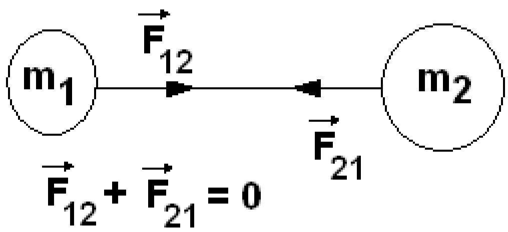
GRAVITATION

Newton's law of gravitation

The law states that every particle of matter in the universe attracts every other particle with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. The direction of the force is along the line joining the line of particles

Gravitational force is always attractive.

Consider two bodies of masses m_1 and m_2 with their centres separated by a distance r. The gravitational force between them is



Therefore from Newton's law of gravitation in vector form is

$$\vec{F} = \frac{Gm_1m_2}{r^2}\hat{r}$$

Here F_{12} represents force on mass 1 due to mass 2

And F_{21} is force of mass 2 due to mass 1

Note F_{12} and F_{21} are equal and opposite. The gravitational force forms between two particles form an action reaction pair.

Solved Numerical

Q) How a mass M be divided so that gravitational force is maximum between the parts

Solution:

Let r be the distance between two parts m and M-m, the gravitational force between them is

$$F = G\frac{m(M-m)}{r^2} = \frac{G}{r^2}[Mm - m^2]$$

For F to be maximum

$$\frac{dF}{dm} = 0$$

$$\frac{d}{dm}\left(\frac{G}{r^2}[Mm - m^2]\right) = 0$$

Or M -2m = 0 [as $G/r^2 \neq 0$]

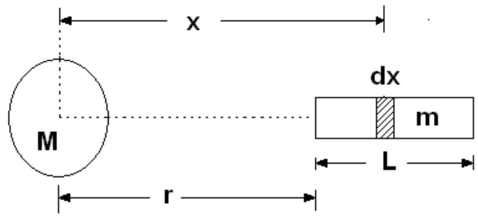
Or m/M = ½

The force is maximum when two parts are equal

Q) Find the gravitational force of attraction between a uniform sphere of mass M and a uniform rod of length L and mass m oriented as shown in figure. (given in solution)

Solution

Since the sphere is uniform its entire mass may be considered at the centre. The force on the elementary mass dm is



$$dF = \frac{GM\,dm}{x^2}$$

$$F = \int_r^{r+L} \frac{GM\,dm}{x^2}$$

$$but\ dm = \frac{m}{L}dx$$

$$F = \int_r^{r+L} \frac{GM}{x^2}\frac{m}{L}dx$$

$$F = -\frac{GMm}{L}\left[\frac{1}{x}\right]_r^{r+L}$$

$$F = -\frac{GMm}{L}\left[\frac{1}{r+L} - \frac{1}{r}\right]$$

$$F = -\frac{GMm}{L}\frac{L}{r(r+L)}$$

$$F = -\frac{GMm}{r(r+L)}$$

Gravitational Field or Intensity (I)

Process of action at a distance in which gravitational force is exerted mutually on two bodies separated by some distance is explained through the field

(i) Every object produces a gravitational field around it, due to mass

(ii) This field exerts a force on another body brought (or lying) in this field

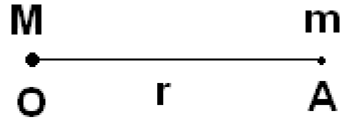
The gravitational force exerted by the given body on a body of unit mass at a given point is called the intesity of gravitational field (I) at that point" It is also known as the gravitational field or gravitational intensity

The gravitational field intensity is a vector quanity and its direction is the direction along which the unit mass ha a tendensy to move. The unit of gravitational field intesity is N/Kg and its dimensions are [LT^{-2}]

Calculation of gravitational field

(a)Gravitational field intensity due to a point mass.

Consider a point mass M at O and let us calculate gravitational intesity at A due to this point mass.



Suppose a test mass is placed at A

By Newton's law of gravitation, force on test mass

$$F = \frac{GMm}{r^2} \quad along\ \overrightarrow{AO}$$

$$I = \frac{F}{m} = -\frac{GM}{r^2}\hat{e}_r \quad ---eq(1)$$

(b) Gravitational firld intensity due to a uniform circular ring at a point on its axis

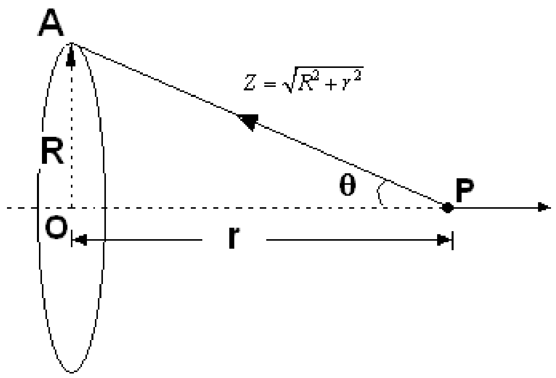


Figure shows a ring of mass M and radius R. Let P is the point at a distance r from the centre of the ring. By symmetry the field must be towards the centre that is along PO

Let us assume that a particle of mass dm on the ring say, at point A. Now the distance APis

$$\sqrt{R^2 + r^2}$$

Again the gravitational field at P due to dm is along PA and the magnitude is

$$dI = \frac{G\,dm}{Z^2}$$

$$\therefore dEcos\theta = \frac{G\,dm}{Z^2}cos\theta$$

Sine components will be canceled when we consider magnetic field due to entire ring and only cos components will be added

Net gravitational field I

$$I = \frac{Gcos\theta}{Z^2}\int dm$$

$$I = \frac{GM}{Z^2}cos\theta$$

But cosθ = r/Z

$$I = \frac{GM}{Z^2}\frac{r}{Z} = \frac{GMr}{Z^3}$$

$$I = \frac{GMr}{(R^2 + r^2)^{3/2}} \quad along\ PO$$

Cases

(i) If r >>R, $r^2 + R^2 = r^2$

$$\therefore I = -\frac{GMr}{r^3} = -\frac{GM}{r^2}[\,negative\ sign\ indicates\ attraction]$$

(ii) If r <<R, $r^2 + R^2 = R^2$

$$\therefore I = -\frac{GMr}{R^3}$$

$$\therefore I \propto r$$

(iii) For maximum I

$$\frac{\partial I}{\partial r} = 0$$

$$\frac{GM\left[(r^2+R^2)^{3/2}-\frac{3}{2}(r^2+R^2)^{1/2}\times 2r^2\right]}{[r^2+R^2]^3}=0$$

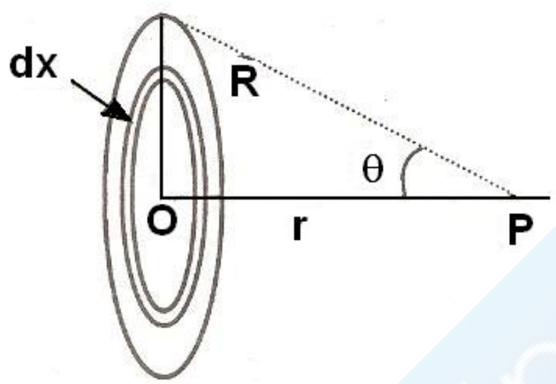
$$\left[(r^2+R^2)^{3/2}-\frac{3}{2}(r^2+R^2)^{1/2}\times 2r^2\right]=0$$

$$\left[(r^2+R^2)-\frac{3}{2}\times 2r^2\right]=0$$

$$[(r^2+R^2)-3r^2]=0$$

$$r=\pm\frac{R}{\sqrt{2}}$$

(c) Gravitational field intensity due to a uniform disc at a point on its axis



Let the mass of disc be M and its radius is R and P is the point on its axis where gravitational field is to be calculated

Let us draw a ring of radius x and thickness dx

O is the centre of circle. Area of ring is 2πxdx

The mass of ring

$$dm=\frac{M}{\pi R^2}2\pi x dx=\frac{2Mxdx}{R^2}$$

Gravitational field at p due to ring is

$$dI=\frac{G\left(\frac{2Mxdx}{R^2}\right)r}{(r^2+x^2)^{3/2}}$$

$$\int dI=\frac{2GMr}{R^2}\int_0^R\frac{xdx}{(r^2+x^2)^{3/2}}$$

$$I=\frac{2GMr}{R^2}\left[-\frac{1}{\sqrt{r^2+x^2}}\right]_0^R$$

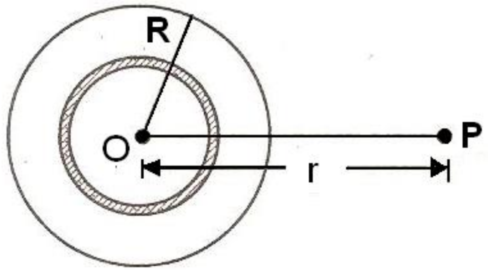
$$I=\frac{2GMr}{R^2}\left[\frac{1}{r}-\frac{1}{\sqrt{r^2+x^2}}\right]$$

In terms of θ

$$I=\frac{2GM}{R^2}\left[\frac{r}{r}-\frac{r}{\sqrt{r^2+x^2}}\right]$$

$$I = \frac{2GM}{R^2}(1 - cos\theta)$$

(d) Gravitational field due to a uniform solid sphere



Case I

Field at an external point

Let the mass of the sphere be M and its radius be R. We have to calculate the gravitational field at P

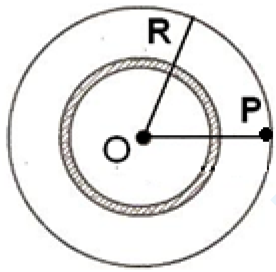
$$\int dI = \int \frac{Gdm}{r^2}$$

$$\int dI = \frac{G}{r^2}\int dm = \frac{Gm}{r^2}$$

Thus, a uniform sphere may be treated as a single particle of equal mass placed at its centre for calculating the gravitational field at an external point

Case II

Find at an internal point



Suppose the point P is inside the solid sphere, in this case r<R the sphere may be divided into thin spherical shells all centered at O. Suppose the mass of such a shell is dm. then gravitational field due to this spherical shell

$$dI = \frac{Gdm}{r^2} \quad along\ PO$$

$$\int dI = \int \frac{Gdm}{r^2}$$

$$\int dI = \frac{G}{r^2}\int dm$$

But dm = density × volume

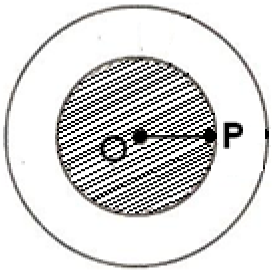
$$\int dm = \frac{M}{\frac{4}{3}\pi R^3}\frac{4}{3}\pi r^3 = \frac{Mr^3}{R^3}$$

$$\therefore I = \frac{GM}{R^3}r$$

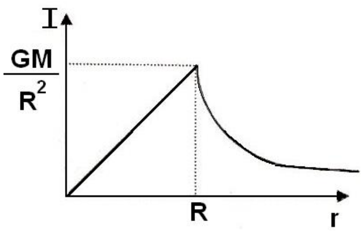
Therefore gravitational field due to a uniform sphere at an internal point is proportional to the distance of the point from the centre of the sphere. At the centre r = 0 the field is zero. At the surface of the sphere r = R

$$I = \frac{GM}{R^2}$$

Note : (I)Gravitational field due to solid sphere is continuous but it is not differentiable function



(ii) Gravitational field at point inside the sphere is only due to the mass enclosed by the surface passing through the point, volume enclosed is shown by shaded portion in diagram and field due to outer volume is zero .



(e) Field due to uniform thin spherical shell

Case I

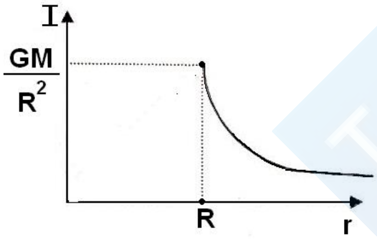
When point lies inside the spherical shell surface passing through point do not enclose any mass thus I = 0

Case II

Point P lies outside the spherical shell

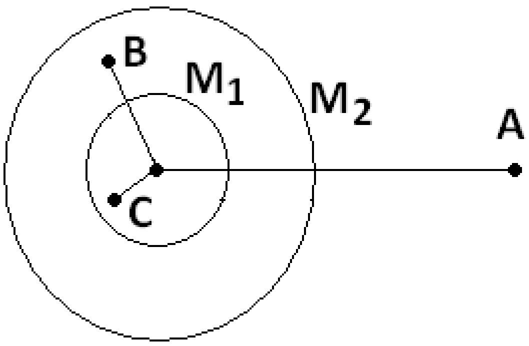
$$I = \int dI = \frac{G}{r^2}\int dm = \frac{GM}{r^2}$$

Note : Gravitational field due to thin spherical shell is both discontinuous and non-differentiable function



Solved Numerical

Q) Two concentric shells of masses M_1 and M_2 are situated as shown in figure. Find the force on a particle of mass m when the particle is located at (a) point A (b) point B (c) point C. The distance r is measured from the centre of the shell



Solution:
We know that attraction at an external point due to spherical shell of mass M is

$$\frac{GM}{r^2}$$

While at an internal point is zero. So

(a) At point A let r = a, the external point for both shells so field intensity

$$I_A = \frac{G(M_1 + M_2)}{a^2}$$
$$\therefore F_A = mI_A = \frac{mG(M_1 + M_2)}{a^2}$$

(b) For point B, let r = b, the point is external to shell of mass M_2 and internal to the shell of mass M_1, so

$$I_B = \frac{GM_2}{b^2} + 0$$
$$\therefore F_B = mI_B = \frac{GmM_2}{b^2}$$

(c) For point C, let r = c, the point is internal to both the shells; so

$$I_C = 0 + 0 = 0$$
$$\therefore F_C = mI_C = 0$$

Gravitational potential

Gravitational potential (V) at a point is defined as the amount of work done in moving unit mass from the point to infinity against the gravitational field. It is a scalar quantity. Its unit is N m kg^{-1}. Or J kg^{-1} dimensional formula $M^0L^2T^{-2}$

Mathematically

V = W/m

By the definition of potential energy U = W

So V = U/m

Or U = mV

Thus gravitational potential at a point represents potential energy of unit point mass at that point

Work is done against gravitational force thus

$$W = -\int \vec{F}_{gravitation} \cdot \overrightarrow{dr}$$

$$\therefore V = \frac{W}{m} = -\int \frac{\vec{F}_{gravitation}}{m} \cdot \overrightarrow{dr}$$

$$But \quad \frac{\vec{F}_{gravitation}}{m} = I$$

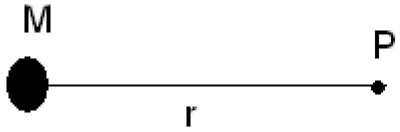
$$\therefore V = -\int \vec{I} \cdot \overrightarrow{dr}$$

i.e dV = _Idr

$$or \quad E = -\frac{dV}{dr}$$

Calculation of Gravitational potential

(a) Gravitational potential at a point (P) due to a point mass(M)



We have gravitational field due to a point mass

$$I = -\frac{GM}{r^2}$$

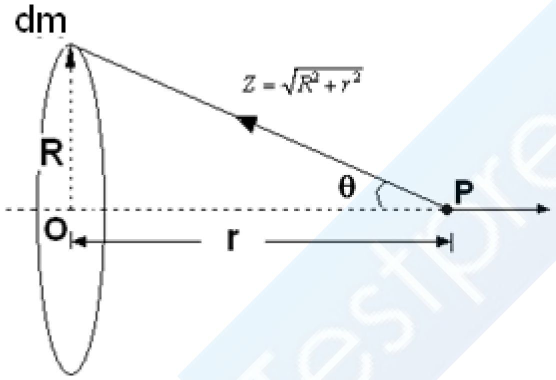
The negative sign is used as gravitational force is attractive

$$\therefore V = -\int_{\infty}^{r} -\frac{GM}{r^2} dr$$

$$V = GM \int_{\infty}^{r} \frac{dr}{r^2}$$

$$V = GM \left[\frac{-1}{r}\right]_{\infty}^{r} = -GM \left[\frac{1}{r} - \frac{1}{\infty}\right] = -\frac{GM}{r}$$

(b) Gravitational potential at a point due to a ring



Let M be the mass and R be the radius of thin ring.
Considering a small element of the ring and treating it as a point mass, the potential at the point P is

$$dV = \frac{-G\, dm}{Z} = \frac{-G\, dm}{\sqrt{R^2 + r^2}}$$

Hence, the total potential at the point P is given by

$$V = -\int \frac{G\, dm}{\sqrt{R^2 + r^2}} = \frac{GM}{\sqrt{R^2 + r^2}}$$

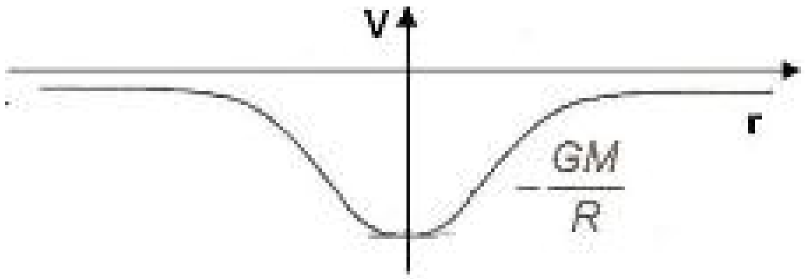
At r = 0

$$V = \frac{-GM}{R} \quad and \quad \frac{dV}{dr} = 0$$

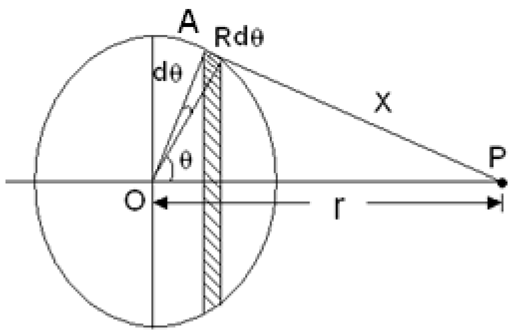
Thus at centre of ring gravitational field is zero but potential is not zero
Also

$$\frac{dV}{dr} = \frac{d}{dr}\left(\frac{-GM}{\sqrt{R^2 + r^2}}\right)$$

$$\frac{dV}{dr} = \frac{GM \times 2r}{R^2 + r^2} = 0 \Longrightarrow V\ is\ minimum\ at\ r = 0$$



(c) Gravitational potential at a point due to a spherical shell (hollow sphere)



Consider a spherical shell of mass M and radius R. P is a point at a distance 'r' from the centre O of the shell.
Consider a ring at angle to OP. Let θ be the angular position of the ring from the line OP.
The radius of the ring = Rsinθ
The width of the ring = Rdθ
Surface area of ring = (2πRsinθ)Rdθ
Surface area of ring = $2\pi R^2 \sin\theta d\theta$
The mass of the ring =

$$(2\pi R^2 sin\theta d\theta)\frac{M}{4\pi R^2} = \frac{Msin\theta d\theta}{2}$$

If 'x' is the distance of the point P from a point on the ring, then the potential at P due to the ring

$$dV = -\frac{GMsin\theta d\theta}{2x} \quad ----eq(1)$$

From cosine property of triangle OAP
$x^2 = R^2 + r^2 - 2Rr\cos\theta d\theta$
Differentiating
$2xdx = 2Rr \sin\theta d\theta$

$$\therefore \ sin\theta d\theta = \frac{x\,dx}{Rr}$$

On substituting above value of sinθdθ

$$dV = -\frac{GM}{2x} \times \frac{x\,dx}{Rr}$$

$$dV = -\frac{GM}{2Rr}dx$$

Case I
When point P lies outside the spherical shell

$$V = -\frac{GM}{2Rr}\int_{r-R}^{r+R} dx = -\frac{GM}{2Rr}[x]_{r-R}^{r+R}$$

$$V = -\frac{GM}{2Rr}[(r+R)-(r-R)] = -\frac{GM}{r}$$

This is the potential at P due to a point mass M at O

For an external point, a spherical shell behaves as a point mass supposed to be placed at its center

Case II

When the point P lies inside the spherical shell

$$V = -\frac{GM}{2Rr}\int_{R-r}^{R+r} dx = -\frac{GM}{2Rr}[x]_{R-r}^{R+r}$$

$$V = -\frac{GM}{R}$$

This expression is independent of r. Thus, the potential at every point inside the spherical shell is the same and is equal to the potential of the surface of the shell

(d) Gravitational potential due to a homogeneous solid sphere

Case (I)

When the point P lies outside the sphere.

For external point, a solid sphere behaves as if its entire mass is concentrated at the centre.

Case(II)

When the point O lies inside the sphere

Let us consider a concentric spherical surface through the point O. The potential at P arises out of the inner sphere and the outer thick spherical shell

$V = V_1 + V_2$, where V_1 = potential due to the inner sphere and V_2 = potential due to outer thick shell

The mass of the inner sphere =

$$\frac{4}{3}\pi r^3 \rho$$

ρ = density of the sphere =

$$\frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$$

The potential at P due to this sphere

$$V_1 = -\frac{G\left[\frac{4\pi r^3}{3}\right]\rho}{r} = -\frac{4\pi G\rho}{3}r^2$$

To find V_2, consider a thin concentric shell of radius x and thickness dx

The volume of the shell = $4\pi x^2\, dx$

The mass of the shell = $4\pi x^2\, dx\, \rho$

The potential at P due to this shell

$$V_2 = -\int_r^R 4\pi G\rho x dx$$

$$V_2 = -4\pi G\rho\left[\frac{x^2}{2}\right]_r^R$$

$$V_2 = -4\pi G\rho\left[\frac{R^2}{2} - \frac{r^2}{2}\right]$$

$$V_2 = -2\pi G\rho[R^2 - r^2]$$

$$V = V_1 + V_2 = -\frac{4\pi G\rho}{3}r^2 = -2\pi G\rho[R^2 - r^2]$$

$$V = -\frac{4\pi G\rho}{3}\left[r^2 + \frac{3R^2}{2} - \frac{3r^2}{2}\right]$$

$$V = -\frac{4\pi G\rho}{3}\left[\frac{3R^2}{2} - \frac{r^2}{2}\right]$$

$$V = -\frac{4\pi G}{3}\frac{3M}{4\pi R^3}\left[\frac{3R^2}{2} - \frac{r^2}{2}\right]$$

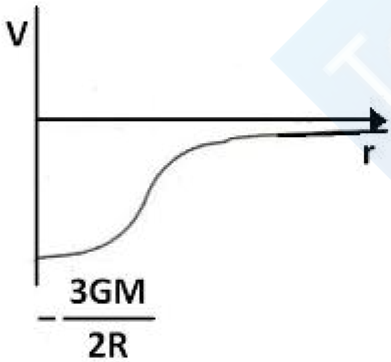
$$V = -\frac{GM}{2R^3}[3R^2 - r^2]$$

At r = 0

$$V = \frac{-3GM}{2R}$$

$$\frac{dV}{dr} = 0$$

Hence gravitational field is 0 at the centre of a solid sphere



Gravitational potential energy

The gravitational potential energy of a mass m at a distance r from another mass M is defined as the amount of work done against gravitational force in moving the mass m from infinity to a distance r

$$U_{(r)} = -\int_{\infty}^{r}\vec{F}\cdot\overrightarrow{dr}$$

Work is done against gravitational force so negative sign

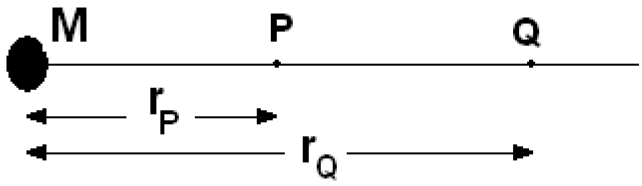
$$U_{(r)} = -\int_{\infty}^{r} \frac{-GMm}{r^2} dr$$

Gravitational force is attractive hence negative sign taken

$$U_{(r)} = GMm \left[\frac{-1}{r}\right]_{\infty}^{r}$$

$$U_{(r)} = -\frac{GMm}{r}$$

Gravitational potential difference



If we take point at P at a distance r_p and other point Q at a distance r_Q . Object of mass m is moved from P to Q then, Work done

$$U = -\int_{r_P}^{r_Q} \frac{-GMm}{r^2} dr$$

$$U = GMm \left[\frac{-1}{r}\right]_{r_P}^{r_Q}$$

$$U = U_P - U_Q = -GMm \left[\frac{1}{r_Q} - \frac{1}{r_P}\right]$$

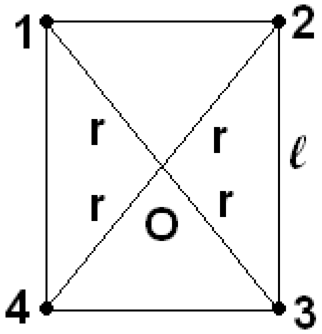
Or

$$U = U_Q - U_P = GMm \left[\frac{1}{r_P} - \frac{1}{r_Q}\right]$$

Solved Numerical

Q) A particle of mass m is placed on each vertex of a square of side l . Calculate the gravitational potential energy of this system of four particles. Also calculate the gravitational potential at the centre of the square

Solution:



Here we can write energy due to every pair of particles as

$$U_{ij} = \frac{-Gm_i m_j}{r_{ij}}$$

Where m_i and m_j respectively are the masses of the particles I and j respectively and r_{ij} is the distance between them. $m_i = m_j = m$

Therefore potential energy

$$U = -Gm^2\left[\sum_{i<j}\frac{1}{r_{ij}}\right]$$

$$U = -Gm^2\left[\frac{1}{r_{12}}+\frac{1}{r_{13}}+\frac{1}{r_{14}}+\frac{1}{r_{23}}+\frac{1}{r_{24}}+\frac{1}{r_{34}}\right]$$

$$U = -Gm^2\left[\frac{1}{l}+\frac{1}{\sqrt{2}l}+\frac{1}{l}+\frac{1}{l}+\frac{1}{\sqrt{2}l}+\frac{1}{l}\right]$$

$$U = -Gm^2\left[\frac{4+\sqrt{2}}{l}\right]$$

Gravitational potential at the centre, due to each particle is same
The total gravitational potential at the centre of the square is

V= 4(potential due to every particle)

$$V = 4\left(\frac{-Gm}{r}\right)$$

Where $r = \frac{\sqrt{2}l}{2}$

$$V = \frac{-4\sqrt{2}Gm}{l}$$

Q) Two objects of masses 1kg and 2kg respectively are released from rest when their separation is 10m. Assuming that on it mutual gravitational force act on them, find the velocity of each of them when separation becomes 5m (Take G = 6.66×10^{-11} Nm^2/kg^2)
Solution:
Let v_1 and v_2 be the final velocity of masses, m_1 = 1kg, m_2 = 2 initial velocity is zero
From law of conservation of momentum

$$m_1\vec{v}_1 + m_2\vec{v}_2 = 0$$

$$\vec{v}_1 = -\frac{m_2}{m_1}\vec{v}_2$$

$$|\vec{v}_1| = 2|\vec{v}_2| \qquad ----eq(1)$$

Initial potential energy

$$U_i = \frac{-Gm_1m_2}{r_i} = \frac{-(6.67\times10^{-11})(1\times2)}{10}$$

$$U_i = -13.32\times10^{-12}J$$

Final potential energy

$$U_f = \frac{-Gm_1m_2}{r_f} = \frac{-(6.67\times10^{-11})(1\times2)}{5}$$

$$U_f = -26.64\times10^{-12}J$$

Change in Potential energy = -13.32×10^{-12} J
According to law of conservation of energy

ΔK = -ΔU

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_1 v_2^2 = 13.32 \times 10^{-12}$$

From equation (1)

$$\frac{4}{2}v_2^2 + \frac{1}{2}(2)v_2^2 = 13.32 \times 10^{-12}$$

$$3v_2^2 = 13.32 \times 10^{-12}$$

$$v_2 = 21.07 \times 10^{-5} m/s$$

$$v_1 = 42.14 \times 10^{-5} m/s$$

Variation in acceleration due to gravity

(a) With altitude

At the surface of earth

$$g = \frac{GM_e}{R_e^2}$$

At height 'h' above the surface of earth

$$g' = \frac{GM_e}{(R_e + h)^2}$$

$$\frac{g'}{g} = \frac{R_e^2}{(R_e + h)^2} = \frac{1}{\left(1 + \frac{h}{R_e}\right)^2}$$

$$g' = \frac{g}{\left(1 + \frac{h}{R_e}\right)^2}$$

So, with increase in height, g decreases. If h <<R, then for binomial theorem

$$g' = g\left[1 + \frac{h}{R_e}\right]^{-2} = g\left[1 - \frac{2h}{R_e}\right]$$

(b) With depth

At the surface of the earth

$$g = \frac{GM_e}{R_e^2}$$

For a point at the depth 'd' below the surface

Mass the earth enclosed by the surface passing through point P as shown in figure be m then

$$g' = \frac{Gm}{(R_e - d)^2}$$

We know that gravitational at point P due to shaded portion is zero thus

$$m = \frac{4}{3}\pi(R_e - d)^3 \times \frac{M_e}{\frac{4}{3}\pi R_e^3}$$

$$m = \frac{M_e}{R_e^3}(R_e - d)^3$$

$$g' = \frac{G}{(R_e - d)^2}\frac{M_e}{R_e^3}(R_e - d)^3$$

$$g' = \frac{GM_e}{R_e^3}(R_e - d)$$

Thus

$$\frac{g'}{g} = \frac{\frac{GM_e}{R_e^3}(R_e - d)}{\frac{GM_e}{R_e^2}} = \frac{R_e - d}{R_e}$$

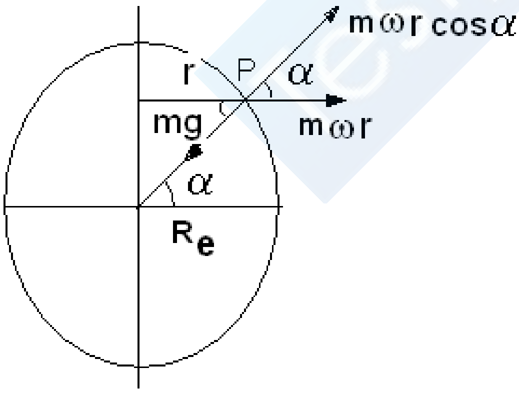
$$g' = g\left[1 - \frac{d}{R_e}\right]$$

So with increase in depth below the surface of the earth, g decreases and at the center of the earth it becomes zero

It should be noted that value of g decreases, if we move above the surface or below the surface of the earth

(C) Due to rotation of the earth

The earth is rotating about its axis from west to east. So, the earth is a non-inertial frame of reference. Everybody on its surface experiences a centrifugal force. Consider a point P . Perpendicular distance from point with axis of rotation is r . Then centrifugal force at point is $m\omega^2 r\cos\alpha$, going ouward where α is the latitude of the place.



Here α is the angle made by the line joining a given place on the Earth's surface to the centre of the Earth with the equatorial line is called latitude of the place. Hence for equator latitude is and for poles latitude is 90^O

Gravitational force mg is acting towards the centre of earth.

Thus resultant force

Is $mg' = mg - m\omega^2 r\cos\alpha$ ----eq(1)

From the geometry of figure $r = R_e\cos\alpha$ theus from equation (1)

$g' = g - \omega^2 R_e\cos^2\alpha$

Cases

(i) At equator $\alpha = 0 \therefore \cos\alpha = 1$

$\therefore g' = g - \omega^2 R_e$

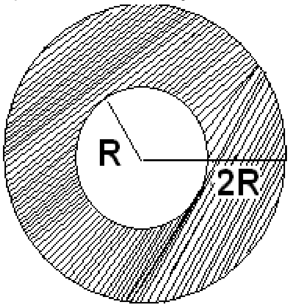
Which shows minimum value of the effective gravitational acceleration

(ii) At poles, $\alpha = 90 \therefore \cos\alpha = 0$

$\therefore g' = g$, which shows the maximum value of the effective gravitational acceleration

Solved Numerical

Q) The density of the core of planet is ρ_1 and that of the outer shell is ρ_2. The radii of the core and that of the planet are R and 2R respectively. Gravitational acceleration at the surface of the planet is same as at a depth R. Find the ratio ρ_1/ρ_2



Solution:

Mass of inner sphere M_1

$$M_1 = \frac{4}{3}\pi R^3 \rho_1$$

Volume of outer shell

$$\frac{4}{3}\pi(2R)^3 - \frac{4}{3}\pi R^3 = \frac{4}{3}\pi(7R^3)$$

Mass of outer shell M_2

$$\frac{4}{3}\pi(7R^3)\rho_2$$

Gravitational acceleration at surface of planet:

$$a = \frac{G}{(2R)^2}(M_1 + M_2)$$

$$a = \frac{G}{(2R)^2}\left(\frac{4}{3}\pi R^3\rho_1 + \frac{4}{3}\pi 7R^3\rho_2\right)$$

$$a = \frac{GR\pi}{3}(\rho_1 + 7\rho_2) \quad -----eq(1)$$

Gravitational acceleration at depth R

$$a' = \frac{G\frac{4}{3}\pi R^3\rho_1}{R^2} = \frac{4G\pi R\rho_1}{3} \quad ----eq(2)$$

Given a = a' thus

$$\frac{GR\pi}{3}(\rho_1 + 7\rho_2) = \frac{4G\pi R\rho_1}{3}$$

$$(\rho_1 + 7\rho_2) = 4\rho_1, \qquad \frac{\rho_1}{\rho_2} = \frac{7}{3}$$

Satellite

(a) Orbital speed of satellite

The velocity of a satellite in its orbit is called orbital velocity. Let v_o be the orbital velocity

Gravitational force provides necessary centripetal acceleration

$$\therefore \frac{GM_e m}{r^2} = \frac{mv_o^2}{r}$$

$$v_0 = \sqrt{\frac{GM_e}{r}}$$

As $r = R_e + h$

$$v_0 = \sqrt{\frac{GM_e}{R_e + h}}$$

Notes

Orbital velocity is independent of the mass of the body and is always along the tangent to the orbit

Close to the surface of the earth, r = R as h=0

$$v_0 = \sqrt{\frac{GM}{R}} = \sqrt{gR} = \sqrt{10 \times 6.4 \times 10^6} = 8\ km/s$$

(b) Time period of a Satellite

The time taken by a satellite to complete one revolution is called the time period (T) of the satellite

It is given by

$$T = \frac{2\pi}{v_o} = 2\pi r\sqrt{\frac{r}{GM}}$$

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3$$

$$\Rightarrow T^2 \propto r^3$$

(c) Angular momentum of a satellite (L)

In case of satellite motion, angular momentum will be given by

$$L = mvr = mr\sqrt{\frac{GM}{r}}$$

Or

$$L = (m^2 GMr)^{1/2}$$

In the case of satellite motion, the net force on the satellite is centripetal force. The torque of this force about the centre of the orbit is zero. Hence, angular momentum of the satellite is conserved. i.e L is constant

(d) Energy of satellite

The P.E. of a satellite is

$$U = -\frac{GMm}{r}$$

The kinetic energy of the satellite is

$$K = \frac{1}{2} m v_0^2$$

$$But\ v_0 = \sqrt{\frac{GM}{r}}$$

$$K = \frac{GMm}{2r}$$

Total mechanical energy of the satellite

$$E = -\frac{GMm}{r} + \frac{GMm}{2r} = -\frac{GMm}{2r}$$

Note

We have K = -E

Also U = 2E

Total energy of a satellite in its orbit is negative. Negative energy means that the satellite is bounded to the central body (earth) by an attractive force and energy must be supplied to remove it from the orbit to infinity.

(e) Binding energy of the satellite

The energy required to remove the satellite from its orbit to infinity is called binding energy of the satellite. i.e.

$$Binding\ energy = -E = \frac{GMm}{2r}$$

Solved Numerical

Q) An artificial satellite is moving in a circular orbit around the earth with a speed equal to half the magnitude of escape velocity from the earth

a) Determine the height of satellite above the earth's surface

b) If the satellite is suddenly stopped, find the speed with which the satellite will hit the earth's surface after falling down

Solution:

Escape velocity = √(2gR), where g is the acceleration due to gravity on the surface of earth and R is the radius

Orbital velocity =

$$\frac{1}{2} v_e = \frac{1}{2}\sqrt{2gR} = \sqrt{\frac{gR}{2}} \qquad ----eq(1)$$

(1/2)V_e = (1/2) √(2gR) =√(gR/2)

a) If h is the height of satellite above earth's surface, the gravitational force provides the centripetal force for circular motion

$$\frac{mv_0^2}{R+h} = \frac{GMm}{(R+h)^2}$$

$$\Rightarrow v_0^2 = \frac{GM}{R+h} = \frac{gR^2}{(R+h)}$$

$$\therefore \left(\frac{1}{2}v_0\right)^2 = \frac{gR^2}{R+h}$$

From equation (1)

$$\frac{gR}{2} = \frac{gR^2}{R+h}$$

R+h = 2R

H = R

b) If the satellite is stopped in orbit, the kinetic energy is zero and its potential energy is

$$\frac{-GMm}{2R}$$

When it reaches the earth, let v be its velocity

Hence kinetic energy = (1/2) mv^2

Potential energy =

$$\frac{-GMm}{R}$$

By the law of conservation of energy

$$\therefore \frac{1}{2}mv^2 - \frac{GMm}{R} = -\frac{GMm}{2R}$$

$$v^2 = 2GM\left(\frac{1}{R} - \frac{1}{2R}\right) = \frac{2gR^2}{2R} = gR$$

Velocity with which the satellite will hit the earth's surface after falling down is

$$v = \sqrt{gR}$$

Q) Two satellites of same masses are launched in the same orbit around the earth so as to rotate opposite to each other. They collide inelastically and stick together as wreckage. Obtain the total energy of the system before and after collisions. Describe the subsequent motion of wreckage.

Solution

Potential energy of satellite in orbit

$$-\frac{GMn}{r}$$

If v is the velocity in orbit, we have

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$v^2 = \frac{GM}{r}$$

Kinetic energy

$$\frac{1}{2}mv^2 = \frac{GMm}{2r}$$

Total energy

$$\frac{GMm}{2r} - \frac{GMm}{r} = -\frac{GMm}{2r}$$

For the two satellites, the total energy before collision

$$2\left(-\frac{GMm}{2r}\right) = -\frac{GMm}{r}$$

After collision, let v' be the velocity of the wreckage. By the law of conservation of momentum, since they are approaching each other

$$m\vec{v} - m\vec{v} = 2mv'$$
$$\therefore v' = 0$$

The wreckage has no kinetic energy after collision but has potential energy

$$P.E. = \frac{-GM(2m)}{r}$$

Total energy after collision = $\frac{-2GMm}{r}$

After collision, the centripetal force disappears and the wreckage falls down under the action of gravity.

Geostationary satellite

If there is a satellite rotating in the direction of earth's rotation. i.e. from west to east, then for an observer on the earth the angular velocity of the satellite will be same as that of earth $\omega_S = \omega_E$

However, if $\omega_S = \omega_E = 0$, satellite will appear stationary relative to the earth. Such a satellite is called 'Geostationary satellite' and is used for communication purposes

The orbit of geostationary satellite is called 'Parking Orbit'

We know that

$$T^2 = \frac{4\pi^2}{GM} r^3$$

$$For\ geostationary\ satellite, T = 24\ Hours$$

Putting this value of T in the above equation, we get

R = 42000Km

Or h= 3600.0 km

Where h is height of the satellite from the surface of the earth

Weightlessness in a satellite

When the astronaut is in an orbiting satellite, both the satellite and astronaut have the same acceleration towards the centre of the Earth. Hence, the astronaut does not exert any force on the floor of the satellite. So, the floor of the satellite also does not exert any force of reaction on the astronaut. As there is no reaction, the astronaut has a feeling of weightlessness.

The radial acceleration of the satellite is given by

$$a_r = \frac{F_r}{m} = \frac{GMm}{r^2} \times \frac{1}{m} = \frac{GM}{r^2}$$

For an astronaut of mass m_a inside the satellite, we have following forces

$$\text{Downward force} = \frac{GMm_a}{r^2}$$

Upward pseudo force as motion of satellite is accelerated motion

$$\text{Upward force} = \frac{GMm_a}{r^2}$$

Thus resultant force on Astronaut is zero, or normal force is zero

Hence, the astronaut feels weightlessness