OSCILLATIONS

Periodic Motion and Oscillatory motion

If a body repeats its motion along a certain path, about a fixed point, at a definite interval of time, it is said to have a *periodic motion*

If a body moves to and fro, back and forth, or up and down about a fixed point in a definite interval of time, such motion is called an *oscillatory motion.* The body performing such motion is called an *oscillator*.

Simple Harmonic Motion

The periodic motion of a body about a fixed point, on a linear path, under the influence of the force acting towards the fixed point and proportional to displacement of the body from the fixed point is called a *simple harmonic motion* (SHM)

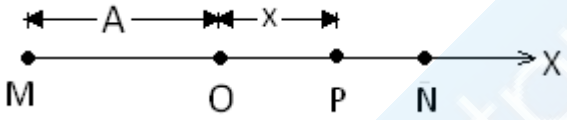
A body performing simple harmonic motion is known as *Simple Harmonic Oscillator (SHO)*

Simple harmonic motion is a special type of periodic motion in which

(i) The particle oscillates on a straight line.

(ii) The acceleration of the particle is always directed towards a fixed point on the straight line.

(iii) The magnitude of acceleration is proportional to the displacement of the particle from fixed point.



This fixed point is called the centre of the oscillation or *mean position*. Taking this point as origin "O".

The maximum displacement of oscillator on either side of the mean position is called *amplitude* denoted by A

The time required to complete one oscillation is known as *periodic time* (T) of oscillator

In other words, the least time interval of time after which the periodic motion of an oscillator repeat itself is called a periodic time of the oscillator. Distance travelled by oscillator is 4A in *periodic time*

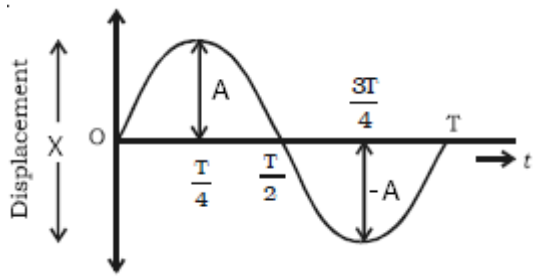
The number of oscillations completed by simple harmonic oscillator in one second is defined as *frequency*. SI unit is s^{-1} or hertz (H)

It is denoted by f and f = 1/T

2π times the frequency of an oscillator is called the *angular frequency* of the oscillator

It is denoted by ω. Its SI unit is rad s^{-1}, $\omega = 2\pi f$.

If we draw the graph of displacement of SHO against time as shown in figure, which is as shown in figure



Mathematical equation is the function of time
x(t) = Asin(ωt +ϕ) -----eq(1)
Here X(t) represents displacement at time t
A = Amplitude
(ωt + ϕ) = Phase
ϕ = Initial phase (epoch) for a given graph ϕ= 0.
If oscillation have started from negative x end (M) then initial phase is –π/2
If oscillation stated from positive end (N) then initial phase is π/2
ω = angular frequency

Velocity

Velocity of oscillator is

$$v(t) = \frac{dx(t)}{dt}$$

v(t) = ωAcos(ωt + φ)

$$v(t) = \pm\omega A\left(\sqrt{1 - sin^2(\omega t + \varphi)}\right)$$

$$v(t) = \pm\omega\left(\sqrt{A^2 - A^2 sin^2(\omega t + \varphi)}\right)$$

From eq(1)

$$v(t) = \pm\,\omega\left(\sqrt{A^2 - x^2}\right)$$

Velocity is maximum at x = 0 or equilibrium position v = ±ωA
Velocity is minimum at x = A or extreme positions v= 0
Note that velocity is out of phase of displacement by π/2

Acceleration

$$a(t) = \frac{dv(t)}{dt}$$

a(t) = -ω^2Asin(ωt + ϕ)
a(t) = -ω^2 x(t)
At x = 0 ,acceleration is a = 0

And maximum at x = $\pm$A , a = $\mp$ ω^2A
Note velocity is out of phase of displacement by π

Solved Numerical

Q) A particle moving with S.H.M in straight line has a speed of 6m/s when 4m from the centre of oscillations and a speed of 8m/s when 3m from the centre. Find the amplitude of oscillation and the shortest time taken by the particle in moving from the extreme position to a point midway between the extreme position and the centre

Solution

From the formula for velocity

$$v(t) = \pm\,\omega\left(\sqrt{A^2 - x^2}\right)$$

$$6 = \pm\,\omega\left(\sqrt{A^2 - 4^2}\right)$$

And

$$8 = \pm\,\omega\left(\sqrt{A^2 - 3^2}\right)$$

By taking ratio of above equations

$$\frac{6}{8} = \frac{\sqrt{A^2 - 4^2}}{\sqrt{A^2 - 3^2}}$$

On simplifying we get A = 5 m
On substituting value of A in equation

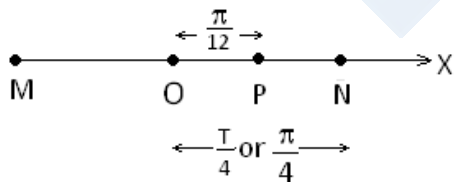
$$6 = \pm\,\omega\left(\sqrt{A^2 - 4^2}\right)$$

$$6 = \pm\,\omega\left(\sqrt{5^2 - 4^2}\right)$$

ω = 2 rad
From the equation for oscillation
x(t) = Asin(ωt +ϕ)

$$\frac{A}{2} = Asin2t$$

0.5 = sin2t
⇒ 2t = π/6



t= π/12 this is the time taken by oscillator to move from centre to midway
Now Time taken by oscillator to move from centre to extreme position is T/4

As T = 2π /ω thus
Time taken to reach to extreme position from the centre =

$$\frac{T}{4} = \frac{2\pi}{4\omega} = \frac{2\pi}{4 \times 2} = \frac{\pi}{4}$$

Now time taken to reach oscillator to reach middle from extreme point is =

$$\frac{\pi}{4} - \frac{\pi}{12} = \frac{\pi}{6}\ sec$$

Q) A point moving in a straight line SHM has velocities v_1 and v_2 when its displacements from the mean position are x_1 and x_2 respectively. Show that the time period is

$$2\pi\sqrt{\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}$$

<u>Solution</u>

From the formula for velocity

$$v(t) = \pm\,\omega\left(\sqrt{A^2 - x^2}\right)$$

$$v_1^2 = \omega^2(A^2 - x_1^2)$$

$$A^2 = \frac{v_1^2}{\omega^2} + x_1^2 \quad ---\ eq(1)$$

$$A^2 = \frac{v_2^2}{\omega^2} + x_2^2 \quad ---\ eq(2)$$

From equation (1) and (2)

$$\frac{v_1^2}{\omega^2} + x_1^2 = \frac{v_2^2}{\omega^2} + x_2^2$$

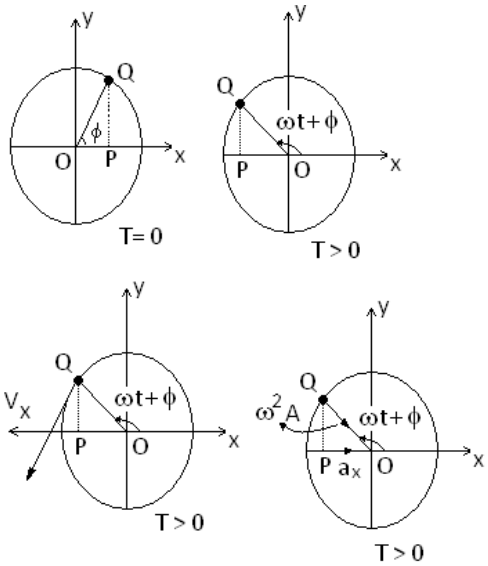
$$\frac{v_2^2}{\omega^2} - \frac{v_1^2}{\omega^2} = x_1^2 - x_2^2$$

$$\frac{1}{\omega^2}(v_2^2 - v_1^2) = x_1^2 - x_2^2$$

$$\frac{T^2}{(2\pi)^2}(v_2^2 - v_1^2) = x_1^2 - x_2^2$$

$$T = 2\pi\sqrt{\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}$$

Relation between simple harmonic motion and uniform circular motion



In figure shown Q is a point moving on a circle of radius A with constant angular speed ω (in rad/sec). P is the perpendicular projection of Q on the horizontal diameter, along the x-axis.
Let us take Q as the reference point and the circle on which it moves the reference circle. As the reference point revolves, the projected point P moves back and forth along the horizontal diameter

Let the angle between the radius OQ and the x-axis at the time t=0 be called ϕ. At any time t later the angle between OQ and the x-axis is (ωt+ ϕ), the point Q moving with constant angular speed ω. The x-coordinate of Q at any time is, therefore

$X = A\cos(\omega t + \phi)$

i.e. P moves with simple harmonic motion

Thus, when a particle moves with uniform circular motion, its projection on a diameter moves with simple harmonic motion. The angular frequency ω of simple harmonic motion is the same as the angular speed of the reference point.

The velocity of Q is $v = \omega A$. The component of v along the x-axis is

$V_x = -v\sin(\omega t + \phi)$

$V_x = -\omega A\sin(\omega t + \phi)$,

Which is also the velocity of p. The acceleration of Q is centripetal and has a magnitude,

$a = \omega^2 A$

The component of 'a' along the x-axis is

$A_x = -a\cos(\omega t + \phi)$,

$A_x = -\omega^2 A\cos(\omega t + \phi)$,

Which is the acceleration of P

The force law for simple harmonic motion

We know that

$F = ma$

As $a = -\omega^2 x(t)$

$F = -m\omega^2 x(t)$

This force is restoring force

According to Hook's law, the restoring force is given by

$F = -kx(t)$

With k as spring constant

Thus $k = m\,\omega^2$

∴ angular frequency

$$\omega = \sqrt{\frac{k}{m}}$$

And frequency of oscillation

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

In many cases, the simple harmonic motion can also occur even without spring. In that case k is called the force constant of SHM and it is restoring force per unit displacement (K = -F/x)

Solved numerical

Q) A spring balance has a scale that reads 50kg. The length of the scale is 20cm. A body suspended from this spring, when displaced and released, oscillates with period of 0.6s. Find the weight of the body

Solution:

Here m = 50kg

Maximum extension of spring x = 20- 0 = 20 cm = 0.2 m

Periodic time T = 0.6s

Maximum force F = mg

F = 50× 9.8 = 490 N

K = F/x = 490/0.2 = 2450 N m^{-1}

As

$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$m = \frac{(0.6)^2 \times 2450}{4 \times (3.14)^2} = 22.36\ kg$$

∴ Weight of the body = mg = 22.36 ×9.8 = 219.1 N = 22.36 kgf

Examples of simple harmonic motion

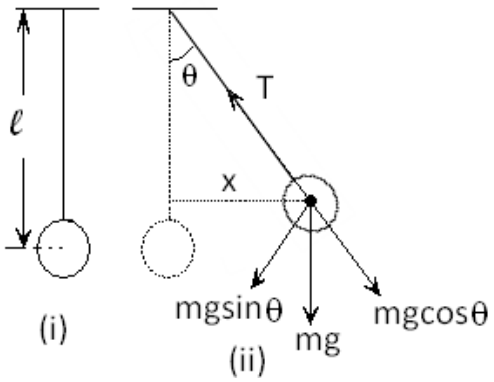
Simple pendulum

A simple pendulum consists of a heavy particle suspended from a fixed support through a light, inextensible and torsion less string

The time period of simple pendulum can be found by force or torque method and also by energy method

(a)Force method: the mean position or the equilibrium position of the simple pendulum is when θ = 0 as shown in figure(i). the length of the string is l, and mass of the bob is m

When the bob is displaced through distance 'x', the forces acting on it are shown in figure(ii)



The restoring force acting on the bob to bring it to the mean position is

F = - mgsinθ (-ve sign indicates that force is directed towards the mean position)

For small angular displacement

Sinθ ≈θ = x/l

$$\therefore F = -mg\frac{x}{l}$$

$$\therefore a = -g\frac{x}{l}$$

Comparing it with equation of simple harmonic motion a = $-\omega^2 x$

$$\omega^2 = \frac{g}{l}$$

$$\therefore \omega = \sqrt{\frac{g}{l}}$$

Time period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

(b)Torque method:

Now taking moment of force acting on the bob about O

$$\tau = -(mgsin\theta)l \quad --- eq(1)$$

Also from Newton's second law

$$\tau = I\alpha \quad --- eq(2)$$

From equation(1) and (2)

$$I\alpha = -(mgsin\theta)l$$

Since θ is small

$$I\alpha = -(mg\theta)l$$

But Moment of inertia I = ml^2

$$ml^2\alpha = -(mg\theta)l$$
$$\therefore \alpha = -\left(\frac{g}{l}\right)\theta$$

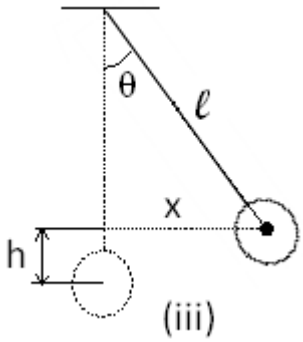
Comparing with simple harmonic motion equation ,$\alpha = -\omega^2\theta$

$$\omega = \sqrt{\frac{g}{l}}$$
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

Note: component of force mgcosθ cannot produce torque because it passes through fixed point

(c) Energy method:

Let the potential energy at the mean position be zero. Let the bob is displaced through an angle 'θ'. Let its velocity be 'v'



Then potential energy at the new position

U = mgl(1-cosθ)

Kinetic energy at this instant K = (1/2) mv^2

Total mechanical energy at this instant

E = U + K

E = mgl(1-cosθ) + (1/2) mv^2

We know, in simple harmonic motion E = constant

$$\frac{dE}{dt} = 0$$

$$\Rightarrow mgl\left[sin\theta\frac{d\theta}{dt}\right] + mv\frac{dv}{dt} = 0$$

Bur v = ωl

$$\therefore v = l\frac{d\theta}{dt}$$
$$\therefore mgl\left[sin\theta\frac{d\theta}{dt}\right] + ml\frac{d\theta}{dt}\frac{dv}{dt} = 0$$
$$\therefore g[sin\theta] + \frac{dv}{dt} = 0$$

$$\frac{dv}{dt} = a = -g[sin\theta] \approx g\theta$$

But θ = x/l

$$\therefore a = -g\frac{x}{l}$$

But $a=-\omega^2 x$

$$\therefore \omega^2 = \frac{g}{l}$$

$$\omega = \sqrt{\frac{g}{l}}$$

Periodic time

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

(a)Simple pendulum in a lift:

If the simple pendulum is oscillating in a lift moving with acceleration a, then the effective g of the pendulum is

$g_{eff} = g \pm a$

+ sign is taken when lift is moving upward

- ve sign is taken when lift is moving downward

Hence periodic time of pendulum

$$T = 2\pi\sqrt{\frac{l}{g \pm a}}$$

(b)Simple pendulum in the compartment of a train

If the simple pendulum is oscillating in a compartment of a train accelerating or retarding horizontally at the rate 'a' then the effective value of g is

$$g_{eff} = \sqrt{g^2 + a^2}$$

Hence periodic time of pendulum

$$T = 2\pi\sqrt{\frac{l}{\sqrt{g^2 + a^2}}}$$

(c)Seconds pendulum

The pendulum having the time-period of two seconds, is called the second pendulum. It takes one second to go from one end to the other end during oscillation. It also crosses the mean position at every one second

Solved Numerical

Q) Length of a second's pendulum on the surface of earth is l_1 and l_2 at height 'h' from the surface of earth. Prove that the radius of the earth is given by

$$R_e = \frac{h\sqrt{l_2}}{\sqrt{l_1} - \sqrt{l_2}}$$

Solution:
Period of second's pendulum is 2sec

$$2 = 2\pi\sqrt{\frac{l_1}{g}}$$

And g at surface of earth is given by

$$g = \frac{GM_e}{R_e^2}$$

$$\therefore 2 = 2\pi\sqrt{\frac{l_1 R_e^2}{GM_e}} \quad ---eq(1)$$

g at height h from surface of earth is

$$g = \frac{GM_e}{(R_e + h)^2}$$

Thus at height h

$$2 = 2\pi\sqrt{\frac{l_2(R_e + h)^2}{GM_e}} \quad ---eq(2)$$

From equation(1) and (2) we get

$$2\pi\sqrt{\frac{l_2(R_e + h)^2}{GM_e}} = 2\pi\sqrt{\frac{l_1 R_e^2}{GM_e}}$$

$$l_2(R_e + h)^2 = l_1 R_e^2$$

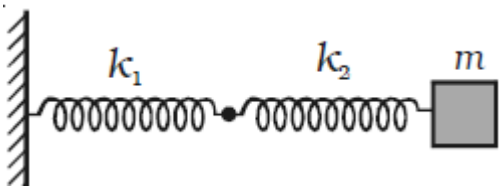
$$\sqrt{l_2}\,(R_e + h) = \sqrt{l_1}\,R_e$$

$$\sqrt{l_2}\,h = R_e\left(\sqrt{l_1} - \sqrt{l_2}\right)$$

$$R_e = \frac{h\sqrt{l_2}}{\sqrt{l_1} - \sqrt{l_2}}$$

Combinations of springs

Series combination



As show in figure consider a series combination of two massless spring of spring constant k_1 and k_2

In this system when the combination of two springs is displaced to a distance *y*, it produces extension x_1 and x_2 in two springs of force constants k_1 and k_2.
$F = -k_1 x_1$; $F = -k_2 x_2$

where *F* is the restoring force.

$$x = x_1 + x_2 = -F\left[\frac{1}{k_1} + \frac{1}{k_2}\right]$$

We know that $F = -kx$
$x = -F/k$
From the above equations,

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$k = \frac{k_1 k_2}{k_1 + k_2}$$

Time period T

$$T = 2\pi\sqrt{\frac{m(k_1 + k_2)}{k_1 k_2}}$$

Frequency f

$$f = \frac{1}{2\pi}\sqrt{\frac{k_1 k_2}{m(k_1 + k_2)}}$$

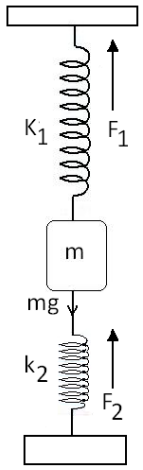
If both the springs have the same spring constant,
$k_1 = k_2 = k$. then equivalent spring constant k' = k/2

$$T = 2\pi\sqrt{\frac{m(2)}{k}}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k'}{m}} = \frac{1}{2\pi}\sqrt{\frac{k}{2m}}$$

Parallel combination

(i) Consider a situation as shown in figure where a body of mass m is attached in between the two massless springs of spring constant k_1 and k_2. let the body is left free for SHM in vertical plane after pulling mass m In this situation when a body is pulled lower through small displacement y. lower spring gets compressed by y, while upper spring elongate by y. hence restoring forces F_1 and F_2 set up in both these springs will act in the same direction.



Net restoring force will be

$F = F_1 + F_2$

$F = - k_1y \ -k_2y$

$F= - (k_1 + k_2)\ y$

If k' is the is equivalent spring constant then

$F = -k'y$ thus

$k' = k_1 + k_2$

Now periodic time T

$$T = 2\pi\sqrt{\frac{m}{k'}} = \sqrt{\frac{m}{k_1 + k_2}}$$

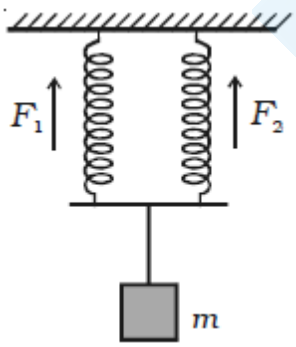
Frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{k'}{m}} = \frac{1}{2\pi}\sqrt{\frac{k_1 + k_2}{m}}$$

If $k_1 = k_2 = k$ then

$$T = 2\pi\sqrt{\frac{m}{2k}}$$

(ii) Two massless springs of equal lengths having force constants k_1 and k_2 respectively are suspended vertically from a rigid support as shown in figure. At their free ends, a block of mass m having non-uniform density distribution is suspended so that spring undergoes equal extension



In this situation two bodies are pulled down through a small distance y and the system is made to perform SHM in vertical plane

Here, the springs have different force constants. Moreover the increase in their length is same. Therefore, the load is distributed equally between the springs. Hence, the restoring force developed in each spring is different.

If F_1 and F_2 are the restoring forces set up due to extension of springs, then

$F_1 = -k_1\ y$ and $F_2 = k_2y$

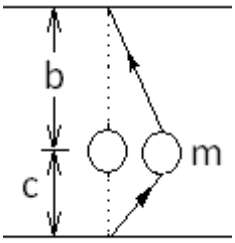
Also the total restoring force

$F = F_1 + F_2$

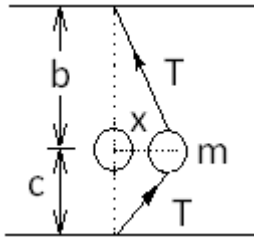
$F = F_1 + F_2$

$F = - k_1y - k_2y$
$F= - (k_1 + k_2) y$
If k' is the is equivalent spring constant then
$F = -k'y$ thus
$k' = k_1 + k_2$
Frequency and periodic time will be same as given in (i)

Q) A small mass m is fastened to a vertical wire, which is under tension T. What will be the natural frequency of vibration of the mass if it is displaced laterally a slight distance and then released?



Solution



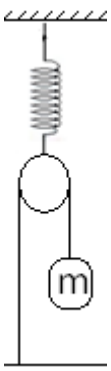
Sphere is displaced by a very small distance x thus angle formed is also small
Using trigonometry we can find Restoring force $F_b = T\sin\theta = T(x/b)$
Now restoring force per unit displacement $k_1 = F_b /x = T/b$
$F_c = T\sin\theta = T(x/c)$
Now restoring force per unit displacement $k_2 = F_c /x = T/c$
Since both restoring forces are parallel combination

$$f = \frac{1}{2\pi}\sqrt{\frac{\frac{T}{b}+\frac{T}{c}}{m}}$$

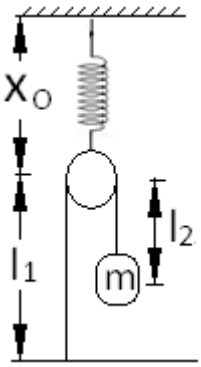
$$f = \frac{1}{2\pi}\sqrt{\frac{T}{m}\left(\frac{1}{b}+\frac{1}{c}\right)}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{T}{m}\left(\frac{b+c}{bc}\right)}$$

Q) The spring has a force constant k. the pulley is light and smooth while the spring and the string are light. If the block of mass 'm' is slightly displaced vertically and released, find the period of vertical oscillation



Solution: When mass m is pulled by distance of x, increase in length of spring is x/2 as explained below



$x_0 + l_1$ = constant
dx_0 be the increase in length of spring
$dx_0 + dl_1 = 0$
and $dx_0 + dl_2 = x$
Thus $2dx_0 + dl_1 + dl_2 = x$
As string is inextensible $dl_1 + dl_2 = 0$
Thus $2dx_0 = x$ or $dx_0 = x/2$

Before pulling mass m spring was in equilibrium
$2T_0 = kx_0$
And $T_0 = mg$
Thus $2mg = kx_0$
When spring stretch by x/2 then tension in spring is T

$$F = k\left(x_0 + \frac{x}{2}\right)$$

$$\text{Or } 2T = k\left(x_0 + \frac{x}{2}\right)$$

$$2T = kx_0 + \frac{kx}{2}$$

$$2T = 2T_0 + \frac{kx}{2}$$

$$2T - 2T_0 = \frac{kx}{2}$$

$$T - T_0 = \frac{kx}{4}$$

Restoring force on mass m is $T-T_0$ which is proportional to displacement

Thus restoring force constant k' = k/4

Period of oscillation

$$T = 2\pi\sqrt{\frac{m}{k'}} = 2\pi\sqrt{\frac{m}{\frac{k}{4}}} = 4\pi\sqrt{\frac{m}{k}}$$

Differential equation of simple harmonic motion

According to Newton's second law of motion

$$F = ma = m\frac{d^2y(t)}{dt^2}$$

Comparing this with F = -ky(t)

$$m\frac{d^2y(t)}{dt^2} = -ky(t)$$

$$\frac{d^2y(t)}{dt^2} = -\frac{k}{m}y(t)$$

$$\frac{d^2y(t)}{dt^2} = -\omega^2 y(t)$$

$$\frac{d^2y(t)}{dt^2} + \omega^2 y(t) = 0$$

This is the second order differential equation of the simple harmonic motion. The solution of this equation is of the type

Y(t) = A sinωt or y(t)=Bcosωt

Or any linear combination of sine and cosine function

Y(t) = A sinωt + Bcosωt

Solved Numerical

Q) The SHM is represented by y = 3sin314t + 4cos314t. y in cm and in t in second. Find the amplitude, epoch the periodic time and the maximum velocity of SHO

Solution

Y = A sin(ωt + φ)

Y = A cosφsin ωt + Asinφcosωt

Comparing with 3sin314t + 4cos314t

3 = Acosφ and 4 = Asinφ

∴ $A^2 \cos^2\varphi + A^2\sin^2\varphi = 3^2 + 4^2$

$A^2 = 25$
A = 5 cm
The initial phase (epoch) is obtained as

$$tan\varphi = \frac{sin\varphi}{cos\varphi} = \frac{4}{3}$$
$$\varphi = tan^{-1}\left(\frac{4}{3}\right)$$
$$\varphi = 53^o 8'$$

Now

$$T = \frac{2\pi}{\omega\pi} = \frac{2}{314} = 0.02\ s$$

Maximum velocity
$V_{max} = \omega A = 314 \times 5 = 1570$ cm/s

Q) The vertical motion of a ship at sea is described by the equation $\frac{d^2x}{dt^2} = -4x$ where x in metre is the vertical height of the ship above its mean position. If it oscillates through a total distance of 1m in half oscillation, find the greatest vertical speed and the greatest vertical acceleration.
Solution
Comparing given with standard equation for oscillation we get
$\omega^2 = 4$ or $\omega = 2$
given amplitude A = 1/2
$V_{max} = \omega A = 2\times (1/2) = 1$m/s
$A_{max} = \omega^2 A = 2^2 \times (1/2) = 2$ m/s^2

Total Mechanical Energy in Simple Harmonic Oscillator

The total energy (*E*) of an oscillating particle is equal to the sum of its kinetic energy and potential energy if conservative force acts on it.

Kinetic energy

Kinetic energy of the particle of mass *m* is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(A^2 - x^2) \quad --- eq(1)$$

Potential energy

From definition of SHM $F = -kx$ the work done by the force during the small displacement dx is $dW = -F.dx = -(-kx)\ dx = kx\ dx$
∴ Total work done for the displacement *x* is,

$$W = \int dw = \int_0^x kx\ dx$$

$k=\omega^2 m$

$$W = \int_0^x m\omega^2 x\, dx = \frac{1}{2}m\omega^2 x^2$$

This work done is stored in the body as potential energy

$$U = \frac{1}{2}m\omega^2 x^2 \quad ---(2)$$

Total energy $E = K + U$

$$E = \frac{1}{2}m\omega^2(A^2 - x^2) + \frac{1}{2}m\omega^2 x^2$$

$$E = \frac{1}{2}m\omega^2 A^2$$

Special cases

(i) When the particle is at the mean position x= 0, from eqn (1) it is known that kinetic energy is maximum total energy is wholly kinetic $K_{max} = \frac{1}{2}m\omega^2 A^2$
and from eqn. (2) it is known that potential energy is zero.

(ii) When the particle is at the extreme position y = $+a$, from eqn. (1) it is known that kinetic energy is zero and from eqn. (2) it is known that Potential energy is maximum. Hence the total energy is wholly potential. $U_{max} = \frac{1}{2}m\omega^2 A^2$

(iii)When y = A/2

$$K = \frac{1}{2}m\omega^2\left[A^2 - \left(\frac{A}{2}\right)^2\right]$$

$$K = \frac{3}{4}\left(\frac{1}{2}m\omega^2 A^2\right)$$

$$K = \frac{3}{4}(E)$$

$$U = \frac{1}{2}m\omega^2\left(\frac{A}{2}\right)^2$$

$$U = \frac{1}{4}\left(\frac{1}{2}m\omega^2 A^2\right)$$

If the displacement is half of the amplitude K and U are in the ratio 3 : 1,

Solved numerical

Q) If a particle of mass 0.2kg executes SHM of amplitude 2cm and period of 6 sec find(i0the total mechanical energy at any instant (ii) kinetic energy and potential energies when the displacement is 1cm

Solution:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{6}$$

(i)Total mechanical energy at any instant is given by

$$E = \frac{1}{2} m\omega^2 A^2$$

$$E = \frac{1}{2}(0.2)\left(\frac{2\pi}{6}\right)^2 (2 \times 10^{-2})^2$$

$$E = 0.1 \times \frac{4\pi^2}{36} \times 4 \times 10^{-4}$$

$$E = 4.39 \times 10^{-5} J$$

(ii)K.E. at the instant when displacement x is given by

$$K = \frac{1}{2} m\omega^2 (A^2 - x^2)$$

$$K = \frac{1}{2}(0.2)\left(\frac{2\pi}{6}\right)^2 (4 \times 10^{-4} - 1 \times 10^{-4})\, J$$

$$K = 3.29 \times 10^{-5} J$$

P.E. energy at that instant = Total energy – K.E

= (4.39 -3.29) $\times 10^{-5}$

=1.1$\times 10^{-5}$J

Angular Simple Harmonic Motion

A body to rotate about a given axis can make angular oscillations. For example, a wooden stick nailed to a wall can oscillate about its mean position in the vertical plane

The conditions for an angular oscillation to be angular harmonic motion are

(i)When a body is displaced through an angle from the mean position, the resultant torque is proportional to the angle displaced

(ii)This torque is restoring in nature and it tries to bring the body towards the mean position

If the angular displacement of the body at an instant is θ , then resultant torque on the body

τ = -kθ

if the momentum of inertia is I, the angular acceleration is

$$\alpha = \frac{\tau}{I} = -\frac{k}{I}\theta$$

Or

$$\frac{d^2\theta}{dt^2} = -\omega^2\theta \quad --- eq(1)$$

Here

$$\omega = \sqrt{\frac{k}{I}}$$

Solution of equation (1) is

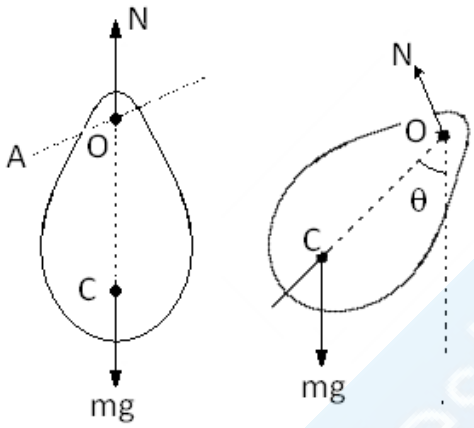
$$\theta = \theta_0 \sin(\omega t + \varphi)$$

Where θ_0 is the maximum angular displacement on either side. Angular velocity at time 't' is given by

$$\omega = \frac{d\theta}{dt} = \theta_0 \omega cos(\omega t + \varphi)$$

Physical pendulum

An rigid body suspended from a fixed support constitutes a physical pendulum.



As shown in figure is a physical pendulum. A rigid body is suspended through a hole at O. When the centre of mass C is vertically below O at a distance of 'l', the body may remain at rest.

The body is rotated through an angle θ about a horizontal axis OA passing through O and perpendicular to the plane of motion The torque of the forces acting on the body, about the axis OA is $\tau = mgl\sin\theta$, here $l = OC$

If momentum of inertia of the body about OA is I, the angular acceleration becomes

$$\alpha = \frac{\tau}{I} = -\frac{mgl}{I} sin\theta$$

For small angular displacement $\sin\theta = \theta$

$$\alpha = -\left(\frac{mgl}{I}\right)\theta$$

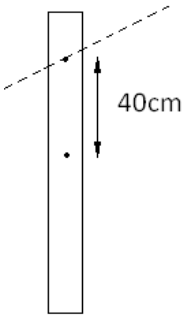
Comparing with $\alpha = -\omega^2\theta$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgl}}$$

Solved Numerical

Q) A uniform meter stick is suspended through a small-hole at the 10cm mark. Find the time period of small oscillations about the point of suspension

Solution:



Let the mass of stick be M. The moment of inertia of the stick about the axis of rotation through the point of suspension is

$$I = \frac{Ml^2}{12} + Md^2 = M\left(\frac{l^2}{12} + d^2\right)$$

Centre of mass of the stick is at 50 cm from top thus it is at distance 40 cm from the small hole thus d = 40 cm, given length of stick = 1m

Time period

$$T = 2\pi\sqrt{\frac{I}{Mgd}}$$

$$T = 2\pi\sqrt{\frac{M\left(\frac{l^2}{12} + d^2\right)}{Mgd}}$$

$$T = 2\pi\sqrt{\frac{\left(\frac{l^2}{12} + d^2\right)}{gd}}$$

$$T = 2 \times 3.14\sqrt{\frac{\frac{1}{12} + (0.4)^2}{9.8 \times 0.4}}$$

T=1.56 sec

Torsional pendulum

In torsional pendulum, an extended body is suspended by a light thread or wire. The body is rotated through an angle about the wire as the axis of rotation.

The wire remains vertical during this motion but a twist 'θ' is produced in the wire. The twisted wire exerts a restoring torque on the body, which is proportional to the angle of the twist.

$\tau \propto -\theta$; $\tau = -k\theta$; k is proportionality constant and is called torsional constant of the wire. If l be the moment of inertia of the body about vertical axis, the angular acceleration is

$$\alpha = \frac{\tau}{I} = \frac{-k}{I}\theta = -\omega^2\theta$$

$$\omega = \sqrt{\frac{k}{I}}$$

Time period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{k}}$

Q) The moment of inertia of the disc used in a torsional pendulum about the suspension wire is 0.2 kg-m^2. It oscillates with a period of 2s. Another disc is placed over the first one and the time period of the system becomes 2.5 s. Find the moment of inertia of the second disc about the wire

Solution

Let the torsional constant of the wire be k

$$T = 2\pi\sqrt{\frac{I}{k}}$$

$$2 = 2\pi\sqrt{\frac{0.2}{k}} \quad ---eq(1)$$

When a second disc having moment of inertia I_1 about the wire is added, the time period is

$$2.5 = 2\pi\sqrt{\frac{0.2 + I_1}{k}} \quad ---eq(2)$$

From eq(1) and (2)

$$I_1 = 0.11 \text{ kg-m}^2$$

Two Body System

In a two body oscillations, such as shown in the figure, a spring connects two objects, each of which is free to move. When the objects are displaced and released, they both oscillate. The relative separation x_1- x_2 gives the length of the spring at any time. Suppose its unscratched length is L; then $x = (x_1 - x_2) - L$ is the change in length of the spring, and F = kx is the magnitude of the force exerted on each particle by the spring as shown in figure.

Applying Newton's second law separately to the two particles, taking force component along the x-axis, we get

$$m_1\frac{d^2x_1}{dt^2} = -kx \quad and \quad m_2\frac{d^2x_2}{dt^2} = +kx$$

Multiplying the first of these equations by m_2 and the second by m_1 and then subtracting,

$$m_1m_2\frac{d^2x_1}{dt^2} - m_1m_2\frac{d^2x_2}{dt^2} = -m_2kx - m_1kx$$

This can be written as

$$\frac{m_1m_2}{(m_1+m_2)}\frac{d^2}{dt^2}(x_1 - x_2) = -kx$$

$$\mu\frac{d^2}{dt^2}(x_1 - x_2) = -kx \quad ---eq(1)$$

Here μ is known as reduced mass and has dimension of mass

$$\mu = \frac{m_1m_2}{(m_1+m_2)}$$

Since L is constant

$$\frac{d^2}{dt^2}(x_1 - x_2) = \frac{d^2}{dt^2}(x + L) = \frac{dx}{dt}$$

Equation (i) becomes

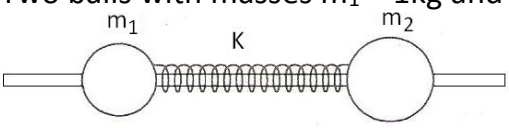
$$\mu\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} = -\frac{k}{\mu}x$$

Thus periodic time

$$T = 2\pi\sqrt{\frac{\mu}{k}}$$

Q) Two balls with masses m_1 = 1kg and m_2 = 2kgkg are slipped on a thin smooth horizontal rod. The balls are interconnected by a light spring of spring constant 24 N/m. the left hand ball is imparted the initial velocity v_1= 12 cm/s.



Find (a) the oscillation frequency of the system (b) the energy and amplitude of oscillation

Solution

Reduced mass

$$\mu = \frac{m_1 m_2}{(m_1 + m_2)} = \frac{1 \times 2}{1 + 2} = \frac{2}{3}\ kg$$

Frequency

f = 1/T

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}}$$

$$f = \frac{1}{2 \times 3.14}\sqrt{\frac{24 \times 3}{2}} = 0.955\ sec$$

Initial velocity given to mass m_1 is v_1

For undamped oscillation, this initial energy remains constant

Hence total energy of S.H.M. of two balls is given as

$$E = \frac{1}{2}\mu v_1^2$$

If amplitude of oscillation is A then

$$\frac{1}{2}\mu v_1^2 = \frac{1}{2}kA^2$$

$$A = \sqrt{\frac{\mu}{k}} v_1$$

$$A = \sqrt{\frac{2}{3 \times 24}} \times 0.12 = 0.02\, m$$

Or A = 2 cm

Damped oscillation

Experimental studies showed that the resistive force acting on the oscillator in a fluid medium depends upon the velocity of the oscillator

Thus resistive force or damping force acting on the oscillator is

$F_d \propto v$

$\therefore F_d = -nv$

Here b is damping constant and has SI units kg/second. The negative sign indicates that the force F_d opposes the motion

Thus, a damped oscillator oscillate under the influence of the following forces

(i)Restoring force $F_x = -kx$ and

(ii) Resistive force $F_d = -bv$

Net force $F = F_x + F_d$

According to second law of motion

$ma = -ky - bv$

$$m\frac{d^2x}{dt^2} = -\mathrm{ky} - \mathrm{b}\frac{\mathrm{dx}}{\mathrm{dt}}$$

$$\frac{d^2x}{dt^2} + \frac{\mathrm{b}}{\mathrm{m}}\frac{\mathrm{dx}}{\mathrm{dt}} + \frac{\mathrm{k}}{\mathrm{m}}\mathrm{y} = 0 \quad ---\ \mathrm{eq}(1)$$

Solution of this equation is

$$x(t) = A\, e^{-bt/2m}\, sin(\omega' t + \varphi)$$

Here $A\, e^{-bt/2m}$ is the amplitude of the damped oscillation and decreases exponentially with time

The angular frequency ω' of the damped oscillator is given by

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

Energy of oscillator

$$E = \frac{1}{2} k A^2 e^{-bt/m}$$

Above equation is valid only if $b << \sqrt{(km)}$

Natural Oscillations

When a system capable of oscillating is given some initial displacement from its equilibrium position and left free (i.e. in absence of any external force) it begins to oscillate. Thus the oscillations performed by it in absence of any resistive forces are known as natural oscillations. The frequency of natural oscillations is known as natural frequency f_0 and corresponding angular frequency is denoted by ω^0

Forced Oscillation

Oscillations of the system under the influence of an external periodic force are forced oscillation

Consider an external periodic force $F = F_0 \sin\omega t$ acting on the system which is capable to oscillate

Equation for oscillation can be witten as

$$m\frac{d^2x}{dt^2} + \mathrm{b}\frac{\mathrm{dx}}{\mathrm{dt}} + \mathrm{ky} = \mathrm{F_0}\sin\omega\mathrm{t}$$

$$\frac{d^2x}{dt^2} + \frac{\mathrm{b}}{\mathrm{m}}\frac{\mathrm{dx}}{\mathrm{dt}} + \frac{\mathrm{k}}{\mathrm{m}}\mathrm{y} = \frac{\mathrm{F_0}}{\mathrm{m}}\sin\omega\mathrm{t}$$

The solution of equation is given by

$$X = A\sin(\omega t + \varphi)$$

Here, A and φ are the constants of the solution they are found as,

$$A = \frac{F_0}{[m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2]^{1/2}}$$

$$\varphi = tan^{-1}\frac{\omega x}{v_0}$$

Here m is the mass of oscillator, v_0 is velocity of oscillator, x is the displacement of oscillator.

(i) for small damping factor

$$m(\omega_0^2 - \omega^2) \gg b\omega$$

Equation for amplitude becomes

$$A = \frac{F_0}{m^2(\omega_0^2 - \omega^2)^2}$$

(ii) For large damping factor

$$b\omega \gg m(\omega_0^2 - \omega^2)$$

$$A = \frac{F_0}{b\omega}$$

If when value of ω approaches $\omega_0$the amplitude becomes maximum. This phenomenon is known as resonance. The value of ω for which resonance occurs is known as the resonant frequency.

Solved numerical

Calculate the time during which the amplitude becomes $A/2^n$ in case of damped oscillations, where A = initial amplitude

Solution:

$$A(t) = Ae^{-bt/2m}$$

But $A(t) = A/2^n$

$$\therefore \frac{A}{2^n} = Ae^{-bt/2m}$$

Taking log to the base e on both sides

$$\frac{bt}{2m} = nln2$$

$$t = \frac{2mnln2}{bt} = \frac{2mn}{bt}(0.693)$$

----------END----------