to the surface of earth

*i.e.*, $h << R_e$ then $E = -\dfrac{GM_em}{2R_e}$.

- Kinetic energy of a satellite is equal to negative of total energy while potential energy is equal to twice the total energy.

*i.e.* $K = -E$, $U = 2E$

- **Binding energy of a satellite**

$$E_B = -E = \frac{GM_em}{2r} = \frac{GM_em}{2(R_e+h)}.$$

- **Angular momentum of a satellite**

$$L = mv_or = mr\sqrt{\frac{GM_e}{r}} = [m^2rGM_e]^{1/2}$$

- Angular momentum of a satellite depends on both, mass of the satellite ($m$) and mass of the earth ($M_e$). It also depends upon the radius of the orbit ($r$) of the satellite.
- Angular momentum is conserved in the motion of satellite.

**Illustration 12**

Imagine a light planet revolving around a very massive star in a circular orbit of radius $R$ with a period of revolution $T$. If the gravitational force of attraction between the planet and the star is proportional to $R^{-5/2}$, then show that $T^2$ will be proportional to $R^{7/2}$.

**Soln.:** The gravitational force is proportional to $R^{-\frac{5}{2}}$.

$\Rightarrow F = K\cdot R^{-\frac{5}{2}}$, where $K$ is a constant. This is the force that provides the centripetal force for the planet's revolution around the star.

$$\Rightarrow \quad K\cdot R^{-\frac{5}{2}} = m\omega^2\cdot R \quad \text{But } \omega = \frac{2\pi}{T}, \text{ so}$$

$$K\cdot R^{-\frac{5}{2}} = m\left(\frac{2\pi}{T}\right)^2\cdot R \Rightarrow T^2 = \left(\frac{4\pi^2\cdot m}{K}\right)R^{7/2}$$

Hence, $T^2 \propto R^{7/2}$

**Illustration 13**

The magnitude of gravitational potential energy of the moon-earth system is $U$ with zero potential energy at infinite separation. The kinetic energy of the moon with respect to the earth is $K$. Then show that $U > K$.

**Soln. :** $U(r) = \dfrac{-GMm}{r}$

Where $M$ is mass of the earth, $m$ is the mass of the moon and $r$ is their distance of separation.

$$F = \frac{-U(r)}{r} = -GMm\left(\frac{-1}{r^2}\right) = \frac{GMm}{r^2}$$

The force $F$ provides the centripetal force for the moon to go around.

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \Rightarrow mv^2 = \frac{GMm}{r} \quad \text{or } K = \frac{1}{2}mv^2 = \frac{GMm}{2r}$$

If we take $U$ as the magnitude of potential energy

$$\Rightarrow \quad U = \frac{GMm}{r} \Rightarrow K = \frac{U}{2}$$

Hence, $U > K$.

**Illustration 14**

Which quantity is conserved for a satellite revolving around the earth in a fixed orbit?

**Soln.:** In motion of satellite, angular momentum is conserved.

$L = mvr = \text{constant}$