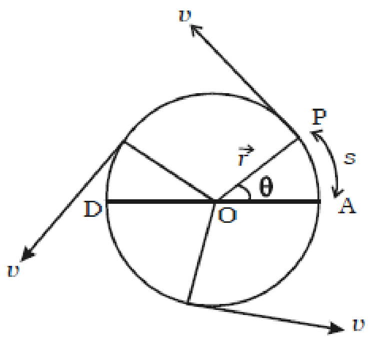
7.1 Circular Motion

When a particle moves on a circular path with a constant speed, then its motion is known as uniform circular motion in a plane. The magnitude of velocity in circular motion remains constant but the direction changes continuously.

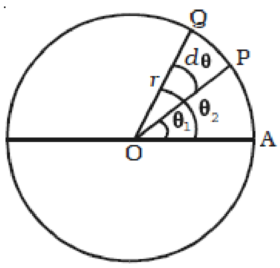


Let us consider a particle of mass *m* moving with a velocity *v* along the circle of radius *r* with centre O as shown in Fig

P is the position of the particle at a given instant of time such that the radial line OP makes an angle θ with the reference line DA. The magnitude of the velocity remains constant, but its direction changes continuously. The linear velocity always acts tangentially to the position of the particle (i.e) in each position, the linear velocity **v** is perpendicular to the radius vector **r**.

Angular displacement

Let us consider a particle of mass *m* moving along the circular path of radius *r* as shown in Fig.



Let the initial position of the particle be A. P and Q are the positions of the particle at any instants of time *t* and *t* + *dt* respectively. Suppose the particle traverses a distance *ds* along the circular path in time interval *dt.* During this interval, it moves through an angle $d\theta = \theta_2 - \theta_1$. *The angle swept by the radius vector at a given time is called the angular displacement of the particle.*

If *r* be the radius of the circle, then the angular displacement is given by $d\theta = ds / r$

The angular displacement is measured in terms of radian.

Angular velocity

The rate of change of angular displacement is called the angular velocity of the particle

Let $d\theta$ be the angular displacement made by the particle in time *dt* , then the angular velocity of the particle is

$$\omega = \frac{d\theta}{dt}$$

Its unit is rad s^{-1} and dimensional formula is T^{-1}.

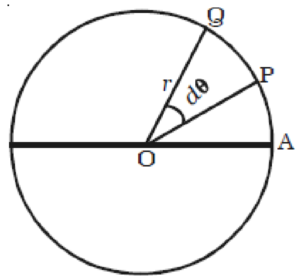
For one complete revolution, the angle swept by the radius vector is 360° or 2π radians. If *T* is the time taken for one complete revolution, known as period, then the angular velocity of the particle is

$$\omega = \frac{2\pi}{T}$$

If particle makes n revolution per second then frequency is n

7.2 Relation between linear velocity and angular velocity

Let us consider a body P moving along the circumference of a circle of radius *r* with linear velocity *v* and angular velocity ω as shown in Fig.. Let it move from P to Q in time *dt* and *d*θ be the angle swept by the radius vector.



Let PQ = *ds,* be the arc length covered by the particle moving along the circle, then the angular displacement *d* θ is expressed as

$$d\theta = \frac{ds}{r}$$

But ds = v dt

$$\therefore d\theta = \frac{vdt}{r}$$

$$\therefore \frac{d\theta}{dt} = \frac{v}{r}$$

$$\therefore \omega = \frac{v}{r} \; or \; v = \omega r$$

In vector notion

$$\vec{v} = \vec{\omega} \times \vec{r}$$

Thus, for a given angular velocity ω, the linear velocity *v* of the particle is directly proportional to the distance of the particle from the centre of the circular path (i.e) for a body in a uniform circular motion, the angular velocity is the same for all points in the body but linear velocity is different for different points of the body.

7.2 Angular acceleration

If the angular velocity of the body performing rotatory motion is non-uniform, then the body is said to possess angular acceleration.

The rate of change of angular velocity is called angular acceleration.

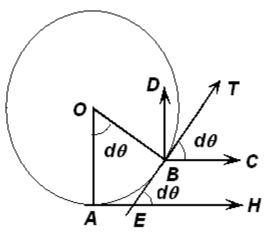
If the angular velocity of a body moving in a circular path changes from ω_1 to ω_2 in time *t* then its angular acceleration is

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}$$

$$\alpha = \frac{\omega_2 - \omega_1}{t}$$

The angular acceleration is measured in terms of rad s^{-2} and its dimensional formula is T^{-2}.

7.3 Centripetal acceleration



The speed of a particle performing uniform circular motion remains constant throughout the motion but its velocity changes continuously due to the change in direction (i.e) the particle executing uniform circular motion is said to possess an acceleration.

Consider a particle executing circular motion of radius *r* with linear velocity *v* and angular velocity ω. The linear velocity of the particle acts along the tangential line. Let *d*θ be the angle described by the particle at the centre when it moves from A to B in time *dt*.

At A and B, linear velocity *v* acts along

AH and BT respectively. In Fig. ∠*AOB* = *d*θ = ∠*HET* (∵ angle subtended by the two radii of a circle = angle subtended by the two tangents).

The velocity *v* at B of the particle makes an angle dθ with the line BC and hence it is resolved horizontally as *v cos d*θ along BC and vertically as *v* sin *d* θ along BD.

∴ The change in velocity along the horizontal direction = *v cos d*θ −*v*

If *d*θ is very small, *cos d*θ = 1

∴Change in velocity along the horizontal direction = *v* − *v* = 0

(i.e) there is no change in velocity in the horizontal direction.

The change in velocity in the vertical direction (i.e along AO) is

dv = *v sin d*θ − *0* = *v sin d*θ

If *d*θ is very small, *sin d*θ = *d*θ

∴ The change in velocity in the vertical direction (i.e) along radius of the circle *dv* = *v.d*θ

But linear acceleration

$$a = \frac{dv}{dt} = v\frac{d\theta}{dt} = v\omega$$

We know that *v* = *r* ω

$$a = \frac{v^2}{r}$$

Hence, the acceleration of the particle producing uniform circular motion is along AO (i.e) directed towards the centre of the circle. This acceleration is directed towards the centre of the circle along the radius and perpendicular to the velocity of the particle. This acceleration is known as centripetal or radial or normal acceleration.

Solved Numerical

Q1) A particle moves in a circle of radius 20cm. Its linear speed at any time is given by v = 2t where v is in m/s and t is in seconds. Find the radial and tangential acceleration at t = 3 sec and hence calculate the total acceleration at this time

Solution

The linear speed at 3 sec is v = 2×3 = 6 m/s

The radial acceleration at 3 sec

$$a_r = \frac{v^2}{r} = \frac{6 \times 6}{0.2} = 180\ ms^{-2}$$

The tangential acceleration is given by dv/dt

Thus tangential acceleration $a_t = 2$ as v = 2t

Total acceleration

$$\sqrt{a_r^2 + a_t^2} = \sqrt{180^2 + 2^2} = \sqrt{32404}\ ms^{-2}$$

7.4 Centripetal force

According to Newton's first law of motion, a body possesses the property called directional inertia (i.e.) the inability of the body to change its direction. This means that without the application of an external force, the direction of motion cannot be changed. Thus when a body is moving along a circular path, some force must be acting upon it, which continuously changes the body from its straight-line path (Fig 2.40). It makes clear
that the applied force should have no component in the direction of the motion of
the body or the force must act at every point perpendicular to the direction of motion of the body.
This force, therefore, must act along the radius and should be directed towards the
centre. Hence for circular motion, a constant force should act on the body, along the radius towards the centre and perpendicular to the velocity of the body. This force is known as centripetal force.
If m is the mass of the body, then the magnitude of the centripetal force is given by
F = mass × centripetal acceleration

$$F = \frac{mv^2}{r} = mr\omega^2$$

Examples

Any force like gravitational force, frictional force, electric force, magnetic force etc. may act as a centripetal force. Some of the examples of centripetal force are :
(i) In the case of a stone tied to the end of a string whirled in a circular path, the centripetal force is provided by the tension in the string.
(ii) When a car takes a turn on the road, the frictional force between the tyres and the road provides the centripetal force.
(iii) In the case of planets revolving round the Sun or the moon revolving round the earth, the centripetal force is provided by the gravitational force of attraction between them
(iv) For an electron revolving round the nucleus in a circular path, the electrostatic force of attraction between the electron and the nucleus provides the necessary centripetal force.

7.5 Non uniform circular motion

If the speed of the particle moving in a circle is not constant the acceleration has both radial and tangential components. The radial and tangential accelerations are

$$a_r = \frac{v^2}{r} \quad and\ a_t = \frac{dv}{dt}$$

The magnitude of the resultant acceleration will be

$$a = \sqrt{a_r^2 + a_t^2} = \sqrt{\left(\frac{v^2}{r}\right)^2 + \left(\frac{dv}{dt}\right)^2}$$

Solved numerical

Q2) A certain string which is 1m long will break if the load on it is more than 0.5kg. A mass of 0.05kg is attaché to one end of it and the particle is whirled round in a horizontal circle by holding the free end of the string by one hand. Find the greatest number of revolutions per minute possible without breaking the string

Solution

Centrifugal force = breaking force

$Mr\omega^2$ = breaking force (mg)

$0.05\times1\times \omega^2 = 0.5\times9.8$

$\omega = \sqrt{98}$

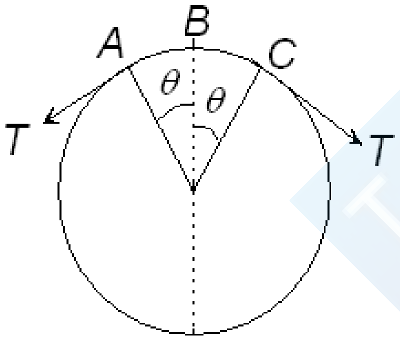
Now $\omega = 2\pi f$ here f = number of revolution per second

$2\pi f = 9.899$

f =1.567 rev/sec

f = 1.567×60 = 94.02 rev/min

Q3) A metal ring of mass m and radius R is placed on a smooth horizontal table and is set to rotate about its own axis in such a way that each part of ring moves with velocity v. Find tension in the ring



Solution

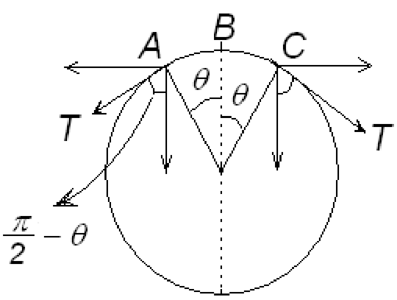
If we cut the ring along the imaginary diameter then left half will go left and right half will go on right side . Thus tension on left and right are as shown in figure

Now let the mass of small segment ABC be Δm

$$\Delta m = \frac{m}{2\pi R} R(2\theta) \quad ---\text{eq}(1)$$

As the elementary portion ACB moves in a circle of radius R at speed v its acceleration towards centre is

$$\frac{(\Delta m)v^2}{R}$$



Resolving the tension along x and y axis. We have considered BAC as very small segment then vertically down ward components will be along the radius. Producing tension on the ring

Thus

$$T\cos\left(\frac{\pi}{2} - \theta\right) + T\cos\left(\frac{\pi}{2} - \theta\right) = \frac{(\Delta m)v^2}{R}$$

$$2Tsin\theta = \frac{(\Delta m)v^2}{R} \quad ----eq(2)$$

Putting value of Δm from equation (1) in equation(2)

$$2Tsin\theta = \frac{\mathrm{m}}{2\Pi\mathrm{R}}\mathrm{R}(2\theta)\frac{v^2}{R}$$

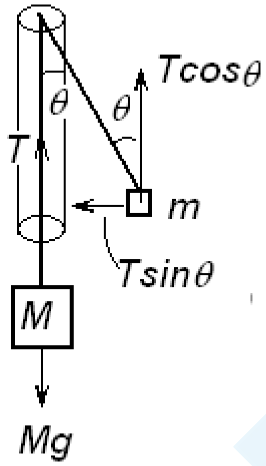
$$T = \frac{\mathrm{m}}{2\Pi}\left(\frac{2\theta}{2\sin\theta}\right)\frac{v^2}{R}$$

Since θ is very small

$$\frac{2\theta}{2\sin\theta} = 1$$

$$T = \frac{\mathrm{m}v^2}{2\Pi R}$$

Q4) A large mass M and a small mass m hang at the two ends of the string that passes through a smooth tube as shown in figure. The mass m moves around in a circular path, which lies in the horizontal plane. The length of the string from the mass m to the top of the tube is l and θ is the angle this length makes with vertical. What should be the frequency of rotation of mass m so that M remains stationary



Solution

Let m follows a circular path of radius r and angular frequency ω

The force acting on M is Mg downwrds

Thus tension in string T = Mg -----(1)

Force acting on m for circular motion is $mr\omega^2$ from diagram

Component of tension in string towards centre is Tsinθ

Thus $T\sin\theta = mr\omega^2$ -------(2)

From eq(1) and eq(2)

$$Mg\sin\theta = mr\omega^2$$

$$\omega = \sqrt{\frac{Mgsin\theta}{mr}} \quad ----eq(3)$$

From the geometry of figure r = lsinθ thus

$$\omega = \sqrt{\frac{Mgsin\theta}{mlsin\theta}} = \sqrt{\frac{Mg}{ml}}$$

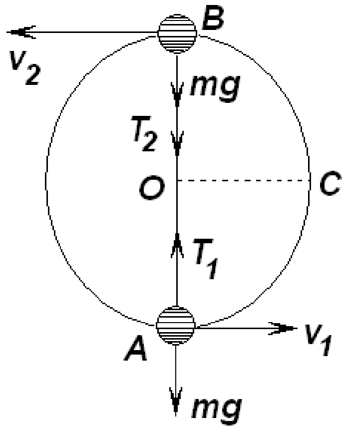
Now ω=2πf

$$2\pi f = \sqrt{\frac{Mg}{ml}}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{Mg}{ml}}$$

7.6 Motion in vertical circle

Body is suspended with the help of a string.



Imagine an arrangement like a simple pendulum where a mass m is tied to a string of length r, the other end of the string being attached to a fixed point O.

Let the body initially lie at the equilibrium position A. Let it be given a velocity v_1 horizontally. If the velocity is small then the body would make oscillations in the vertical plane like a simple pendulum

Let us now find the velocity of projection v_1 at the lowest point A required to make the body move in a vertical circle.

The following points can be noted while considering the motion

(i)The motion along the vertical circle is non-uniform since velocity of body changes along the curve. Hence the centripetal acceleration must also change.

(ii)The resultant force acting on the body (in the radial direction) provides the necessary centripetal force.

(iii) The velocity decreases as the body up from A to B, the topmost point

When the body is at A, the resultant force acting on it is T_1 – mg, where T, is the tension in the string The centripetal force

$$T_1 - mg = \frac{mv_1^2}{r}$$

$$T_1 = mg + \frac{mv_1^2}{r} \quad --- eq(1)$$

The tension is always positive (even if v_1 is zero). Therefore, the string will be taut when is at A. The condition for the body to complete the vertical circle is that the string should be taut all time. i.e. the tension is greater than zero

The region where string is most likely become slack is about the horizontal radius OC. Also the tension would become the least when it is the topmost point B. Let it be T_2 at B and the velocity of the body be v_2. The resultant force on the body at B is

$$T_2 + mg = \frac{mv_2^2}{r}$$

$$T_2 = \frac{mv_2^2}{r} - mg \quad ---- eq(2)$$

If the string is to taut at B, $T_2 \geq 0$

$$\frac{mv_2^2}{r} - mg \geq 0$$

$$v_2^2 \geq rg$$

$$v_2 \geq \sqrt{rg} \quad ---eq(3)$$

At A, the Kinetic Energy of the body

$$\frac{1}{2}mv_1^2$$

Potential energy of the body = 0

Total energy at point A

$$\frac{1}{2}mv_1^2$$

At B, the kinetic energy of the body =

$$\frac{1}{2}mv_2^2$$

Potential energy at B = mg(2r)

Total energy at B

$$2mgr + \frac{1}{2}mv_2^2$$

Using the principle of conservation of energy, we have

$$\frac{1}{2}mv_1^2 = 2mgr + \frac{1}{2}mv_2^2$$

$$v_1^2 = v_2^2 + 4gr \quad ----eq(4)$$

Combining equations (3) and (4)

$$v_1^2 \geq rg + 4gr$$

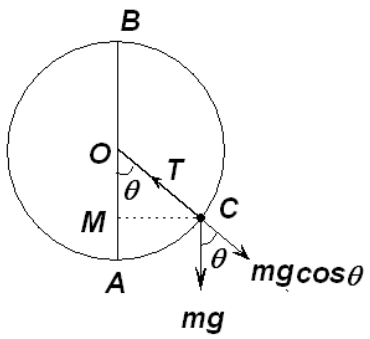
$$v_1 \geq \sqrt{5gr}$$

Hence, if the body has minimum velocity of √(5gr) at the lowest point of vertical circle, it will complete the circle.

The particle will describe complete circle if both v_2 and T_2 do not vanish till the particle reaches the highest point.

Important points

(i)If the velocity of projection at the lowest point A is less than √(2gr), the particle will come to instantaneously rest at a point on the circle which lies lower than the horizontal diameter. It



will then move down to reach A and move on to an equal height on the horizontal diameter. It will then move down to reach A and move on to an equal height on the other side of A. Thus the particles execute oscillations. In case v vanishes before T does We may find an expression for tension in the string when it makes an angle θ with the vertical. At C, the weight of the body acts vertically downwards, and the tension in the string is towards the centre O.

The weight mg is resolved radially and tangentially.

The radial component in mgcosθ and the tangential component is mgsinθ

The centripetal force is T - mgcosθ

$$T - mg\cos\theta = \frac{mv^2}{r}$$

Where v is the velocity at C

$$\therefore T = m\left(\frac{v^2}{r} + g\cos\theta\right) \quad ----eq(1)$$

The velocity can be expressed in terms of v_1 at A

The total energy at A

$$\frac{1}{2}mv_1^2$$

The kinetic energy at C

$$\frac{1}{2}mv^2$$

Potential energy at C = mg(AM)

= mg (AO-MO)

=mg(r – r cosθ)

=mgr(1-cosθ)

Total energy at C =

$$\frac{1}{2}mv^2 + mgr(1 - \cos\theta)$$

From conservation of energy

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv^2 + mgr(1 - \cos\theta)$$

$$v_1^2 = v^2 + 2gr(1 - \cos\theta)$$

$$v^2 = v_1^2 - 2gr(1 - \cos\theta)$$

Substituting value of v in equation (1)

$$T = m\left(\frac{v_1^2 - 2gr(1 - \cos\theta)}{r} + g\cos\theta\right)$$

$$T = \frac{mv_1^2}{r} + 3mg\left(\cos\theta - \frac{2}{5}\right) \quad -----eq(2)$$

This expression gives the value of the tension in the string in terms of the velocity at the lowest point and the angle θ

Equation (1) shows that tension in the string decreases as θ increases, since the term gcosθ decreases as θ increases

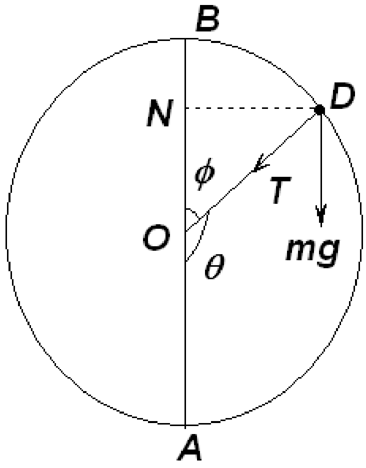
When θ is 90^{O}, cosθ = 0 and

$$T_H = \frac{mv^2}{r}$$

This is obvious because, the weight is vertically downwards where as the tension is horizontal. Hence the tension alone is the centripetal force

(ii) If the velocity of projection is greater than √(2gr) but less than √(5gr), the particle rises above the horizontal diameter and the tension vanishes before reaching the highest point.

(iii) if we make θ an obtuse angle.



At point D, the string OD makes an angle φ with vertical. The radial component of the weight is mgcosφ towards the centre O

$$T + mgcos\varphi = \frac{mv^2}{r}$$

$$T = m\left(\frac{v^2}{r} - gcos\varphi\right) \quad - - - - - eq(1)$$

Kinetic energy at D

$$\frac{1}{2}mv^2$$

Potential energy at D = mg(AN)

=mg(AO+ON)

= mg(r+rcosφ)

= mgr(1+cosφ)

From conservation of energy

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv^2 + mgr(1 + cos\varphi)$$

$$v^2 = v_1^2 - 2mgr(1 + cos\varphi)$$

Substituting value of v² in eq(1) we get

$$T = m\left(\frac{v_1^2 - 2mgr(1 + cos\varphi)}{r} - gcos\varphi\right)$$

$$T = m\left[\frac{v_1^2}{r} - 2g(1 + cos\varphi) - gcos\theta\right]$$

$$T = m\left[\frac{v_1^2}{r} - 3g\left(cos\varphi + \frac{2}{3}\right)\right]$$

This equation shows that the tension becomes zero if

$$\frac{v_1^2}{r} = 3g\left(cos\varphi + \frac{2}{3}\right) \quad - - - - eq(2)$$

If the tension is not to become zero

$$> 3g\left(cos\varphi + \frac{2}{3}\right)$$

Equation (2) gives the value of φ at which the string becomes slack

$$cos\varphi + \frac{2}{3} = \frac{v_1^2}{3rg}$$

$$cos\varphi = \frac{v_1^2}{3rg} - \frac{2}{3}$$

7.7 A body moving inside a hollow tube or sphere

The same discussion holds good for this case, but instead of tension in the sting we have the normal reaction of the surface. If N is the normal reaction at the lowest point, then

$$N - mg = \frac{mv_1^2}{r}$$

$$N = m\left(\frac{v_1^2}{r} + g\right)$$

At highest point of the circle

$$N + mg = \frac{mv_1^2}{r}$$

$$N = m\left(\frac{v_1^2}{r} - g\right)$$

The condition $v_1 \geq \sqrt{(5rg)}$ for the body to complete the circle hold for this also

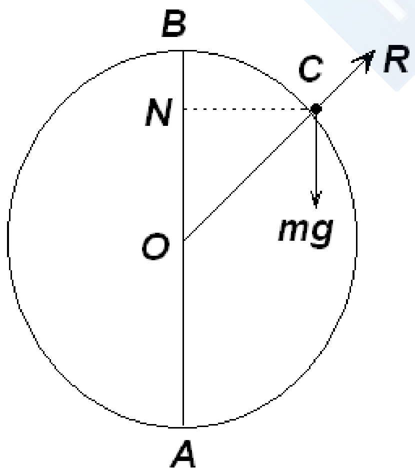
All other equations (can be) similarly obtained by replacing T by reaction R

7.8 Body moving on a spherical surface

The small body of mass m is placed on the top of a smooth sphere of radius r.

If the body slides down the surface, at what point does it fly off the surface?

Consider the point C where the mass is, at a certain instant. The forces are the normal reaction R and the weight mg.



The radial component of the weight is mgcosφ acting towards the centre. The centripetal force is

$$mgcos\varphi - R = \frac{mv^2}{r}$$

Where v is the velocity of the body at O

$$R = m\left(gcos\varphi - \frac{v^2}{r}\right)$$

The body flies off the surface at the point where R becomes zero

$$gcos\varphi = \frac{v^2}{r}$$

$$cos\varphi = \frac{v^2}{gr} \qquad -----eq(1)$$

To find v, we can use conservation of energy

$$\frac{1}{2}mv^2 = mg(BN)$$

$$= mg(OB - ON)$$

$$= mg(r - r\cos\varphi)$$

$$v^2 = 2rg(1-\cos\varphi)$$

$$\frac{v^2}{gr} = 2(1 - \cos\varphi) \quad ----eq(2)$$

From equation(1) and (2)

$$\cos\varphi = 2-2\cos\varphi$$

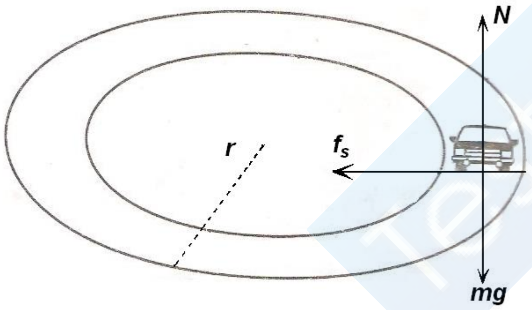
$$\cos\varphi = 2/3$$

This gives the angle at which the body goes of the surface. The height from the ground to that point = AN = $r(1+\cos\varphi)$

Height from ground = $r(1+2/3) = (5/3)\ r$

7.9 Motion of a vehicle on a level circular path

Vehicle can move on circular path safely only if sufficient centripetal force is acting on the vehicle. Necessary centripetal force is provided by friction between tyres of the vehicle



Let v be the velocity of vehicle.

Forces on vehicle are

(1)Weight of vehicle (mg in downward direction)

(2) The normal reaction N by road in upward

(3) The frictional force f_S by road – parallel to the surface of the road

Since the vehicle has no acceleration in vertical direction

N-mg = 0

N = mg

The required centripetal force for the circular motion of the vehicle on this road must be provided by the friction f_S

$$f_s = \frac{mv^2}{r}$$

But $f_s = \mu N = \mu mg$

From this we can say that for safe speed on the road

Centripetal force ≤ maximum friction

$$\frac{mv^2}{r} \le \mu mg$$
$$v^2 \le \mu rg$$

And maximum safe speed

$$v_{max} = \sqrt{\mu rg}$$

If the speed of the vehicle is more than this v_{max} it will be thrown away from the road. Note that safe speed is independent of mass of vehicle

Solved Numerical

Q5) A car goes on a horizontal circular road of radius R, the speed increases at a rate dv/dt. The coefficient of friction between road and tyre is μ. Find the speed at which the car will skid

Solution

Here at any time t , the speed of car becomes v, the net acceleration in the plane of road is

$$\sqrt{\left(\frac{v^2}{r}\right)^2 + a^2}$$

This acceleration provided by frictional force

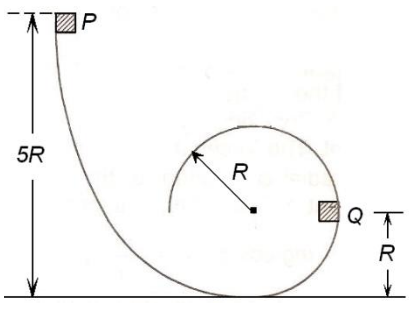
At the moment car will side if

$$M\sqrt{\left(\frac{v^2}{r}\right)^2 + a^2} = \mu Mg$$

$$v = [R^2(\mu^2 g^2 - a^2)]^{1/4}$$

Q6) A small block of mass m slides along the frictionless loop-to-loop track shown in figure. a)It starts from rest at P, what is the resultant force acting on it at Q (b) At what height above the block be released so that the force it exerts against the track at the top of the loop equals its weight?

Solution



Difference between Point P and Q is 4R

Thus according to law of conservation of energy

K.E. at Q = Loss of P.E

$$\frac{1}{2}mv^2 = mg\,(4R)$$

$$V^2 = 8gR$$

At Q, the only force acting on the block is mg downward and Normal of the track which is in radial direction provides centripetal force for circular motion.

$$N = \frac{mv^2}{R} = \frac{m \times 8gR}{R} = 8mg$$

The loop must exert a force on the block equal to eight time the block's weight

(b) For the block to exert a force equal to its weight against the track at the top of the loop

$$\frac{mv'^2}{R} = 2mg$$

$$v'^2 = 2gR$$

(b) block exerts the force equal to mass of the block on at the highest point of circular loop

At highest point centrifugal force acts outward and gravitational force acts down wards. Thus to get force exerted by the block on loop

F = centrifugal – gravitational force

$$mg = \frac{mv'^2}{R} - mg$$

$$2mg = \frac{mv'^2}{R}$$

$$v'^2 = 2gR$$

Thus Kinetic energy

$$\frac{1}{2}mv'^2 = mgR$$

Let the block be released from height H then loss of potential energy = mg(H-R)

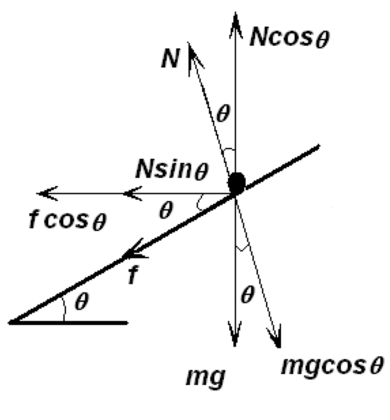
$$\therefore mg(H - R) = mgR$$

$$H = 2R$$

Thus block must be released from the height of 2R

7.10 Motion of vehicle on Banked road

To take sharp turns friction of the road may not be sufficient. If at the curvature of road if the road is banked then in required centripetal force for circular motion, a certain contribution can be obtained from the normal force (N) by the road and the contribution of friction can be decreased to certain extent.



In figure the section of road with the plane of paper is shown. This road is inclined with the horizontal at an angle θ. The forces acting on the vehicle are also shown in figure.

Forces on vehicle

(1) Weight (mg) downward direction

(2) Normal force(N) perpendicular to road

(3) frictional force parallel to road

As the acceleration of vehicle in the vertical direction is zero

$N\cos\theta = mg + f\sin\theta$

$\therefore mg = N\cos\theta - f\sin\theta$ --------eq(1)

In the horizontal direction, the vehicle performs a circular motion. Hence it requires centripetal force., which is provided by the horizontal components of f and N

$$\frac{mv^2}{r} = Nsin\theta + fcos\theta \qquad -----eq(2)$$

Dividing equation (2) by (1) we get

$$\frac{v^2}{rg} = \frac{Nsin\theta + fcos\theta}{Ncos\theta - fsin\theta}$$

Now f = μN thus

$$\frac{v^2}{rg} = \frac{Nsin\theta + \mu Ncos\theta}{Ncos\theta - \mu Nsin\theta}$$

$$\frac{v^2}{rg} = \frac{sin\theta + \mu cos\theta}{cos\theta - \mu sin\theta}$$

Dividing numerator and denominator by cosθ

$$\frac{v^2}{rg} = \frac{tan\theta + \mu}{1 - \mu tan\theta}$$

$$v = \sqrt{rg\left[\frac{tan\theta + \mu}{1 - \mu tan\theta}\right]}$$

Cases: (i) If μ = 0 then

$$v_0 = \sqrt{rgtan\theta}$$

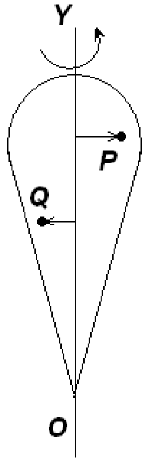
If we drive the vehicle at this speed on the banked curved road the contribution of friction becomes minimum in the enquired centripetal force, and hence wear and tear of the tyres can be minimized. This speed v_o is called the optimum speed

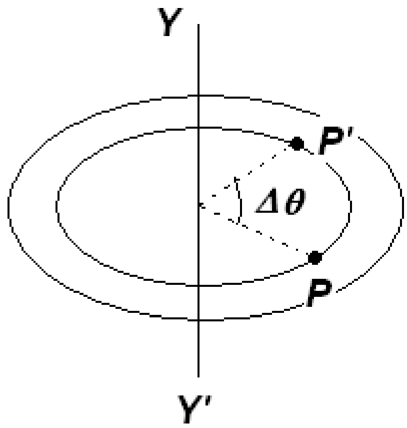
(ii) If $v < v_O$ then the frictional force will act towards the higher (upper edge of the banked road. The vehicle can be kept stationary. It can be parked on banked road, only if $tan\theta \leq \mu$

7.11 ROTATIONAL MOTION OF RIGID BODIES

A rigid body is a body whose deformation is negligible when suspended to external force , in a rigid body the distance between any two points remains constant. A rigid body can undergo various types of motion. It may translate, rotate or any translate and rotate at the same time When a rigid body translates each particle of rigid body undergoes same displacement, has same velocity and same acceleration. To apply equation of translation, all its mass can be considered at centre of mass and we can use $\mathbf{F}_{ext} = M\mathbf{a}_{CM}$

Rotation





If all the particles of a rigid body perform circular motion and the centres of these circles are steady on a definite straight line called axis of rotation it is a geometrical line and the motion of the rigid body is called the rotational motion

In figure two particles P and Q of rigid body are shown. The rigid body rotates about axis OY. The circular paths of particles P and Q are in the plane perpendicular to axis of motion OY

Consider a rigid body undergoes rotation about axis YY'. A point P of the rigid body is undergoing circular motion of radius OP = r. If during a time interval Δt , the body rotates through an angle Δθ, the arc PP' will subtend an angle Δθ at the centre of motion of its circular path. Δθ is angular displacement of rigid body as well as of point P.

Average angular speed during a time interval Δt is defined as

$$\overline{\omega} = \frac{\Delta\theta}{\Delta t}$$

Instantaneous angular speed is defined as

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

If the body rotates through equal angles in equal intervals of time, it is said as rotating uniformly. But if it's angular speed changes with time it is said to be accelerated and its angular acceleration is defined as

Average angular acceleration

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t}$$

Instantaneous angular acceleration

$$\alpha = \frac{d\omega}{dt}$$

In case of uniform rotation, angular displacement Δθ = ωt

In case of uniformly accelerated rotation following kinematic relations we use

$$\omega = \omega_0 + \alpha t$$

$$\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha\Delta\theta$$

Linear velocity and linear acceleration of a particle in rigid body

Linear velocity and angular velocity is related by following relation

$$\vec{v} = \vec{\omega} \times \vec{r}$$

Direction of ω is along the axis of rotation

On taking derivative of above equation we get

$$\vec{a} = \vec{\omega} \times \frac{d\vec{r}}{dt} + \frac{d\vec{\omega}}{dt} \times \vec{r}$$

$$\vec{a} = \vec{\omega} \times \vec{v} + \vec{\alpha} \times \vec{r}$$

Thus linear acceleration have two components we find the direction using right hand screw rule

Radial component $\vec{\omega} \times \vec{v}$

Tangential component $\vec{\alpha} \times \vec{r}$

Magnitude Since angle between ω and v, α and r is π/2 and v = ωr

$$a = \sqrt{(\omega^2 r)^2 + \alpha^2 r^2}$$

If resultant acceleration makes an angle β with Op, the

$$tan\beta = \frac{\alpha r}{\omega^2 r}$$

If the rigid body is rotating with constant angular velocity, that is, its angular acceleration α=0, then the tangential component of its linear acceleration becomes zero, but radial component remains non-zero. This condition is found in the uniform circular motion

Solved Numerical

Q7) Grind stone is rotating about axis have radius 0.24 m its angular acceleration is 3.2 rad/s^2. (a)Starting from rest what will be angular velocity after 2.7 s (b) Linear tangential speed of point of the rim (c) the tangential acceleration of a point on the rim and (d) the radial acceleration of point on the rim at the end of 2.7 s.

Solution

(a) Angular speed after 2.7s

$$\omega = \omega_0 + \alpha t$$

ω =0 + 3.2 × 2.7 = 8.6 rad/s = 1.4/s

(b) Linear tangential speed

V_t = ωr

V_t = 8.6 × 0.24 = 2.1 m/s

(c) Tangential acceleration

a_t = αr

$a_t = 3.2 \times 0.24 = 0.77$ m/s^2

(d) $a_r = \omega^2 r$

$a_r = (8.6)^2 \times 0.24 = 18$ m/s^2

7.12 Moment of force

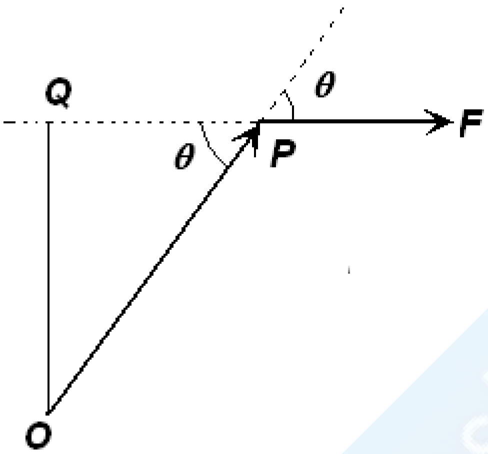
To translate a body, we need to apply a force on a body. i.e. cause of translation is force and it is related with linear acceleration of the body as **F** = M**a**

But for rotation, not only the magnitude of the force but its line of action and point of application is also important. Turning effect of the force depends on

(i) Magnitude of the force

(ii) Direction of the force

(iii) The distance of force from the axis of rotation



Taking consideration of all these we define the torque of a force which gives measure of a turning effect of a force Consider a force F acting on a body at point P, then turning effect of this force, torque, about point O is defined as

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Where r is vector joining O to any point on the line of action of force

$$\tau = rF\sin\theta\hat{n}$$

Here $\hat{n}$ is a unit vector perpendicular to plane formed by r and F and direction is given by right hand screw rule

Magnitude is given by $\tau = rF\sin\theta$

$\tau = F(OQ)$

Hence torque about a point can also be calculated by multiplying force with the perpendicular distance from the point on the line of action of the force. Direction of torque can be obtained by the definition of cross product.

To calculate torque of a force about an axis, we consider a point on the axis and then we define $\vec{\tau} = \vec{r} \times \vec{F}$ about point O

The component of vector **τ along** the axis gives the torque about the axis. If force is parallel to the axis or intersects the axis, its torque about the axis becomes zero. If a force is perpendicular to the axis, we calculate torque as product of magnitude of force and perpendicular distance of line of action of force from axis.

Torque acting on the system of particles:

The mutual internal forces between the particles of system are equal and opposite, the resultant torques produced due to them becomes zero. Hence we will not consider the internal force in our discussion

Suppose for a system of particles the position vectors of different particles are $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, … $\mathbf{r}_n$. and respective forces acting on them are $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$,……, $\mathbf{F}_n$

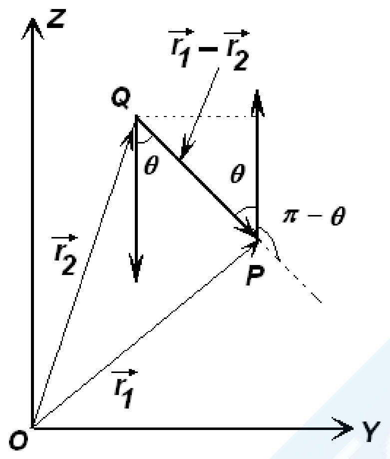
The resultant torque on the system means the vector sum of the torque acting on every particle of the system

$$\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 + \cdots + \vec{\tau}_n$$

Resultant torque

$$\tau = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i$$

7.13 Couple



Two forces of equal magnitude and opposite directions which are not collinear form couple. A shown in figure F_1 and F_2 act on two particles P and Q of the rigid body having position vectors r_1 and r_2 respectively. Here $|F_1| = |F_2|$ and the directions are mutually opposite. The resultant torques τ_1 and τ_2 produce moment of couple τ

$$\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2$$

$$\vec{\tau} = (\vec{r}_1 \times \vec{F}_1) + (\vec{r}_2 \times \vec{F}_2)$$

$$\vec{\tau} = (\vec{r}_1 \times \vec{F}_1) - (\vec{r}_2 \times \vec{F}_1)$$

As $F_2 =- F_2$

$$\vec{\tau} = (\vec{r}_1 - \vec{r}_1) \times \vec{F}_1$$

$$\vec{\tau} = |\vec{r}_1 - \vec{r}_1|(F_1) sin(\pi - \theta)$$

Where $(\pi - \theta)$ is the angle between ($\mathbf{r}_1 - \mathbf{r}_2$) and F_1

$$\vec{\tau} = |\vec{r}_1 - \vec{r}_1|(F_1) sin\theta$$

From figure $|r_1 - r_2|\sin\theta$ = perpendicular distance between the two forces

∴ Moment of couple = (magnitude of any one of the two forces) (perpendicular distance between the forces)

Equilibrium

If the external forces acting on a rigid body are $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$… $\mathbf{F}_n$. and if resultant force

$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 +$ ……. $+ \mathbf{F}_n = 0$, then rigid body remains in translational equilibrium.

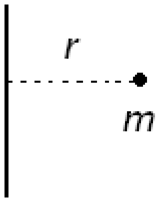
If the torques produced by the above mentioned forces are $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, $\boldsymbol{\tau}_3$… $\boldsymbol{\tau}_n$. Then the rigid body remains in rotational equilibrium when $\tau = \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 + \boldsymbol{\tau}_3 +$ ……$+ \boldsymbol{\tau}_n = 0$

That is if rigid body is stationary, it will remain stationary and if it is performing rotational motion, it will continue rotational motion with constant angular velocity.

7.14 Moment of inertia

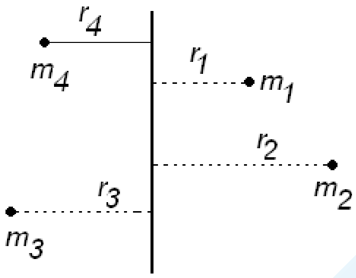
The state of motion of a body can undergo change in rotation if torque is applied. The resulting angular acceleration depends partly on the magnitude of the applied torque, however the same torque applied different bodies produce different angular acceleration, indicating that each body has an individual amount of rotational inertia is called moment of inertia and it is represented by I.

The moment of inertia of body is a function of the mass of the body, the distribution of the mass and the position of the axis of rotation



Consider a particle of mass m situated at a distance r from the axis as shown in the figure. Its moment of inertia I is defined as

$I = mr^2$



If a system of particles is made of number of particles of masses $m_1, m_2, m_3, \ldots, m_n$ at distances $r_1, r_2, r_3, \ldots, r_n$ from the axis of rotation its momentum of inertia is defined as

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \cdots + m_n r_n^2$$

$$I = \sum_{i=1}^{n} m_i r_i^2$$

7.15 Moment of inertia of continuous body

For calculating moment of inertia of continuous body, we first divide the body into suitably chosen infinitesimal elements. The choice depends on symmetry of the body. Consider an element of the body at a distance r from the axis of rotation. The moment of inertia of this element about the axis can be defined as $(dm)r^2$ and the discrete sum over particles becomes integral over the body $I = \int (dm) r^2$

Radius of gyration:

Suppose a rigid body has mass M. It is made up of n particles each having mass m

$\therefore m_1 = m_2 = \ldots = m_n = m$

$\therefore M = nm$

As shown in figure, the moment of inertia of the body the given axis

$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \cdots + m_nr_n^2$$

Here r_1, r_2, r_3 , ….,r_n are perpendicular distances of the respective particles of the body from the axis

$$I = mr_1^2 + mr_2^2 + mr_3^2 + \cdots + mr_n^2$$

$$I = \frac{nm(r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2)}{n}$$

$$I = MK^2$$

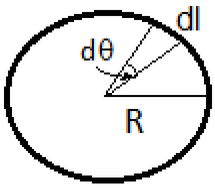
Where

$$K^2 = \frac{r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2}{n}$$

$$K = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2}{n}}$$

K^2 show the mean (average) value of the squares of the perpendicular distance of the particles of the body from the axis. K is called the radius of gyration of the body about the given axis. SI unit is 'm'

7.15.01 Moment of inertia of circular ring about axis passing through Centre of mass and perpendicular to plane of ring



Let M be mass of ring having radius R, then mass per unit length $\lambda = M/2\pi R$

Length of small segment $dl = Rd\theta$.

Mass of small segment $dm = \lambda\, dl$

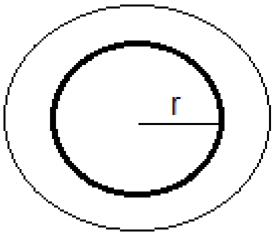
$$dm = \frac{M}{2\pi R} Rd\theta = \frac{M}{2\pi} d\theta$$

Momentum of inertia of small element about center $dI = dmR^2$

$$dI = \frac{MR^2}{2\pi} d\theta$$

$$I = \frac{MR^2}{2\pi} \int_0^{2\pi} d\theta = MR^2$$

7.15.02 Moment of inertia of disc about axis passing through Centre of mass and perpendicular to plane of disc



Let M be the mass of disc having radius R.

Mass per unit area $\sigma = \frac{M}{\pi R^2}$

Disc can be imagined as composed of concentric rings of thickness dr.

One such ring is show in figure. Let its radius be r and mass dm.

Volume of ring = $2\pi rdr$

Mass of ring = Volume × density

Mass of ring $dm = \frac{M}{\pi R^2} 2\pi \mathrm{rdr} = \frac{2M}{R^2} rdr$

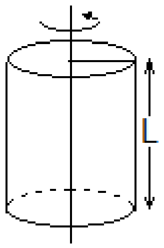
We know that moment of inertia of ring dmr^2

Thus moment of inertia of disc

$$I = \frac{2M}{R^2}\int_0^R rdrr^2 = \frac{2M}{R^2}\int_0^R r^3 dr$$

$$I = \frac{2M}{R^2}\left[\frac{r^4}{4}\right]_0^R = \frac{1}{2}MR^2$$

7.15.03 Moment of inertia of hollow cylinder about geometric axis



Consider a hollow cylinder of radius R and length L, and mass M

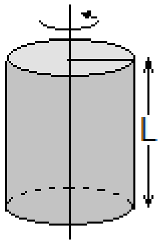
Hollow cylinder may be consider as stack of rings. If mass of each ring is dm then moment of inertia is dmR^2

If we integrate over the length as R^2 is constant integration of dm is M

Thus moment of inertia of hollow cylinder is same as ring

$$I = MR^2$$

7.15.04 Moment of inertia of Solid cylinder about geometric axis



Consider a solid cylinder of radius R and length L, and mass M

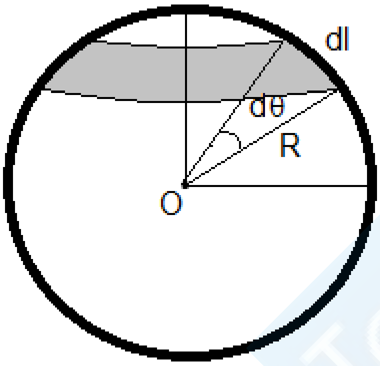
Solid cylinder may be consider as stack of rings. If mass of each ring is dm then moment of inertia is $\frac{1}{2}dmR^2$

If we integrate over the length as R^2 is constant integration of dm is M

Thus moment of inertia of solid cylinder is same as ring

$$I = \frac{1}{2}MR^2$$

7.15.05 Moment of inertia of hollow sphere about diameter



Let M be the mass of hollow sphere having radius R, rotating around the diameter. In figure axis is passing through O and is perpendicular to plane of paper. Surface density of mass $\frac{M}{4\pi R^2}$

Consider a very small element of length dl will trace a ring of radius r shown in figure have area A = dl×2πr , have mass $dm = \frac{M}{4\pi R^2}dl \times 2\pi r = \frac{M}{4\pi R^2}Rd\theta \times 2\pi r$

Moment of inertia of ring dmr^2

$$I = \int_0^{\pi/2} \frac{M}{4\pi R^2} Rd\theta \times 2\pi r \times r^2$$

$$I = \int_0^{\pi/2} \frac{M}{2R} d\theta \times r^3$$

Ring makes an angle of θ with O, then radius r= Rsinθ

Now ring makes angle of 0 to π/2 with center for upper hemisphere. For full sphere it will be twice of integration over 0 to π/2

$$I = 2\int_0^{\pi/2} \frac{M}{2R} d\theta d\theta R^3\, sin^3\theta$$

$$I = 2 \times \frac{MR^2}{2}\int_0^{\pi/2} sin^3\theta d\theta$$

As $\sin^3\theta = (1-\cos^2\theta)\sin\theta$

$$I = 2 \times \frac{MR^2}{2}\int_0^{\pi/2} (1 - cos^2\theta) sind\theta$$

Let cosθ = t

dt =sinθdθ

$$I = 2 \times \frac{MR^2}{2}\int_0^{\pi/2} (1 - t^2) dt$$

$$I = 2 \times \frac{MR^2}{2}\left[t - \frac{t^3}{3}\right]_0^{\pi/2}$$

$$I = 2 \times \frac{MR^2}{2}\left[cos\theta - \frac{cos^3\theta}{3}\right]_0^{\pi/2}$$

As cos(π/2)=0

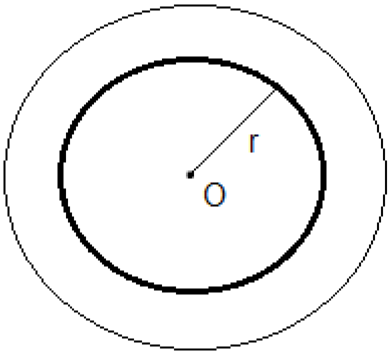
$$I = 2 \times \frac{MR^2}{2}\left[cos0 - \frac{cos^3 0}{3}\right]$$

$$I = 2 \times \frac{MR^2}{2}\left[1 - \frac{1}{3}\right]$$

$$I = 2 \times \frac{MR^2}{2}\left[\frac{2}{3}\right] = \frac{2}{3}MR^2$$

$$I = \frac{2}{3}MR^2$$

7.15.06 Moment of inertia of solid sphere about diameter



Let M be the mass of solid sphere having radius R, rotating around the diameter. In figure axis is passing through O and is perpendicular to plane of paper. Volume density of sphere is

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$$

Solid sphere may be considered as composition of concentric small hollow sphere. Consider a hollow sphere as shown in figure of radius r and thickness dr. mass of sphere dm= volume × density.= $4\pi r^2 \times dr \times \rho$

$$dm = 4\pi r^2 dr \times \frac{3M}{4\pi R^3} = \frac{3M}{R^3} r^2 dr$$

Moment of inertia of hollow sphere =

$$dI = \frac{2}{3} dmr^2$$

By integrating over radius 0 to R we get I of solid sphere

$$I = \int_0^R \frac{2}{3} dmr^2$$

$$I = \frac{2}{3}\int_0^R r^2 dm = \frac{2}{3}\int_0^R r^2 \frac{3M}{R^3} r^2 dr$$

$$I = \frac{2M}{R^3}\int_0^R r^4 dr$$

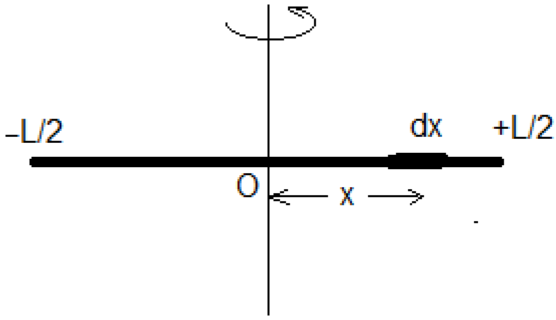
$$I = \frac{2M}{R^3}\int_0^R r^4 dr$$

$$I = \frac{2M}{R^3}\left[\frac{r^5}{5}\right]_0^R$$

$$I = \frac{2M}{R^3}\frac{R^5}{5}$$

$$I = \frac{2}{5}MR^2$$

7.15.07 Moment of inertia of rod about axis passing through center and perpendicular to length



Let length of thin rod be L and mass be M, rotating around axis perpendicular to length L and passing through center O. Liner mass density of rod λ= M/L

Consider a small segment of length dx , Mass of segment dm =Mdx/L

Moment of inertia of segment $dI \;= \frac{M}{L}x^2dx$

By integrating dm over −L/2 to +L/2 we get

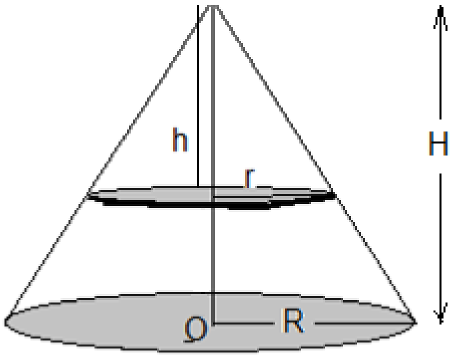
$$I = \int_{-L/2}^{+L/2}\frac{M}{L}x^2dx \;= \frac{M}{L}\int_{-L/2}^{+L/2}x^2dx$$

$$I = \frac{M}{L}\left[\frac{x^3}{3}\right]_{-L/2}^{+L/2}$$

$$I = \frac{M}{3L}\left[\frac{L^3}{8}+\frac{L^3}{8}\right]$$

$$I = \frac{1}{12}ML^2$$

7.15.08 Moment of inertia of solid cone about axis passing through vertex and perpendicular to base of radius R and height H



Let M be the mass of cone and H and R bet the height and radius of cone.

Volume of cone

$$V = \pi R^2 \frac{H}{3}$$

Volume density of cone ρ

$$\rho = \frac{M}{V} = \frac{3M}{\pi R^2 H}$$

Consider a segmental disc of radius r and thickness dh mass of disc

$$dm = V \times \rho = \pi r^2 dh \times \rho$$

$$dm = \pi r^2 dh \times \frac{3M}{\pi R^2 H} = r^2 dh \times \frac{3M}{R^2 H} \dots (i)$$

Moment of inertia of disc about axis perpendicular to plane

$$dI = \frac{1}{2} dm \times r^2 \dots . (ii)$$

Substituting (i) in (ii)

$$dI = \frac{1}{2} r^2 dh \times \frac{3M}{R^2 H} \times r^2$$

$$dI = \frac{3}{2} \frac{M}{R^2 H} r^4 dh \dots (iii)$$

Now

$$\frac{h}{r} = \frac{H}{R}$$

$$r = \frac{R}{H} h \dots . (iv)$$

Substituting (iii) in (iv)

$$dI = \frac{3}{2} \frac{M}{R^2 H} \left(\frac{R}{H} h\right)^4 dh$$

$$dI = \frac{3}{2}\frac{MR^2}{H^5}(h)^4 dh$$

Integrating from 0 to H

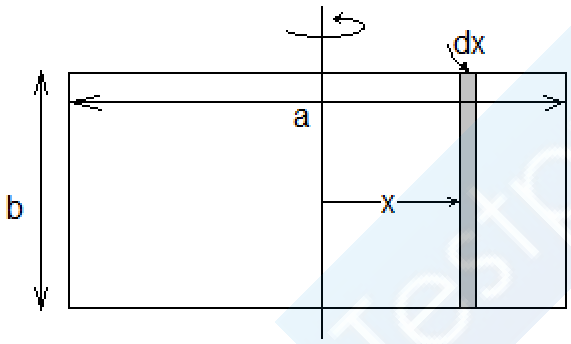
$$I = \frac{3}{2}\frac{MR^2}{H^5}\int_0^H (h)^4 dh$$

$$I = \frac{3}{2}\frac{MR^2}{H^5}\left[\frac{h^5}{5}\right]_0^H$$

$$I = \frac{3}{2}\frac{MR^2}{H^5}\frac{H^5}{5}$$

$$I = \frac{3}{10} MR^2$$

7.15.09 Moment of inertia of rectangular plate about axis passing through center of mass and parallel to plane



Let mass of the plate be M, sides are a and b as shown in figure.

Surface mass density $\sigma = \frac{M}{ab}$

Consider a strip element of thickness dx and height b, at position of x from axis

Mass of the strip dm= area ×density

$$dm = bdx \times \frac{M}{ab} = \frac{M}{a}dx$$

Moment of inertia dI = dmx^2 (as every point of strip is at distance x)

On integrating from −a/2 to a/2 moment of inertia along axis parallel to b

$$I_b = \int_{-a/2}^{a/2} \frac{M}{a} x^2 dx$$

$$I_b = \frac{M}{a}\int_{-a/2}^{a/2} x^2 dx$$

$$I_b = \frac{M}{a}\left[\frac{x^3}{3}\right]_{-a/2}^{a/2}$$

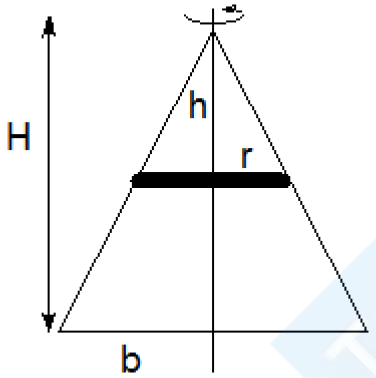
$$I_b = \frac{M}{3a}\left[\frac{a^3}{8} + \frac{a^3}{8}\right]$$

$$I_b = \frac{M}{12}a^2$$

Similarly

$$I_a = \frac{M}{12}b^2$$

7.15.10 Moment of inertia of triangle axis of rotation passing through vertices, and perpendicular to base



Let M be the mass of triangular plate having base b and height H.

Surface mass density of triangle is σ= M/A

$$\sigma = \frac{M}{\frac{1}{2}bH} = \frac{2M}{bH}$$

Consider elemental tin rod of length r and thickness dh as shown in figure area of element =2rdh

Mass of element $dm = 2rdh \times \frac{2M}{bH}$

Momentum of inertia of thin road

$$dI = \frac{1}{12}dm(2r)^2 = \frac{1}{3}dmr^2$$

$$dI = \frac{1}{3}2rdh \times \frac{2M}{bH}r^2$$

Integrating from 0 to H

$$I = \int_0^H \frac{4}{3}\frac{M}{bH} r^3 dh \ldots (i)$$

Now from figure

$$\frac{h}{r} = \frac{H}{b}$$

$$r = \frac{b}{H} h \ldots (ii)$$

Substituting (ii) in (i)

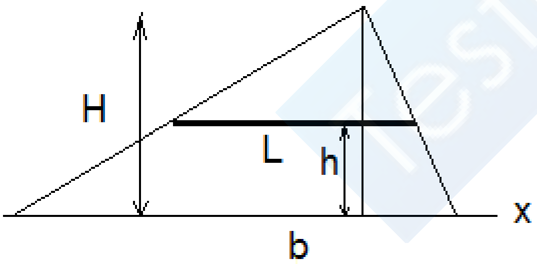
$$I = \int_0^H \frac{4}{3}\frac{M}{bH}\left(\frac{b}{H}h\right)^3 dh$$

$$I = \frac{4}{3}\frac{Mb^2}{H^4}\int_0^H h^3\, dh$$

$$I = \frac{4}{3}\frac{Mb^2}{H^4}\left[\frac{h^4}{4}\right]_0^H$$

$$I = \frac{1}{3}Mb^2$$

7.15.11 Moment of inertia of triangle axis of rotation passing through base



Let M be the mass of triangle and height H.

Surface mass density of triangle is σ= M/A

$$\sigma = \frac{M}{\frac{1}{2}bH} = \frac{2M}{bH}$$

Consider a elemental thin rod of length L and thickness dh. Area of rod =Ldh

Mass of rod $dm = Ldh \times \frac{2M}{bH}$

As every small mass of thin rod is at equal distance of (H-h) thus moment of inertia of rod $dI=dm(H-h)^2$

Integrating over o to H

$$I = \int_0^H dmh^2$$

$$I = \int_0^H Ldh \times \frac{2M}{bH} h^2 \ldots (i)$$

Now

$$\frac{L}{b} = \frac{H-h}{H}$$

$$L = b\frac{H-h}{H} \ldots (ii)$$

Substituting (ii) in (i)

$$I = \int_0^H b\frac{H-h}{H} dh \times \frac{2M}{bH} (h)^2$$

$$I = \frac{2M}{H^2} \int_0^H (H-h)dh \times (h)^2$$

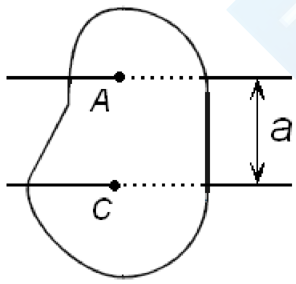
$$I = \frac{2M}{H^2} \int_0^H (Hh^2 - h^3)dh$$

$$I = \frac{2M}{H^2} \left[\frac{H^4}{3} - \frac{H^4}{4}\right]$$

$$I = \frac{MH^2}{6}$$

7.16 CHANGE OF AXIS

(i)The parallel axis theorem

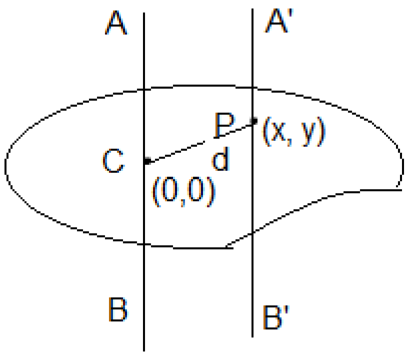


If the moment of inertia of uniform body of mass m about an axis through c, its centre of mass is I_C and I_A is the moment of inertia about a parallel axis through a point A, then

$I_A = I_C + Ma^2$

Where a is the distance between the parallel axes

Proof of theorem



Consider a planar body as shown in figure. Let C be center of mass and AB is axis passing through C which is along z-axis.

Now moment of inertia of point Q about AB is dI= dm×r^2

But perpendicular distance is $r^2 = x^2 + y^2$

Thus dI =dm (x^2+y^2)

Body is composed of many such particles

Thus moment of inertia of body along axis AB = I_C

$$I_C = \int dm(x^2 + y^2)$$

Now if we consider the moment of inertia along axis A'B'

Consider now point P at position (x',y') from origin. Then perpendicular distance of point Q from A'B' = $d^2 = (x' - x)^2 + (y' - y)^2$

Thus moment of inertia about axis A'B'

$$I_{AB} = \int dm[(x' - x)^2 + (y' - y)^2]$$

$$I_{AB} = \int dm(x^2 + y^2 + x'^2 + y'^2 - 2xx' - 2yy')$$

As x' and y' are constant

$$I_{AB} = \int dm(x^2 + y^2) + \int dm(x'^2 + y'^2) - 2x' \int dmx\ - 2y' \int dmy$$

Now first term

$$\int dm(x^2 + y^2)$$

Represent moment of inertia along AB = I_C

Second term

$$\int dm(x'^2 + y'^2)$$

Here x' and y' are the coordinates of point P through which axis of rotation AB pass and is fix

We know that d^2 = x'^{2} + y'^{2} thus integration will give Md^2

Third term

$$2x' \int dmx$$

Here x is the distance from center of mass, and coordinates of center of mass is (0,0) thus integration will be zero as x_{CM} =0

Fourth term

$$2y' \int dmy$$

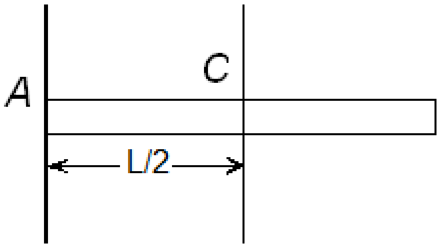
Here y is the distance from center of mass, and coordinates of center of mass is (0,0) thus integration will be zero as y_{CM} =0

From above discussions

$I_{AB} = I_C + Md^2$

Hence proved

7.16.01 Moment of inertia of rod about axis passing through one end perpendicular to length



M be a mass of thin rod and length L. Axis of rotation is passing through point A.

Point C is center of mass of rod. And moment of inertia about axis passing through C.M is

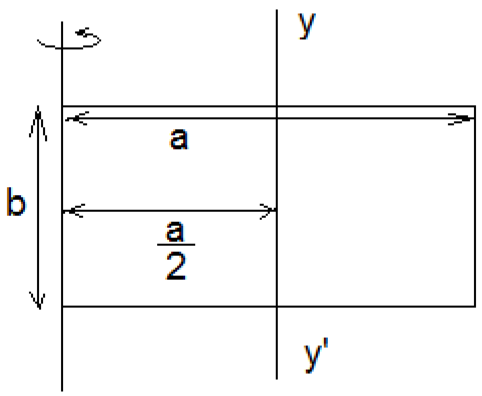
$$I_C = \frac{ML^2}{12}$$

Now by parallel axis theorem $I_A = I_C + M(L/2)^2$

$$I_A = \frac{ML^2}{12} + \frac{L^2}{4}$$

$$I_A = \frac{1}{3} ML^2$$

7.16.02 Moment of inertia of rectangle about axis passing through one side



As shown in figure axis of rotation is passing through side b, Mass of plate is M

Then moment of inertia of plate along axis passing through center of mass as proved

$$I_b = \frac{M}{12}a^2$$

Note that subscript b indicate axis is parallel to side b

Distance between yy' and b is a/2

Thus moment of inertia about axis passing through side b is

$$I = I_b + M\left(\frac{a}{2}\right)^2$$

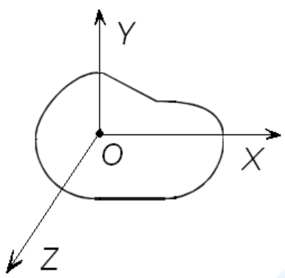
$$I = \frac{M}{12}a^2 + M\left(\frac{a}{2}\right)^2$$

$$I = \frac{1}{3}Ma^2$$

Similarly if axis of rotation is passing through side a the

$$I = \frac{1}{3}Mb^2$$

(ii)The perpendicular axis theorem

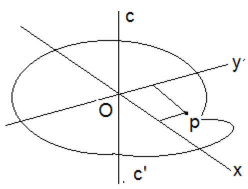


If a plane body has moments of inertia I_{OX} and I_{OY} about two perpendicular axes, OX and OY, in the plane of the body then its moment of inertia about an axis OZ perpendicular to the plane is $I_{OZ} = I_{OX} + I_{OY}$

The three axes under consideration must be mutually perpendicular and concurrent, although they need to pass through the centre of mass of the body

This theorem cannot be applied to three dimensional bodies

Proof



Perpendicular distance of point p of mass dm from x axis is x then moment of inertia about point p about x axis is dmx^2 since planer object consisting of many point total moment of inertia about x axis is $I_X = \int dmx^2$

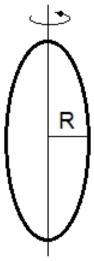
Similarly moment of inertia about y axis is $I_Y = \int dmy^2$

Now perpendicular distance of point p from z axis cc' is $x^2 + y^2$

Thus moment of inertia about Z axis $I_Z = \int dm(x^2+y^2) = \int dmx^2 + \int dmy^2$

$I_Z = I_x + I_y$ hence proved

7.16.03 Moment of inertia of ring about axis passing through any diameter



Let mass of ring is M and radius, rotate about the axis passing through any diameter.

For any two axis passing through two diameter, moment of inertia will be same

Thus $I_X = I_Y = I$

We know that moment of inertia about axis perpendicular to plane of ring is MR^2

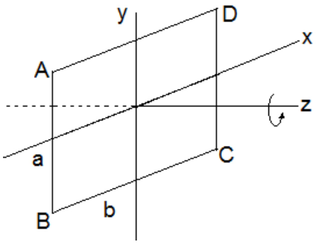
Now by perpendicular axis theorem

$I_Z = I_x + I_Y$

$MR^2 = 2I$

$$I = \frac{1}{2}MR^2$$

7.16.04 Moment of inertia of rectangular plate about axis perpendicular to plane and passing through any diameter



Let M be the mass, a and b sides of rectangular plate

Moment of inertia about x axis which is parallel to side of length b is

$$I_x = \frac{M}{12}a^2$$

Moment of inertia about y axis which is parallel to side of length a is

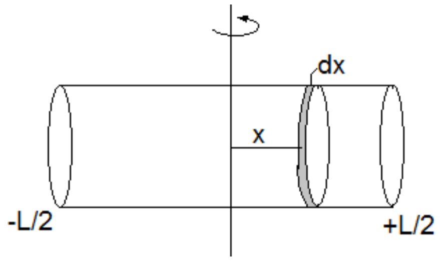
$$I_x = \frac{M}{12}b^2$$

Now according to perpendicular axis theorem

$$I_Z = \frac{M}{12}a^2 + \frac{M}{12}b^2$$

$$I_Z = \frac{M}{12}(a^2 + b^2)$$

7.16.05 Moment of inertia axis passing through its center and perpendicular to its length



Let M be the mass of cylinder , L is length and R is radius

Volume mass density

$$\rho = \frac{M}{\pi R^2 L}$$

Consider a disc of thickness dx at a distance of x from the center of mass

Mass of disc dm= volume × ρ

$$dm = \pi R^2 dx \frac{M}{\pi R^2 L} = \frac{M}{L}dx$$

Now

Moment of inertia of disc along diameter

$$dI = \frac{1}{4}\left(\frac{M}{L}dx\right)R^2$$

By paralle axis theorem moment of inertia about axis of rotation is

$$dI' = \frac{1}{4}\left(\frac{M}{L}dx\right)R^2 + \left(\frac{M}{L}dx\right)\left(\frac{x}{2}\right)^2$$

Now integrating fro –L/2 to +L/2

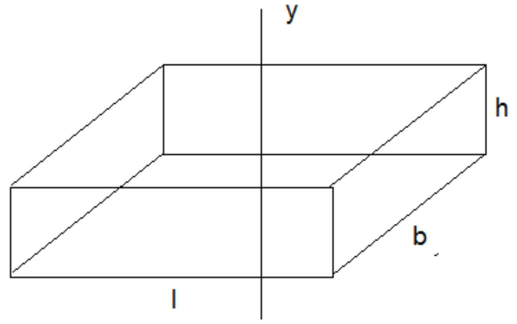
$$I = \int_{-L/2}^{L/2}\left\{\frac{1}{4}\left(\frac{M}{L}dx\right)R^2 + \left(\frac{M}{L}dx\right)\left(\frac{x}{2}\right)^2\right\}$$

$$I = \int_{-L/2}^{L/2}\frac{1}{4}\left(\frac{M}{L}dx\right)R^2 + \int_{-L/2}^{L/2}\left(\frac{M}{L}dx\right)\left(\frac{x}{2}\right)^2$$

$$I = \frac{MR^2}{4L}[x]_{-L/2}^{L/2} + \frac{M}{4L}\left[\frac{x^3}{3}\right]_{-L/2}^{L/2}$$

$$I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$$

7.16.06 Moment of inertia of block



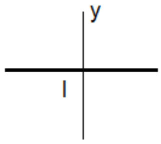
Let M be mass of block of length l, breadth b and height h

Axis Y is passing through the center of mass.

We can use a simple trick to solve this problem **(Note it is applicable to block only, not other shape like cone, prism)**

Momentum of inertia can be added algebraically

If we make h = 0 and b=0, but mass is same then we get is simple thin rod as shown in figure

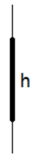


axis is perpendicular passing through center of mass and moment of inertia is

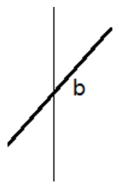
$$I = \frac{M}{12}(l^2) \dots (i)$$

If we make l=0 ,b=0 then object will be like rod along z moment of inertia =0

As shown in figure



If we make h = 0 and l =0, we get rod with length b moment of inertia as shown in figure

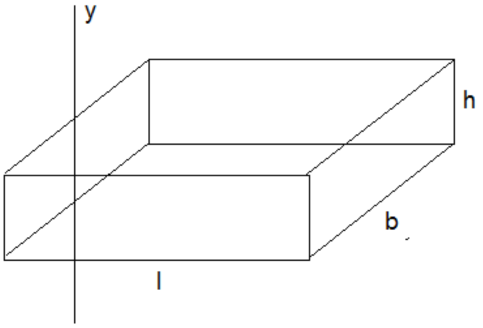


$$I = \frac{M}{12}(b^2) \dots (ii)$$

Thus total is (i)+(ii)

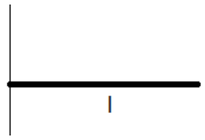
$$I = \frac{M}{12}(l^2 + b^2)$$

Now consider axis is passing through center of side b



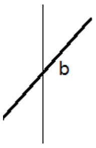
As state earlier we will compress the shape

If b= 0 and h =0, we get thin rod of length l and axis of rotation is at end as shown in figure below



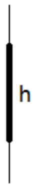
Moment of inertia is $I = \frac{1}{3}Ml^2$

If l=0 and h =0, we get thin rod of length b and axis of rotation is passing through its center as shown in figure



Moment of inertia is $I = \frac{1}{12}Mb^2$

If we make l= 0 and b=0 then we get rod along zxis of rotation having length =h as shown in figure



Moment of inertia is zero

Total momentum (i)+(ii)

$$I = \frac{1}{3}Ml^2 + \frac{1}{12}Mb^2$$

$$I = \frac{M}{12}(4l^2 + b^2)$$

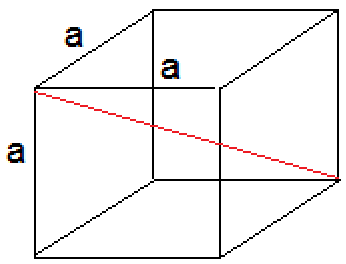
7.17 TABLE OF MOMENT OF INERTIA

Body	Dimension	Axis	Moment of inertia
Circular ring	Radius r	Through its centre and perpendicular to its plane	Mr^2
Circular disc	Radius r	Through its centre and perpendicular to its plane	$\frac{Mr^2}{2}$
Right circular solid cylinder	Radius r and length l	About geometrical axis	$\frac{Mr^2}{2}$
Solid cylinder	Radius r and length l	Through its centre and perpendicular to its length	$M\left[\frac{r^2}{4}+\frac{l^2}{12}\right]$
Uniform solid sphere	Radius R	About a diameter	$\frac{2}{5}MR^2$
Hollow sphere	Radius R	About diameter	$\frac{2}{3}MR^2$
Thin uniform rod	Length 2l	Through its centre and perpendicular to its length	$\frac{Ml^2}{3}$
Thin rectangular sheet	Side a and b	Through its cenre and perpendicular to its plane	$M\left[\frac{a^2}{12}+\frac{b^2}{12}\right]$
Circular disc	Radius R	Through any diameter	$\frac{1}{4}MR^2$
Hollow cylinder	Radius R	Geometrical axis	MR^2

Solved Numerical

Q7) A cube is fixed in sphere of radius R, such that all vertices of cube are on the surface of sphere of mass M. Find moment of inertia of cube around the axis passing through its center of mass and perpendicular to its face

Solution



As shown in figure diagonal shown in red is =2R.

Now diagonal of cube 2R =a√3

Thus $a = \frac{2}{\sqrt{3}}R$

Now mass volume density of sphere = $\frac{M}{V}$

$$\rho = \frac{3M}{4\pi R^3}$$

Mass of cube = Volume ×ρ

$$M' = a^3 \times \frac{3M}{4\pi R^3}$$

$$M' = \left(\frac{2}{\sqrt{3}}R\right)^3 \times \frac{3M}{4\pi R^3}$$

$$M' = \frac{2}{\sqrt{3}} \times \frac{M}{\pi}$$

From the formula

$$I = M'\left[\frac{a^2}{12} + \frac{a^2}{12}\right]$$

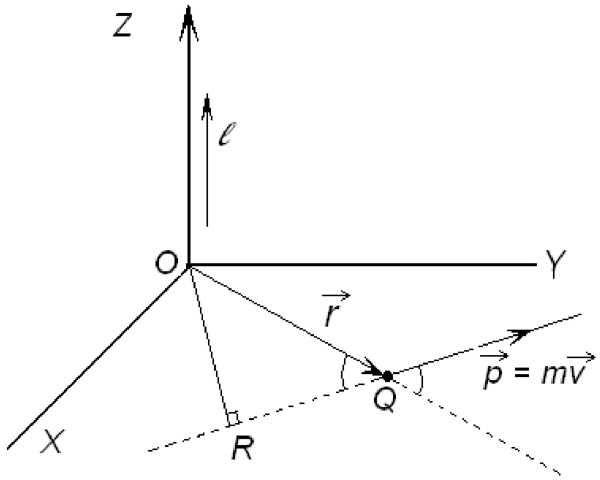
$$I = \frac{2}{\sqrt{3}} \times \frac{M}{\pi}\left[\frac{a^2}{6}\right]$$

$$I = \frac{a^2}{3\sqrt{3}} \times \frac{M}{\pi}$$

7.18 Angular momentum of a particle

The angular momentum in a rotational motion is similar to the linear momentum in translational motion. The linear momentum of a particle moving along a straight line is the product of its mass and linear velocity (i.e) $p = mv$.

The angular momentum of a particle is defined as the moment of linear momentum of



the particle. Let us consider a particle of mass m moving in the *XY* plane with a velocity *v* and linear momentum *p* =*m v* at a distance *r* from the origin The angular momentum L of the particle about an axis passing through O perpendicular to XY plane is defined as the cross product of *r* and *p*

(i.e) *L* =*r* ×*p*

Its magnitude is given by *L* = *r p sin* θ

where θ is the angle between *r* and *p* and L is along a direction perpendicular to the plane containing *r* and *p*

The unit of angular momentum is

kg m^2 s^{-1} or Js

Its dimensional formula is, M L^2 T^{-1}.

Angular momentum of a rigid body

Let us consider a system of n particles of masses m_1, m_2 ….. m_n situated at distances r_1, r_2, …..r_n respectively from the axis of rotation . Let v_1,v_2, v_3 ….. be the linear velocities of the particles respectively, then linear momentum of first particle = m_1v_1.

Since v_1= r_1ω the linear momentum of first particle = $m_1(r_1\ \omega)$

The moment of linear momentum of first particle = linear momentum × perpendicular distance = $(m_1r_1\omega) \times r_1$

angular momentum of first particle = $m_1r_1^2\omega$

Similarly,

angular momentum of second particle = $m_2r_2^2\omega$

angular momentum of third particle = $m_3r_3^2\omega$ and so on.

The sum of the moment of the linear momentum of all the particles of a rotating rigid body taken together about the axis of rotation is known as angular momentum of the rigid body.

∴ Angular momentum of the rotating rigid body = sum of the angular momentum of all the particles.

$$L = m_1r_1^2\omega + m_2r_2^2\omega + m_3r_3^2\omega + \cdots + m_nr_n^2\omega$$

$$L = (m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \cdots + m_nr_n^2)\omega$$

$$L = \omega\sum_{i=1}^{n} m_ir_i^2$$

$$L = I\omega$$

7.19 Relation between angular momentum of a particle and torque on it

By definition angular momentum

$$\vec{l} = \vec{r} \times \vec{p}$$

Differentiating equation with time, we get

$$\frac{d\vec{l}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p}$$

But

$$\frac{d\vec{p}}{dt} = \vec{F} \quad and \quad \frac{d\vec{r}}{dt} = \vec{v}$$

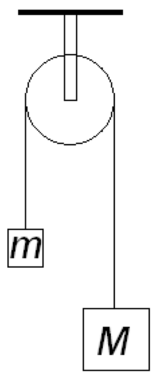
$$\frac{d\vec{l}}{dt} = \vec{r} \times \vec{F} + \vec{v} \times \vec{p}$$

But v and p are the same direction thus $\mathbf{v}\times\mathbf{p} = 0$

$$\frac{d\vec{l}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

Thus the time rate of change of angular momentum is equal to torque.

If total torque on a particle is zero the angular momentum of the particle is conserved. This is known as principle of conservation of angular momentum. Angular momentum of a rotating rigid body about an axis having pure rotation can be written as $L = I\omega$



Solved numerical

Q8) The pulley shown in figure has moment of inertia I about axis and radius R. Find the acceleration of the two blocks. Assume that the string is light and does not slip on the pulley

Solution

Suppose the tension in right string is T_1 and tension in left side string is T_2 . Suppose block M is going down with acceleration a and the other block is going up with same acceleration . This is also tangential acceleration of the rim of the wheel as the string does not slip over the rim of the wheel as the string does not slip over the rim

Tangential acceleration of pulley

$a = \alpha R$

Thus $\alpha = a/R$ -----eq(1)

The equations for mass M

$Ma = Mg - T_1$

$T_1 = M(g-a)$ ------eq(2)

Equation for mass m

$ma = T_2 - mg$ --------eq(3)

$T_2 = m(a-g)$

Torque $\tau = I\alpha$

Also $\tau = (T_1 - T_2)R$

Thus $(T_1 - T_2)R = I\alpha$

Substituting values of T_1 T_2 and α from equation 1, 2, 3 we get

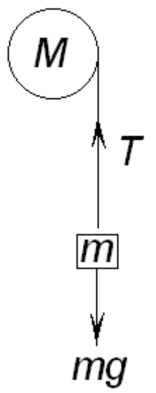
$$[M(g-a) - m(g+a)]R = \frac{Ia}{R}$$

Solving we get

$$a = \frac{(M-m)gR^2}{I + (M+m)R^2}$$

Q9) A string is wound around a disc of radius r and mass M and m is suspended. The body is then allowed to descend. Show that the angular acceleration of disc is

$$\alpha = \frac{mg}{R\left(m + \frac{M}{2}\right)}$$



Solution:

The equation for suspended body is ma = mg – T

T = m(g-a)

Acceleration of body is the tangential acceleration of disc thus

a = αR

Thus

T = m(g-a)

$$T = m(g - \alpha R)$$

This tension produces torque

$\tau = F \times R$

$\tau = m(g - \alpha R)R$

Now torque $\tau = I\alpha$

Thus

$$I\alpha = m(g - \alpha R)R$$

$$I\alpha + m\alpha R^2 = mgR$$

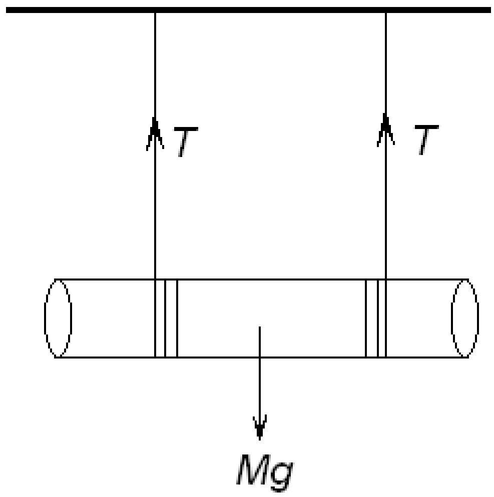
$$\alpha = \frac{mgR}{(I + mR^2)}$$

But I for disc = $MR^2/2$

$$\alpha = \frac{mgR}{\left(\frac{MR^2}{2} + mR^2\right)}$$

$$\alpha = \frac{gm}{\left(\frac{M}{2} + m\right)R}$$

Q10) A cylinder of M is suspended through two string wrapped around it as shown in figure. Fin the tension in the string and the speed of the cylinder as it falls through a distance h



Solution

The portion of the string between the ceiling and the cylinder are at rest. Hence the points of the cylinder where the strings leave it are at rest. The cylinder is thus rolling without slipping on the strings. Suppose the centre of the cylinder falls with an acceleration a. The angular acceleration of the cylinder about its own axis given by

ɑ = a/R

As the cylinder does not slip over the string, the equation of motion for the centre of mass of cylinder is

Mg- 2T = Ma -------eq(1)

And for the notion about the centre of mass is

τ = Iɑ

$$2TR = \left(\frac{MR^2}{2}\right)\alpha$$

$$2TR = \left(\frac{MR^2}{2}\right)\frac{a}{R}$$

$$2T = \frac{Ma}{2} \quad ------eq(2)$$

From equation (1) and (2) we get

$$Mg = \frac{Ma}{2} + Ma$$

$$a = \frac{2g}{3}$$

From eq(2)

$$2T = \frac{M}{2}\frac{2g}{3} = \frac{2Mg}{6}$$

$$T = \frac{Mg}{6}$$

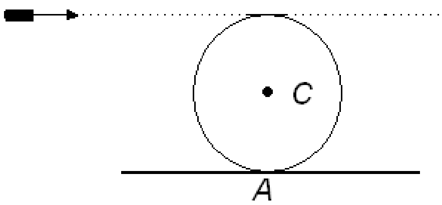
As the centre of the cylinder starts moving from rest, the velocity after it has fallen a height h is given by

$$v^2 = 2\left[\frac{2g}{3}\right]h$$

$$v = \sqrt{\frac{4gh}{3}}$$

Q11) A bullet of mass m moving with velocity u just grazes the top of a solid cylinder of mass M and radius R resting on a rough horizontal surface as shown. Assume that the cylinder rolls without slipping and at the instant the bullet leaves three is no relative motion between the bullet and the cylinder. Find the angular velocity of the cylinder and the velocity of the bullet.

Solution



At the instant the bullet leaves there is no relative motion thus angular momentum is conserved

Let v be the velocity of bullet when it leaves the cylinder and ω be the angular velocity of cylinder

Applying law of conservation of angular momentum at a point A on floor

$$Mu(2R) = mv\,(2R) + (I_{CM} + MR^2)\,\omega$$

Here $v = 2R\omega$ and $I_{CM} = (1/2)MR^2$

Substituting we get

$$2muR = m(2R\omega)(2R) + \frac{1}{2}MR^2\omega + MR^2\omega$$

$$2mu = 4mR\omega + \frac{3}{2}MR\omega$$

$$4mu = 8mR\omega + 3MR\omega$$

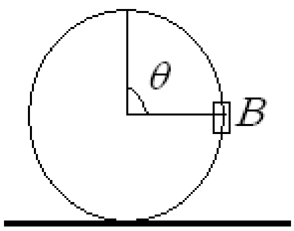
$$\omega = \frac{4mu}{(8m + 3M)R}$$

Solving for v we get as $\omega = v/2R$ angular velocity of bullet with respect to point A

$$\frac{v}{2R} = \frac{4mu}{(8m + 3M)R}$$

$$v = \frac{8muR}{(8m + 3M)}$$

Q12) A small body of mass m is attached at B to a hoop of mass 3m and radius r. the system is released from rest with $\theta = 90^{O}$ and rolls without sliding. Determine



(a) the angular acceleration of the hoop

(b)the horizontal and vertical components of the acceleration of B

(c)normal reaction and frictional force just after the release

Solution

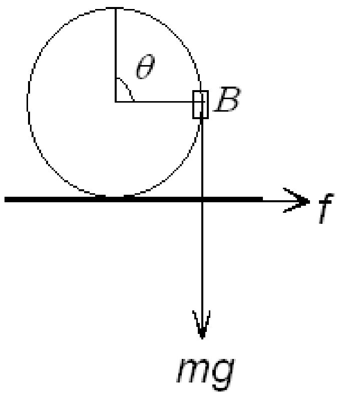
Mass at B will produce torque and will be opposed by torque produced due to friction. Thus direction of frictional force is as shown in figure

Thus resultant torque τ = mgR – fR   -----eq(1)

Also τ = Iα ----eq(2)

From eq(1) and (2)

mgR – fR = Iα



Now I = ($3mR^2 + mR^2$)

∴mgR – fR = ($3mR^2 + mR^2$) α

$$\alpha = \frac{mg - f}{4mR} \quad ------eq(3)$$

Linear acceleration = Friction / total mass

$$a = \frac{f}{4m} \quad -----eq(4)$$

Also for pure rolling a = Rα. Thus from eq(3) and (4)

$$\frac{f}{4m} = \frac{mg - f}{4mR} R$$

$$f = \frac{mg}{2}$$

Therefore

(a) the angular acceleration of the hoop

$$\alpha = \frac{g}{8r}$$

(b) the horizontal and vertical components of the acceleration of B

$$a_h = R\alpha = \frac{g}{8R} R = \frac{g}{8} \qquad a_v = R\alpha = \frac{g}{8R} R = \frac{g}{8}$$

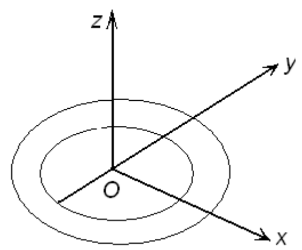
(c) Normal force = gravitational downwards – vertical upward

$$N = 4mg - \frac{mg}{8} = \frac{31mg}{8}$$

Frictional force = mg/2

Solved numerical

Q13) Find the moment of inertia, about a diameter, of a uniform ring of mass M and radius a



Solution:

WE know that moment of inertia I_{OZ}. of the ring about OZ is MA^2. We also know that, from symmetry, the moment of inertia about any one diameter is the same that about any other diameter i.e $I_{OX} = I_{OY}$

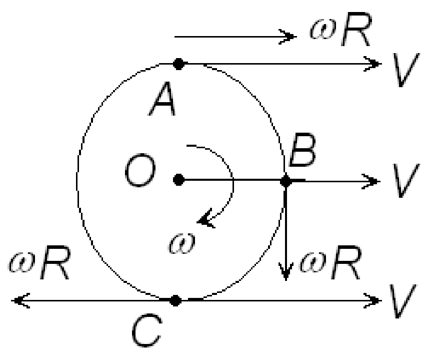
Using the perpendicular axes theorem gives

$I_{OZ} = I_{OX} + I_{OY}$

$Ma^2 = 2I_{OX} = 2I_{OY}$

The moment of inertia of the ring about any diameter = (1/2) Ma^2

7.20 Combined rotational and translational motion of a rigid body: Rolling motion



Translation motion caused by a force and rotational motion about fixed axis is caused by a torque. Rolling motion can be considered as combination of rotational and translational motion. For the analysis of rolling motion we deal translation separately and rotation separately and then we combine the resultant to analyze the overall motion.

Consider a uniform disc rolling on a horizontal surface.

Velocity of centre of mass is V and its angular speed is ω as shown

A, B, C are three points on the disc. Due to the translational motion each point A, B and C will move with centre of mass in horizontal direction with velocity V. Due to pure rotational motion each point will have tangential velocity ωR, R is the radius of the disc. When the two motions are combined, resultant velocities of different points are given by

$$V_A = V + \omega R$$

$$V_B = \sqrt{V^2 + \omega^2 R^2}$$

$$V_C = V - \omega R$$

Similarly, if disc rolls with angular acceleration α and its centre of mass moves with acceleration 'a' different points will have accelerations given by (for ω = 0)

$$a_A = a + \alpha R$$

$$a_B = \sqrt{a^2 + \alpha^2 R^2}$$

$$a_C = a - \alpha R$$

To write equation of motion for rolling motion. We can apply for translation motion

$$\vec{F}_{ext} = M\vec{a}_{CM}$$

And for rotational motion $\tau = I\alpha$

Rolling motions is possible in two ways

Rolling with slipping

Relative motion takes place between contact points

$V_C \neq 0$ and $a_C \neq 0$

Frictional force is μN

Rolling without slipping

$V_C=0 \Rightarrow V=\omega R$

And $a_C = 0 \Rightarrow a = \alpha R$

Frictional force is unknown magnitude it may any value between zero to μN

Kinetic energy of rolling body

If a body of mass m is rolling on a plane such that velocity of its centre of mass is V and the angular speed is ω, its kinetic energy is given by

$$KE = \frac{1}{2}mV^2 + \frac{1}{2}I\omega^2$$

I is moment of inertia of the body about an axis passing through the centre of mass.

In case of rolling without slipping

$$KE = \frac{1}{2}m\omega^2R^2 + \frac{1}{2}I\omega^2 \quad [\because V = \omega R]$$

$$KE = \frac{1}{2}[MR^2 + I]\omega^2$$

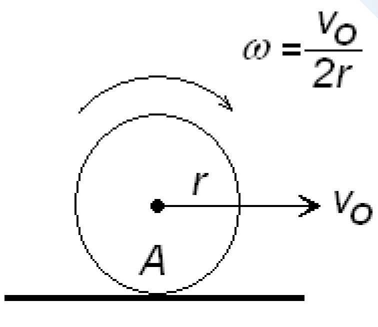
$$KE = \frac{1}{2}I_G\omega^2$$

I_G is the moment of inertia of the body about the axis passing through the point of contact on surface

Solved numerical

Q14) A sphere of mass M and radius r shown in figure slips on a rough horizontal plane. At some instant it has translational velocity v_0 and the rotational velocity $v_0/2r$. Find the translational velocity after the sphere starts pure rolling.

Solution



Let us consider the torque about the initial point of contact A. The force of friction passes through this point and hence its torque is zero. The normal force and the weight balance each other. The net torque about A is zero. Hence the angular momentum about A is conserved. Initial angular momentum

$$L = L_{CM} + Mrv_o$$

$$L = \left(\frac{2}{5}Mr^2\right)\frac{v_0}{2r} + Mrv_0 = \frac{6}{5}Mrv_0$$

Suppose the translational velocity of the sphere, after it starts rolling, is v. The angular velocity is v/r. The angular momentum about A is

$$L = L_{CM} + Mrv$$

$$L = \left(\frac{2}{5}Mr^2\right)\frac{v}{r} + Mrv = \frac{7}{5}Mrv$$

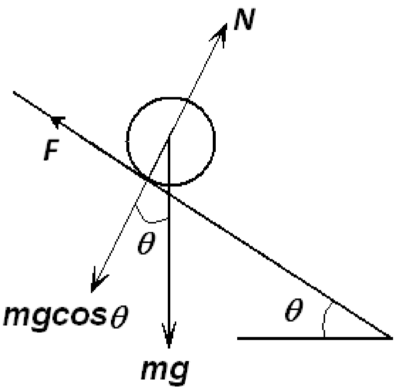
Thus

$$\frac{7}{5}Mrv = \frac{6}{5}Mrv_0$$

$$v = \frac{6}{7}v_0$$

7.21 Rigid bodies rolling without slipping on inclined plane

As stated earlier rolling without slipping is a combination of two motion translator and rotational motion



When object roll on incliner plane its centre of mass undergoes translator motion while other points on the body undergoes rotational motion

As shown in figure, suppose a rigid body rolls down without slipping along an inclined plane of height h and angle θ. Here, the mass of the body is m, moment of inertia I, geometrical radius R and the radius of gyration is K. When body reaches the bottom of the inclined plane, its potential energy decreases by mgh. According to law of conservation of mechanical energy , decrease in potential energy is converted n kinetic energy of the body

Thus

Potential energy = translation kinetic energy + rotational kinetic energy

$$mgh = \frac{1}{2}mV^2 + \frac{1}{2}I\omega^2$$

Now $\omega = V/R$ and $I = mK^2$

$$mgh = \frac{1}{2}mV^2 + \frac{1}{2}mK^2\left(\frac{V}{R}\right)^2$$

$$2gh = V^2 + K^2\left(\frac{V}{R}\right)^2$$

$$2gh = V^2\left(1 + \frac{K^2}{R^2}\right)$$

$$V^2 = \frac{2gh}{1 + \frac{K^2}{R^2}} \qquad ---eq(1)$$

If the length of the slope is d, and the body started from rest, moves with linear acceleration a to reach bottom

$$V^2 = 2ad \qquad --------eq(2)$$

From geometry of figure

$$d = \frac{h}{\sin\theta}$$

$$\therefore V^2 = 2a\frac{h}{\sin\theta}$$

Combining equations

$$2a\frac{h}{\sin\theta} = \frac{2gh}{1+\frac{K^2}{R^2}}$$

$$a = \frac{g\sin\theta}{1+\frac{K^2}{R^2}}$$

Here linear acceleration should be gsinθ. Loss in acceleration is due to frictional force

Decrease in linear acceleration

$$g\sin\theta - \frac{g\sin\theta}{1+\frac{K^2}{R^2}} = g\sin\theta\left[\frac{K^2}{K^2+R^2}\right]$$

Thus frictional force

$$F = mg\sin\theta\left[\frac{K^2}{K^2+R^2}\right]$$

From figure normal reaction and mgcosθ balance each other hence N = mgcosθ

$$\therefore \frac{F}{N} = \left[\frac{K^2}{K^2+R^2}\right]\tan\theta$$

But F/N = μ_S

$$\therefore \mu_S = \left[\frac{K^2}{K^2+R^2}\right]\tan\theta$$

$$\mu_S = \left[\frac{1}{1+\frac{R^2}{K^2}}\right]\tan\theta$$

The work done against the frictional force, resultants in the rotational kinetic energy and hence even in presence of frictional force we have been able to use the law of conservation of mechanical energy.

Now if

$$\mu_S \geq \left[\frac{1}{1+\frac{R^2}{K^2}}\right]\tan\theta \quad ----eq(3)$$

Condition is satisfied object will roll down the slope without slipping.

Cases

(i)Thin ring

For thin ring K = R substituting in above equation(3) we get

$$\mu_S \geq \frac{1}{2}\tan\theta$$

(ii) Circular disc

For circular disc K = R/√2 substituting in above equation(3) we get

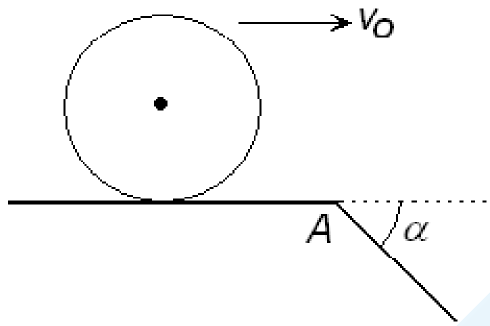
$$\mu_S \geq \frac{1}{3}\tan\theta$$

(iii)Solid sphere $K = \sqrt{\frac{2}{5}}R$

$$\mu_S \geq \frac{2}{7}\tan\theta$$

Solved numerical

Q15) A uniform solid cylinder of radius R = 15cm, rolls over a horizontal plane passing into an inclined plane forming an angle α = 30° with the horizontal as shown in figure. Find the maximum value of the velocity v_o which still permits the cylinder to roll on the inclined plane section without a jump. The sliding is assumed to be zero.



Solution:

Since the cylinder have velocity along x axis. Will follow a projectile motion when it leave the horizontal surface. If projectile motion is prevented then the cylinder is acted upon by centripetal force at the corner , takes a turn and comes on inclined plane. Thus point in contact at A is the centre for circular motion. With radius R. Let v_1 be the velocity at point A . thus centripetal force

$$\frac{mv_1^2}{R}$$

Thus force is provided by gravitational force thus

$$mgcos\alpha \ = \frac{mv_1^2}{R}$$

$$v_1^2 = Rgcos\alpha \qquad ------eq(1)$$

With reference to point A potential energy of cylinder is mgR and when it comes on the inclined plane at A potential energy = mgRcosα (as gravitational acceleration is gcosα)

Let ω_1 be the angular acceleration on inclined plane.

Now according to law of conservation of energy

Potential energy on horizontal + Liner kinetic energy + rotational kinetic energy

= Potential energy at inclined + linear kinetic energy + rotational kinetic energy

$$mgR + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgRcos\alpha + \frac{1}{2}mv_1^2 + \frac{1}{2}I\omega_1^2$$

Now

$$I = \frac{mR^2}{2}, \ \ \omega = \frac{v}{R} \ \ , \ \ \omega_1 = \frac{v_1}{R}$$

Substituting above values

$$mgR + \frac{1}{2}mv^2 + \frac{1}{2}\frac{mR^2}{2}\left(\frac{v}{R}\right)^2 = mgRcos\alpha + \frac{1}{2}mv_1^2 + \frac{1}{2}\frac{mR^2}{2}\left(\frac{v_1}{R}\right)^2$$

From equation(1)

$$mgR + \frac{3}{4}mv^2 = mgRcos\alpha + \frac{3}{4}mRgcos\alpha$$

$$gR + \frac{3}{4}v^2 = gRcos\alpha + \frac{3}{4}Rgcos\alpha$$

$$gR + \frac{3}{4}v^2 = gRcos\alpha\left(1 + \frac{3}{4}\right)$$

$$\frac{3}{4}v^2 = gRcos\alpha\left(\frac{7}{4}\right) - gR$$

$$v^2 = \frac{4}{3}\left[gR\left(\frac{7}{4}cos\alpha - 1\right)\right]$$

$$v^2 = \frac{4}{3}\left[9.8 \times 0.15 \times \left(\frac{7}{4}cos30 - 1\right)\right] = 4m/s$$

v = 2m/s

Q))Two identical trains are moving on rails along the equator on the earth in opposite directions with the same speed. They will exert the same pressure on the rails.

Answer :The question is not so simple as it appears. Actually the entire concept is related to the angular veolcity of the earth and centrifugal forces. Since the earth rotates frm west to east. the train moving along the equatior from west to east will experience a lesser centrifugal force and hence applies greater pressure. On the other hand the one moving from east to west will experiences a larger centriugal force and applies smaller pressures on the track.