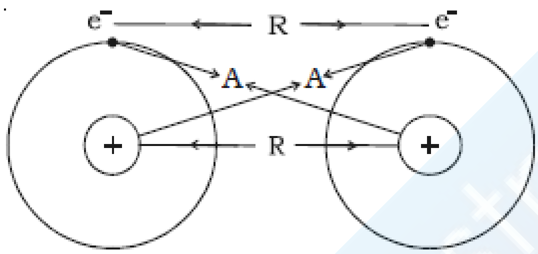
SOLIDS AND FLUIDS

ELASTICITY

In solids, the atoms and molecules are free to vibrate about their mean positions. If this vibration increases sufficiently, molecules will shake apart and start vibrating in random directions. At this stage, the shape of the material is no longer fixed, but takes the shape of its container. This is liquid state. Due to increase in their energy, if the molecules vibrate at even greater rates, they may break away from one another and assume gaseous state. Water is the best example for this changing of states. Ice is the solid form of water. With increase in temperature, ice melts into water due to increase in molecular vibration.
If water is heated, a stage is reached where continued molecular vibration results in a separation among the water molecules and therefore steam is produced. Further continued heating causes the molecules to break into atoms.

Intermolecular or inter atomic forces

Consider two isolated hydrogen atoms moving towards each other as shown in Fig As they approach each other, the following interactions are observed.



(i) Attractive force A between the nucleus of one atom and electron of the other. This attractive force tends to decrease the potential energy of the atomic system.

(ii) Repulsive force R between the nucleus of one atom and the nucleus of the other atom and electron of one atom with the electron of the other atom. These repulsive forces always tend to increase the energy of the atomic system. There is a universal tendency of all systems to acquire a state of minimum potential energy. This stage of minimum potential energy corresponds to maximum stability. If the net effect of the forces of attraction and repulsion leads to decrease in the energy of the system, the two atoms come closer to each other and form a covalent bond by sharing of electrons. On the other hand, if the repulsive forces are more and there is increase in the energy of the system, the atoms will repel each other and do not form a bond. The forces acting between the atoms due to electrostatic interaction between the charges of the atoms are called inter atomic forces. Thus, inter atomic forces are electrical in nature. The inter atomic forces are active if the distance between the two atoms is of the order of atomic size ≈ 10^{-10} m. In the case of molecules, the range of the force is of the order of 10^{-9} m.

Elasticity

When an external force is applied on a body, which is not free to move, there will be a relative displacement of the particles. Due to the property of elasticity, the particles tend to regain their original position. The external forces may produce change in length, volume and shape of the body.

This external force which produces these changes in the body is called deforming force. A body which experiences such a force is called deformed body.
When the deforming force is removed, the body regains its original state due to the force developed within the body. This force is called restoring force.
The property of a material to regain its original state when the deforming force is removed is called elasticity.
The bodies which possess this property are called elastic bodies. Bodies which do not exhibit the property of elasticity are called plastic. The study of mechanical properties helps us to select the material for specific purposes. For example, springs are made of steel because steel is highly elastic

Stress and strain

In a deformed body, restoring force is set up within the body which tends to bring the body back to the normal position. The magnitude of these restoring force depends upon the deformation caused. *This restoring force per unit area of a deformed body is known as stress.* This is measured by the magnitude of the deforming force acting per unit area of the body when equilibrium is established.

$$Stree = \frac{restoring\ force}{Area}$$

Unit of stress in S.I. system is N/m^2. When the stress is normal to the surface, it is called Normal Stress. The normal stress produces a achange in length or a change in volume of the body. The normal stress to a wire or a body may be compressive or tensile (expansive) according as it produces a decrease or increase in length of a wire or volume of the body. When the stress is tangential to the surface, it is called tangential (shearing) stress

Solved Numerical

Q) A rectangular bar having a cross-sectional area of 28 mm^2 has a tensile force of a 7KN applied to it. Determine the stress in the bar
Solution
Cross-sectional area A = 25mm^2 = $28\times(10^{-3})^2$= 28×10^{-6} m^2
Tensile force F = 7KN = 7×10^3N

$$Stree = \frac{7\times 10^3}{28\times 10^{-6}} = 0.25\times 10^9 N/m^2$$

Strain

The external force acting on a body cause a relative displacement of its various parts. A change in length volume or shape takes place. The body is then said to be strained. The relative change produced in the body under a system of force is called strain

$$Strain\ (\varepsilon) = \frac{Change\ in\ dimension}{original\ dimension}$$

Strain has no dimensions as it is a pure number. The change in length per unit length is called linear strain. The change in volume per unit volume is called Volume stain. If there is a change in shape the strain is called shearing strain. This is measured by the angle through which a line originally normal to the fixed surface is turned

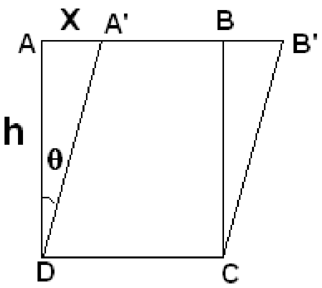
Longitudinal Strain: The ratio of change in length to original length

$$\varepsilon_l = \frac{\Delta l}{l}$$

Volume strain

$$\varepsilon_v = \frac{\Delta v}{v}$$

Shearing strain

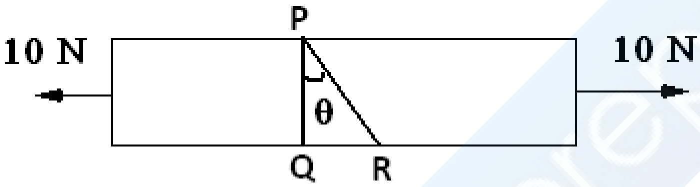


In figure a body with square cross section is shown a tangential force acts on the top surface AB causes shift of Surface by 'X' units shown as surface A'B' ,thus side DA' now mates an angle of θ with original side DA of height h

$$\varepsilon_S = \frac{x}{h} = tan\theta$$

Solved Numerical

Q) As shown in figure 10N force is applied at two ends of a rod. Calculate tensile stress and shearing stress for section PR. Area of cross-section PQ is 10 cm^2 , θ=30^O



Solution

Given cross-section area of PQ = 10 cm^2

Now PQ = PRcosθ

10 = PRcos30

10= PR (√3 /2)

PR = 20/√3 cm^2 or 2/√3 m^2

Now normal force to area PR will be Fcos30 = 10 ×(√3 /2) = 5√3 N

Tangential force to area PR will be Fsin30 = 10×(1/2) = 5 N

∴ Tensile stress for section PR

$$\sigma_l = \frac{normal\ force}{area\ of\ PR} = \frac{5\sqrt{3}}{\frac{2}{\sqrt{3}} \times 10^{-3}} = 7.5 \times 10^3 \frac{N}{m^2}$$

Shearing stress for section PR

$$\sigma_t = \frac{tangential\ force}{area\ of\ PR} = \frac{5}{\frac{2}{\sqrt{3}} \times 10^{-3}} = 2.5\sqrt{3} \times 10^3 \frac{N}{m^2}$$

Testprep Kart

Hooke's Law and types of moduli

According to Hooke's law, *within the elastic limit, strain produced in a body is directly proportional to the stress that produces it.*

$$\frac{stress}{strain} = constant = \lambda$$

Where λ is called modulus of elasticity.
Its unit is N m^{-2} and its dimensional formula is $ML^{-1}T^{-2}$.
Depending upon different types of strain, the following three moduli of elasticity are possible

(i) Young's modulus: When a wire or rod is stretched by a longitudinal force the ratio of the longitudinal stress to the longitudinal strain within the elastic limits is called Young's modulus

$$Young'smodulus\ (Y) = \frac{Longitudinal\ stress}{linear\ strain}$$

Consider a wire or rod of length L and radius r under the action of a stretching force applied normal to its face. Suppose the wire suffers a change in length l then

$$Longitudinal\ stress = \frac{F}{\pi r^2}$$

$$Linear\ strain = \frac{l}{L}$$

$$Young'smodulus\ (Y) = \frac{\frac{F}{\pi r^2}}{\frac{l}{L}} = \frac{FL}{\pi r^2 l}$$

(ii) Bulk modulus: When a solid or fluid (liquid or gas) is subjected to a uniform pressure all over the surface, the shape remains the same, but there is a change in volume. The force perunit area applied normally and uniformly over the surface is called normal stress. The change in volume per unit volume is called volume or bulk strain.

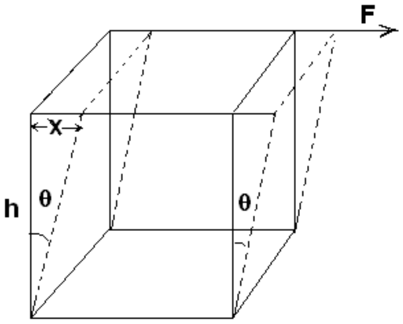
$$Bulk\ modulus\ (B) = \frac{Volume\ stress}{Volume\ strain}$$

$$B = \frac{-\frac{F}{A}}{\frac{\Delta V}{V}} = -\frac{FV}{A\Delta V}$$

Negative sign indicate reduction in volume
The reciprocal of bulk modulus is called compressibility

$$compressibility = \frac{1}{bulk\ modulus}$$

(iii) Modulus of rigidity: According to the definition, the ratio of shearing stress to shearing strain is called modulus of rigidity (η) . in this case the sphape of the body changes but its volume remains unchanged. Consider the case of a cube fixed at its lower face and acted upon by a tangential force F on its upper surface of area A as shown in figure



$$shearing\ stress = \frac{F}{A}$$

$$Shearing\ strain = \theta = \frac{x}{h}$$

$$\eta = \frac{F}{A\theta} = \frac{Fh}{Ax}$$

Solved Numerical

Q) A solid sphere of radius R made of a material of bulk modulus B is surrounded by a liquid in cylindrical container. A massless piston of area A flots on the surface of the liquid. Find the fractional change in the radius of the sphere (dR/R) when a mass M is placed on the piston to compress the liquid

Solution

From the formula of Bulk modulus

$$B = -\frac{FV}{A\Delta V}$$

$$V = \frac{4}{3}\pi R^3$$

$$dV = 4\pi R^2 dR$$

$$B = -\frac{F\frac{4}{3}\pi R^3}{A4\pi R^2 dR}$$

$$\frac{dR}{R} = \frac{Mg}{3AB}$$

Q) Find the natural length of rod if its length is L_1 under tension T_1 and L_2 under tension T_2 within the limits of elasticity

Solution

From the formula of Young's modulus

$$Young's modulus\ (Y) = \frac{\frac{F}{A}}{\frac{l}{L}}$$

Let increase in length for tension T_1 be x and that for tension T_2 be y then

$$\frac{\frac{T_1}{A}}{\frac{x}{L}} = \frac{\frac{T_2}{A}}{\frac{y}{L}}$$

$$\frac{T_1}{x} = \frac{T_2}{y}$$

$$T_1 y = T_2 x$$

But $x = L_1 - L$ and $y = L_2 - L$

$$T_1(L_2 - L) = T_2(L_1 - L)$$

On simplification we get

$$L = \frac{(L_1 T_2 - L_2 T_1)}{(T_2 - T_1)}$$

Q) A copper wire of negligible mass, 1 m length and cross-sectional area $10^{-6}m^2$ is kept on a smooth horizontal table with one end fixed. A ball of mass 1kg is attached to the other end. The wire and the ball are rotated with an angular velocity of 20 rad/s. if the elongation in the wire is 10^{-3}m, obtain the Young's modulus. If on increasing the angular velocity to 100 rad/s the wire breaks down, obtain the breaking stress.

Solution

Given m = 1kg, ω= 20 rad/s, L = 1m $\Delta L = 10^{-3}$ m, $A = 10^{-6}$ m^2

Tension in the thread

$T = m\omega^2 L = 1\times(20)^2\times1 = 400N$

$$Y = \frac{TL}{A\Delta L} = \frac{400 \times 1}{10^{-6} \times 10^{-3}} = 4 \times 10^{11} N/m^2$$

On increasing the angular velocity to 100 rad/s, the wire breaks down then

$$breaking\ stress = \frac{T'}{A} = \frac{m(\omega')^2 L}{A}$$

$$breaking\ stress = \frac{1 \times (100)^2 \times 1}{10^{-6}} = 10^{10} N/m^2$$

Q) A cube is subjected to pressure of 5×10^5 N/m^2. Each side of the cubic is shorteed by 1%. Find volumetric strain and bulk modulus of elasticity of cube

Solution

$V = l^3$

Now $dV = 3l^2\, dl$

Thus

$$\frac{dV}{V} = \frac{3l^2 dl}{l^3} = 3\frac{dl}{l}$$

Sides are reduced by 1% thus dl/l =- 0.01

Thus reduction in volume = -0.03

Normal stress = Increase in pressure

$$B = -\frac{P}{\frac{\Delta V}{V}} = \frac{5 \times 10^5}{0.03} = 1.67 \times 10^7 N/m^2$$

Q) A rubber cube of each side 7cm has one side fixed, while a tangential force equal to the weight of 300kg f is applied to the opposite face. Find the shearing strain produced and the distance through which the strained site moves. The modulus of rigidity for rubber is 2×10^7 dyne/cm^2 g = 10m/s^2

Solution

Here L = 7cm = 7×10^{-2} m

F = 300 kg f = 300×10 N

Modulus of rigidity η = 2×10^7 dynes/cm^2 = 2×10^6 N/m^2

As

$$\eta = \frac{F}{A\theta}$$

$$\theta = \frac{F}{A\eta} = \frac{F}{h^2\eta}$$

$$\theta = \frac{3000}{(7\times10^{-2})^2 \times 2\times10^6} = 0.3\, rad$$

$$\theta = \frac{x}{h}$$

X = hθ

X = 7×0.3 = 2.1 cm

Poisson's Ratio:

It is the ratio of lateral strain to the longitudinal strain. For example, consider a force F applied along the length of the wire which elongates the wire along the length and it contracts radially. Then the longitudinal strain =Δl/l and lateral strain = Δr/r , where r is the radius of the wire

$$Poisson's\ ratio\ (\sigma) = -\frac{\frac{\Delta r}{r}}{\frac{\Delta l}{l}}$$

$$\frac{\Delta r}{r} = -\sigma\frac{\Delta l}{l}$$

For rectangular bar: let b be breadth and h be thickness then

$$\frac{\Delta b}{b} = -\sigma\frac{\Delta l}{l}$$

$$\frac{\Delta h}{h} = -\sigma\frac{\Delta l}{l}$$

The negative sign indicates that change in length and radius is of opposite sign. Change in volume due to longitudinal force

Due to application of tensile force, lateral dimension decreases and length increases. As a result there is a change in volume (usually volume increases). Let us consider the case of a cylindrical rod of length l and radius r.

Since $V=\pi r^2 L$

$$\therefore \frac{\Delta V}{V} = 2\frac{\Delta r}{r} + \frac{\Delta l}{l} \ (for\ very\ small\ change)$$

From above equations or radius and Length

$$\therefore \frac{\Delta V}{V} = -2\sigma\frac{\Delta l}{l} + \frac{\Delta l}{l}$$

$$\therefore \frac{\Delta V}{V} = \frac{\Delta l}{l}(1-2\sigma)$$

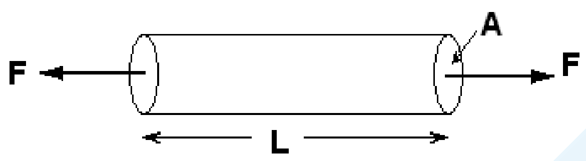
Longitudinal Strain: $\varepsilon_l = \frac{\Delta l}{l}$

$$\therefore \frac{\Delta V}{V} = \varepsilon_l(1-2\sigma)$$

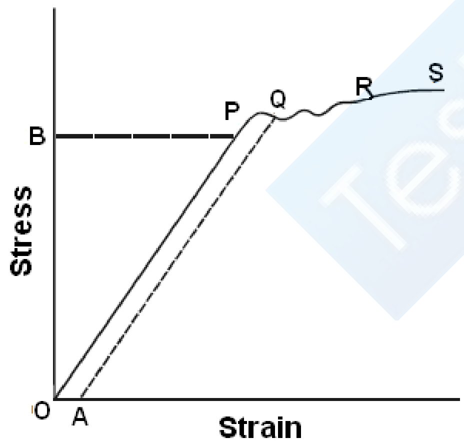
Above equation suggest that since Δv > 0, value of σ cannot exceed 0.5

Stress –Strain relationship for a wire subjected to longitudinal stress

Consider a long wire (made of steel) of cross-sectional area A and original length L in equilibrium under the action of two equal and opposite variable force F as shown in figure. Due to the application of force, the length gets changed to L +l. Then, longitudinal stress = F/A and Longitudinal strain = l/L



The extension of the wire is suitably measured and a stress – strain graph is plotted



(i) In the figure the region OP is linear. Within a normal stress, strain is proportional to the applied stress. This is
Hooke's law. Up to P, when the load is removed the wire
regains its original length along PO. The point P represents the **elastic limit**, PO represents the elastic range of the material and OB is the **elastic strength**.
(ii) Beyond P, the graph is not linear. In the region PQ the material is partly elastic and partly plastic. From Q, if we start decreasing the load, the graph does not come to O via P, but traces a straight line QA.
Thus a permanent strain OA is caused in the wire. This is called **permanent set**.
(iii) Beyond Q addition of even a very small load causes enormous strain. This point Q is called the yield point. The region QR is the **plastic range**.
(iv) Beyond R, the wire loses its shape and becomes thinner and thinner in diameter and ultimately breaks, say at S. Therefore S is the **breaking point**. The stress corresponding to S is called **breaking stress**.

Elastic potential energy or Elastic energy stored in a deformed body

The elastic energy is measured in terms of work done in straining the body within its elastic limit

Let F be the force applied across the cross-section A of a wire of length L. Let l be the increase in length. Then

$$Y = \frac{\frac{F}{A}}{\frac{l}{L}} = \frac{FL}{Al}$$

$$F = \frac{YAl}{L}$$

If the wire is stretched further through a distance of dl, the work done dw

$$dW = F \times dl = \frac{YAl}{L}dl$$

Total work done in stretching the wire from original length L to a length L +l (i.e. from l = 0 to l = l)

$$W = \int_0^l \frac{YAl}{L}dl$$

$$W = \frac{YA}{L}\frac{l^2}{2} = \frac{1}{2}(AL)\left(\frac{Yl}{L}\right)\left(\frac{l}{L}\right)$$

$$W = \frac{1}{2} \times volume \times stress \times strain$$

Solved Numerical

Q) The rubber cord of catapult has a cross-section area 1mm^2 and total unstrtched length 10 cm. It is stretched to 12cm and then released to project a body of mass 5g. taking the Young's modulus of rubber as 5×10^8 N/m^2, calculate the velocity of projection

Solution

It can be assumed that the total elastic energy of catapult is converted into kinetic energy of the body without any heat loss

L = 12cm = 12×10^{-2} m , l = 2cm = 2×10^{-3} m , A = 1mm^2 = 10^{-6}m

$$U = \frac{YA}{L}\frac{l^2}{2} = \frac{5 \times 10^8 \times (1 \times 10^{-6}) \times (2 \times 10^{-2})^2}{2 \times 10 \times 10^{-2}} = 1$$

Now K.E of projectile = elastic energy of catapult

$$\frac{1}{2}mv^2 = U$$

$$\frac{1}{2} \times 5 \times 10^{-3} \times v^2 = 1$$

V = 20 m/s

FLUID STATICS

Thrust and Pressure

A perfect fluid resists force normal to its surface and offers no resistance to force acting tangential to it surface. A heavy log of wood can be drawn along the surface of water with very little effort because the force applied on the log of wood is horizontal and parallel to the surface of water. Thus fluids are capable of exerting normal stress on the surface with it is in contact

Force exerted perpendicular to a surface is called thrust and thrust per unit area is called pressure

Variation of pressure with height

Let *h* be the height of the liquid column in a cylinder of cross sectional area *A*. If ρ is the density of the liquid, then weight of the liquid column *W* is given by

W = mass of liquid column × g = $Ah\rho g$

By definition, pressure is the force acting per unit area.

$$Pressure = \frac{weight\ of\ liquid\ column}{area\ of\ cross\ -section}$$

$$P = \frac{Ah\rho g}{A} = h\rho g$$

$dP = \rho g\,(dh)$

This differential relation shows that the pressure in a fluid increases with depth or decreases with increased elevation. Above equation holds for both liquids and gases.

Liquids are generally treated as incompressible and we may consider their density ρ constant for every part of liquid. With ρ as constant, equation may be integrated as it stands, and the result is

$P = P_0 + \rho gh$

The pressure P_0 is the pressure at the surface of the liquid where h = 0

Force due to fluid on a plane submerged surface

The pressure at different points on the submerged surface varies so to calculate the resultant force, we divide the surface into a number of elementary areas and we calculate the force on it first by treating pressure as constant then we integrate it to get the net force i.e $F_R = \int P\,(dA)$

The point of application of resultant force must be such that the moment of the resultant force about any axis is equal to the moment of the distributed force about the axis

Solved Numerical

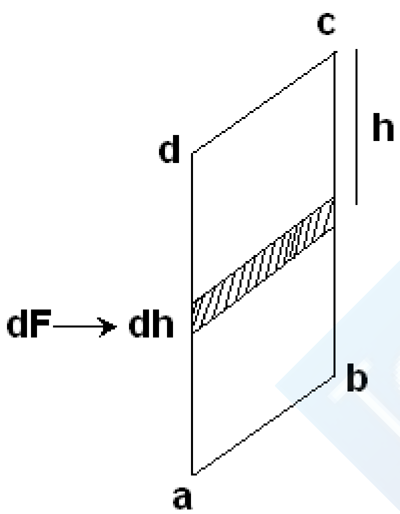
Q) Water is filled upto the top in a rectangular tank of square cross-section. The sides of cross-section is a and height of the tank is H. If density of water is ρ, find force on the bottom of the tank and on one of its wall. Also calculate the position of the point of application of the force on the wall

Solution
Force on the bottom of thank
Area of bottom of tank = a^2
Force = pressure × Area
Force = $H\rho g a^2$
Force on the wall and its point of application
Force on the wall of the tank is different at different heights so consider a segment at depth h of thickness dh
Pressure at depth h = $h\rho g$
Area of strip = a dh
Force on strip dF = $h\rho g$ a dh
Total force at on the wall

$$F = \int_0^H \rho g a h \, dh$$

$$F = \rho g a \left[\frac{h^2}{2}\right]_0^H$$

$$F = \rho g a \frac{H^2}{2}$$



The point of application of the force on the wall can be calculated by equating the moment of resultant force about any line, say dc to the moment of distributioed force about the same line dc
Moment of dF about line cd = dF (h) = ($h\rho g a dh$) h = $\rho g a h^2$ dh
∴ Net moment of distributed forces

$$\rho g a \int_0^H h^2 dh = \rho g a \frac{H^3}{3}$$

Let the point of application of the net force is at a depth ‘x’ from the line cd
Then the torque of the resultant force about the line cd = $Fx = \rho g a \frac{H^2}{2} x$
Now Net moment of distribution of force = Torque

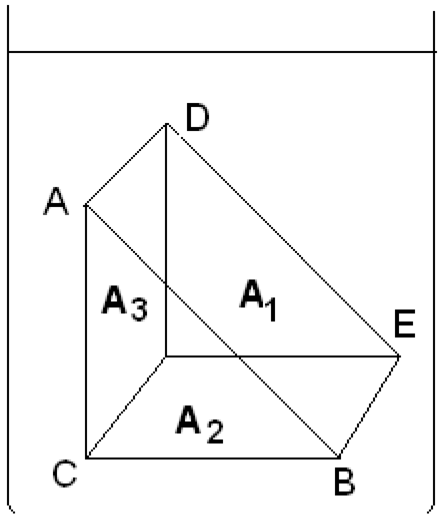
$$\rho g a \frac{H^3}{3} = \rho g a \frac{H^2}{2} x$$

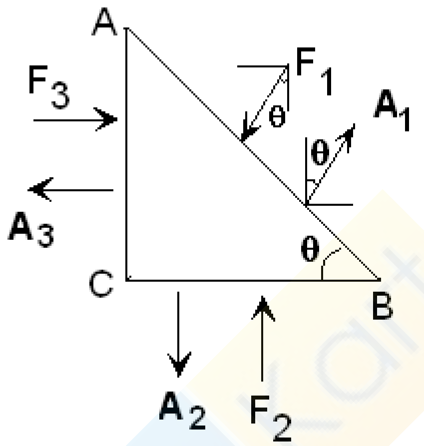
$$x = \frac{2H}{3}$$

Hence, the resultant force on the vertical wall of the tank will act at a depth 2H/3 from the free surface of water or at the height of H/3 from bottom of tank

Pascal's Law

Pascal's law states that if the effect of gravity can be neglected then the pressure in an incompressible fluid in equilibrium is the same everywhere..
This statement can be verified as follows
Consider a small element of liquid in the interior of the liquid at rest. The liquid element is in the shape of prism consisting of two right angled triangle surfaces





Let the areas of surface ADEB, CFEB, ADFC be A_1, A_2, A_3
It is clear from figure that
$A_2 = A_1\cos\theta$ and $A_3 = A_1\sin\theta$
Also, since liquid element is in equilibrium $F_3 = F_1\cos\theta$ and $F_3 = F_1\sin\theta$
now pressure on surface ADEB is $P_1 = F_1 / A_1$
Pressure on the surface CFED is

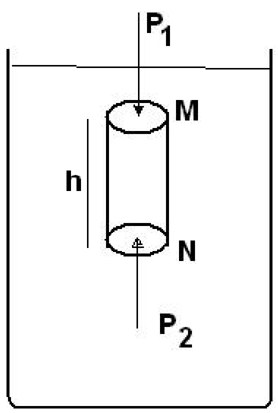
$$P_2 = \frac{F_2}{A_2} = \frac{F_1 cos\theta}{A_1 cos\theta} = \frac{F_1}{A_1}$$

And pressure on the surface ADFC is

$$P_3 = \frac{F_3}{A_3} = \frac{F_1 sin\theta}{A_1 sin\theta} = \frac{F_1}{A_1}$$

So, $P_1 = P_2 = P_3$
Since θ is arbitrary this result holds for any surface. Thus Pascal's law is verified

Pascal's law and effect of gravity



When gravity is taken into account, Pascal's law is to be modified.
Consider a cylindrical liquid column of height *h* and density ρ in a vessel as shown in the Fig.
If the effect of gravity is neglected, then pressure at *M* will be equal to pressure at *N*.
But, if force due to gravity is taken into account, then they are not equal.
As the liquid column is in equilibrium, the forces acting on it are balanced. The vertical forces acting are

(i) Force P_1A acting vertically down on the top surface.
(ii) Weight mg of the liquid column acting vertically downwards.
(iii) Force P_2A at the bottom surface acting vertically upwards. where P_1 and P_2 are the pressures at the top and bottom faces, A is the area of cross section of the circular face and m is the mass of the
cylindrical liquid column.

$$\text{At equilibrium, } P_1A + mg - P_2A = 0 \text{ or } P_1A + mg = P_2A$$
$$P_2 = P_1 + mg\ A$$
$$\text{But } m = Ah\rho$$
$$\therefore P_2 = P_1 + Ah\rho gA$$
$$\text{(i.e) } P_2 = P_1 + h\rho g$$

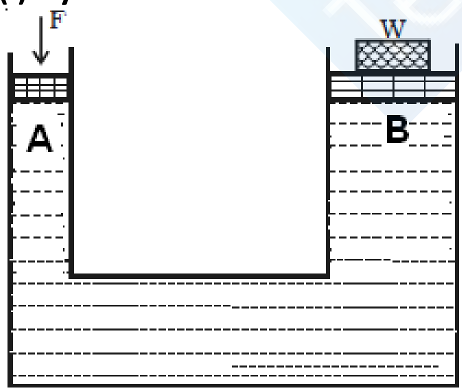
This equation proves that the pressure is the same at all points at the same depth. This results in another statement of *Pascal's law* which can be stated as *change in pressure at any point in an enclosed fluid at rest is transmitted undiminished to all points in the fluid and act in all directions.*

Characteristics of the fluid pressure

(i) Pressure at a point acts equally in all directions
(ii) Liquids at rest exerts lateral pressure, which increases with depth
(iii) Pressure acts normally on any area in whatever orientation the area may be held
(iv) Free surface of a liquid at rest remains horizontal
(v) pressure at every point in the same horizontal line is the same inside a liquid at rest
(vi) liquid at rest stands at the same height in communicating vessels

Application of Pascal's law

(i) Hydraulic lift



An important application of Pascal's law is the hydraulic lift used to lift heavy objects. A schematic diagram of a hydraulic lift is shown in the Fig.. It consists of a liquid container which has pistons fitted into the small and large opening cylinders. If a_1 and a_2 are the areas of the pistons A and B respectively, F is the force applied on A and W is the load on B, then

$$\frac{F}{a_1} = \frac{W}{a_2}$$
$$F = W\frac{a_1}{a_2}$$

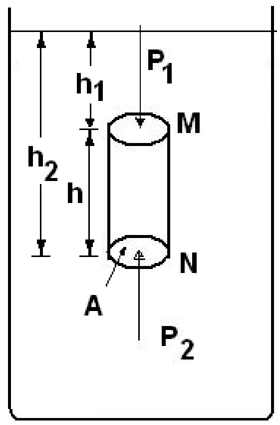
This is the load that can be lifted by applying a force F on A. In the above equation a_2/a_1 is called mechanical advantage of the hydraulic lift. One can see such a lift in many automobile service stations.

Buoyancy and Archimedes principle

If an object is immersed in or floating on the surface of a liquid, it experiences a net vertically upward force due to liquid pressure. This force is called as Buoyant force or force of Buoyancy and it acts from the centre of gravity of the displaced liquid. According to Archimedes principle, “the magnitude of force of buoyancy is equal to the weight of the displaced liquid”

To prove Archimedes principle, consider a body totally immersed in a liquid as shown in the figure.

The vertical force on the body due to liquid pressure may be found most easily by considering a cylindrical volume similar to that one shown in figure



The net vertical force on the element is

dF = (P_2- P_1) A

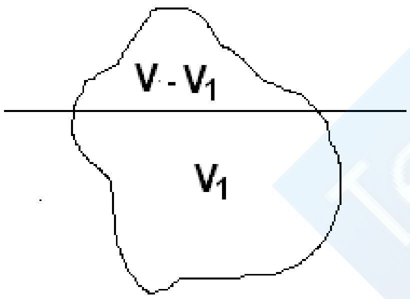
$$F = [(P_0 + h_2\rho g) - (P_0 + h_1\rho g)]A$$
$$F = (h_2 - h_1)\rho g A$$
$$F = h\rho g A$$

But volme V = hA

Thus F = Vρg

∴ force of Buoyancy = Vρg = Weight of liquid displaced

Expression for immersed volume of a floating Body



Let a solid of volume V and density ρ floats in liquid of density ρ_0. Volume V_1 of the body is immersed inside the liquid

The weight of floating body = Vρg

The weight of the displaced liquid = $V_1\rho_0 g$

For the body to float

Weight of body = Weight of liquid displaced

$V\rho g = V_1\rho_0 g$

$$\frac{V_1}{V} = \frac{\rho}{\rho_0}$$
$$V_1 = \frac{\rho V}{\rho_0}$$

∴ Immersed volume = mass of solid / density of liquid

From above it is clear that density of the solid volume must be less than density of the liquid to enable it to float freely in the liquid. How ever a metal vessel floats in water though the density of metal is much higher that the that of eater because floating bodies are hollow inside and hence displaces large volume. When thy float on water, the weight of the displaced water is equal to the weight of the body

Laws of floatation

The principle of Archimedes may be applied to floating bodies to give the laws of flotation

(i) When a body floats freely in a liquid the weight of the floating body is equal to the weight of the liquid displaced

(ii) The centre of gravity of the displaced liquid B (called the centre of buoyancy) lies vertically above or below the centre of gravity of the floating body G

Solved numerical

Q) A stone of mass 0.3kg and relative density 2.5 is immersed in a liquid of relative density 1.2. Calculate the resultant up thrust exerted on the stone by the liquid and the weight of stone in liquid

Solution

Volume of stone V = mass/density

V = 0.3/2.5 = 0.12 m^3

Upward thrust =buoyant force = $V\rho_0 g$ =0.12×1.2×9.8 = 1.41 N

Weight of stone in liquid = Gravitational force – buoyant force

=0.3×9.8 – 1.41 = 1.53 N or 0.156 kg wt

Q) A metal cube floats on mercury with (1/8) th of its volume under mercury. What portion of the cube will remain under mercury if sufficient water is added hust to cover the cube. Assume that the top surface of the cube remains horizontal in both cases. Relative density of mercury = 13.6

From the formula

$$V_1 = \frac{\rho V}{\rho_0}$$

Here V_1 is volume immersed in mercury = V/8 given and ρ_0 density of mercury, ρ density of metal

$$\frac{V}{8} = \frac{\rho V}{13.5}$$

ρ=1.725 is density of metal

Now let V' be the volume immersed in mercury then V-V' is volume immersed in water then

Weight of metal = Buoyant force due to Water + Buoyant force due to mercury

V(1.725) g= (V-V')×1×g +V'×13.6g

V(1.725) = (V-V')×1 + (V'×13.6)

V(0.725) = 12.6V'

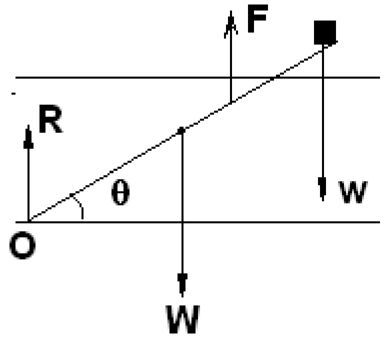
$$\frac{V'}{V} = \frac{0.725}{12.6} = \frac{1}{8}$$

Thus in the second case only (1/18)th of the volume of the cube is under mercury

Q) A rod of length 6 m has a mass of 12kg. If it is hinged at one end at a distance of 3m below a water surface

(i) What weight should be attached to the other end so that 5 m of rod be submerged?

(ii) find the magnitude and direction of the final force exerted on the rod exerted by hinge.
Specific gravity of the material of the rod is 0.5
Solution
Since one end is fixed in water we have to calculate moment of force



Moment of force due to weight of rod about point O= Wg(L/2)cosθ
Moment of force due to additional weight about point O =wL cosθ
Moment of force due to Buoyant force(F) about point O = F(l/2)cosθ
Here l is the length of rod immersed in water = 5m And L is total length of rod

Since rod is at rotational equilibrium at equilibrium
$F(l/2)\cos\theta = wL\cos\theta + Wg(L/2)\cos\theta$
$F(l/2) = wL + W(L/2)g$ --- eq(1)

But F = Vρg
Since 5m is immersed in water thus (5/6) of volume of rod is immersed
Volume of rod = mass/density = 12/0.5 = 24 m^3
Thus F = (5/6)×24×1×g = 20g N
Substituting values in eq(1) we get
∴ 20(2.5)g = w(6) + (12)(3)g
50 = 6w + 36
w = 14g/6 = 2.33g N
w = 2.33 kg wt
Now R = W+w-F
R = 12g +2.33g – 20g
R = -5.67g N
R = -5.67 kg wt
The negative sign indicates that the reaction (vertical) at the hinges acts downwards

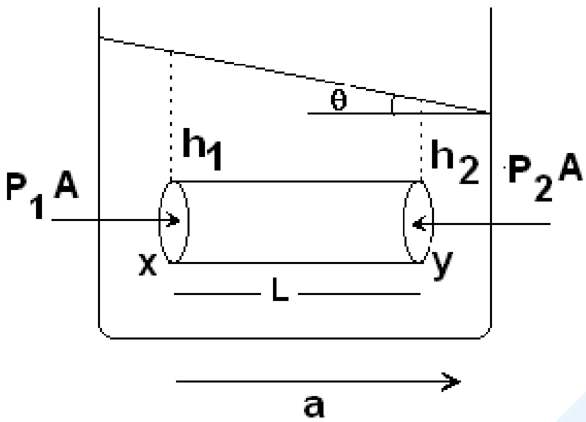
Liquid in accelerated Vessel

Variation of pressure and force of buoyancy in a liquid kept in accelerated vessel
Consider a liquid of density ρ kept in a vessel moving with acceleration **a in upward** direction. Let height of liquid column be h
Then effective gravitational acceleration on liquid = g +a
Thus pressured exerted at depth h $P = P_0 + \rho(g+a)h$
Similarly if liquid in container moves **down with acceleration a**

Then effective gravitational acceleration on liquid = g –a
Thus pressure exerted at depth h, $P = P_0 + \rho(g-a)h$

Also Buoyant force on immersed body when liquid is **moving up**
$F_B = V\rho(g+a)$
Buoyant force on immersed body when liquid is **moving down**
$F_B = V\rho(g+a)$
V is volume of the liquid displaced

Shape of free surface of a liquid in horizontal accelerated vessel



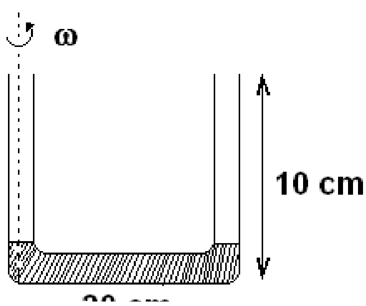
When a vessel filled with liquid accelerates horizontally. We observe its free surface inclined at some angle with horizontal. To find angle θ made by free surface with horizontal, consider a horizontal liquid column including two points x and y at the depth of h_1 and h_2 from the inclined free surface of liquid as shown in figure

Force on area at x = $P_1A = h_1\rho g$
Pseudo force at y= mass of liquid tube of length L and cross sectional area A ×acceleration
Pseudo force at y=ρ(LA)
Total force at y = P_1A + Pseudo force
Force on area at y = $h_1\rho g + \rho(LA)$
Since liquid is in equilibrium
Force on area at x = Force on area at y
$h_1\rho g = h_1\rho g + \rho(LA)$
$(h_1 - h_2)\, g = La$
From geometry of figure

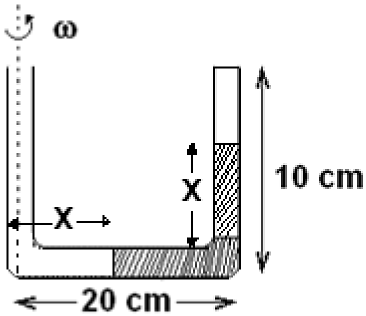
$$\frac{h_1 - h_2}{L} = \frac{a}{g}$$

$$tan\theta = \frac{a}{g}$$

Q) Length of a horizontal arm of a U-tube is 20cm and end of both the vertical arms are open to a pressure 1.01×10^3 N/m^2. Water is poured into the tube such that liquid just fills horizontal part of the tube is then rotated about a vertical axis passing through the other vertical arm with angular velocity ω. If length of water in sealed tube rises to 5cm, calculate ω. Take density of water = 10^3 kg/m^3 and g = 10 m/s^2. Assume temperature to be constant.



Solution



when tube is rotated liquid will experience a centrifugal force thus water moves up in second arm of the U tube.
When centrifugal force + pressure in first arm = force due to pressure in second closed arm +force due to liquid column then equilibrium condition is established ---eq(1)

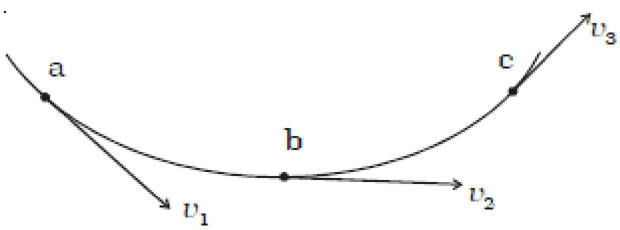
Calculation of force due to pressure in closed tube
Before closing pressure $P_i = 1.01\times10^3$ N/m^2
Volume before closing V_i= 0.1A (A is area of cross-section)
After closing the other arm Pressure P_f and volume V_f = 0.05A
From equation $P_iV_i = P_fV_f$
$(1.01\times10^3)\times 0.1A = P_f \times(0.05A)$
$P_f = 2.02\times10^3$
Force due to pressure = $(2.02\times10^3)\times A$
Pressure in first arm = 1.01×10^3
Calculation of force due to liquid column in second arm
Height of liquid column = 0.05 m
Thus pressure due to column = $h\rho g = 0.05\times10^3\times10 = 500$ N/m^2
Force due to liquid column PA= 0.5A
Calculation of centrifugal force
Mass of the liquid in horizontal part = volume ×density = $(0.2-0.05)A\times10^3=150A$
Centre of mass of horizontal liquid from first arm 'r' $=0.05+\frac{0.2-0.05}{2}=0.125\ m$
Centrifugal force = $m\omega^2 r = 150A\times\omega^2\times0.125 = (18.75A)\omega^2$

Now substituting values in equation 1 we get
$(18.75A)\times\omega^2 + (1.01\times10^3)\times A = (2.02\times10^3)\times A + 500A$
$(18.75)\times\omega^2 + 1.01\times10^3 = (2.02\times10^3) + 500$
ω = 8.97 rad/s

Fluid dynamics

Streamline flow

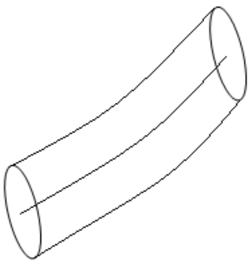
The flow of a liquid is said to be steady, streamline or laminar if every particle of the liquid follows exactly the path of its preceding particle and has the same velocity of its preceding particle at every point.



Let abc be the path of flow of a liquid and v_1, v_2 and v_3 be the velocities of the liquid at the points a, b and c respectively. During a streamline flow, all the particles arriving at 'a' will

have the same velocity v_1 which is directed along the tangent at the point 'a'. A particle arriving at b will always have the same velocity v_2. This velocity v_2 may or may not be equal to v_1.
Similarly all the particles arriving at the point 'c' will always have the same velocity v_3. In other words, in the streamline flow of a liquid, the velocity of every particle crossing a particular point is the same.
The streamline flow is possible only as long as the velocity of the fluid does not exceed a certain value. This limiting value of velocity is called critical velocity.

Tube of flow



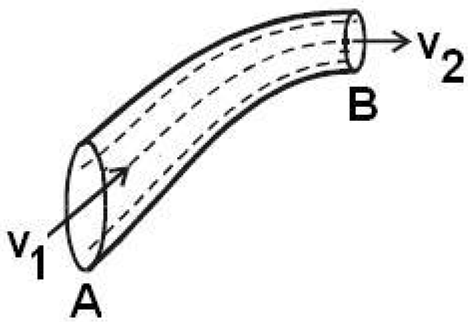
In a fluid having a steady flow, if we select a finite number of streamlines to form a bundle
like the streamline pattern shown in the figure, the tubular region is called a tube of flow.
The tube of flow is bounded by a streamlines so that by fluid can flow across the boundaries of the tube of flow and any fluid that enters at one end must leave at the other end.

Turbulent flow

When the velocity of a liquid exceeds the critical velocity, the path and velocities of the liquid become disorderly. At this stage, the flow loses all its orderliness and is called turbulent flow. Some examples of turbulent flow are :
(i) After rising a short distance, the smooth column of smoke
from an incense stick breaks up into irregular and random patterns.
(ii) The flash - flood after a heavy rain.
Critical velocity of a liquid can be defined as that velocity of liquid upto which the flow is streamlined and above which its flow becomes turbulent.

Equation of continuity

Consider a non-viscous liquid in streamline flow through a tube AB of varying cross section as shown in Fig. Let a_1 and a_2 be the area of cross section, v_1 and v_2 be the velocity of flow of the liquid at A and B respectively.

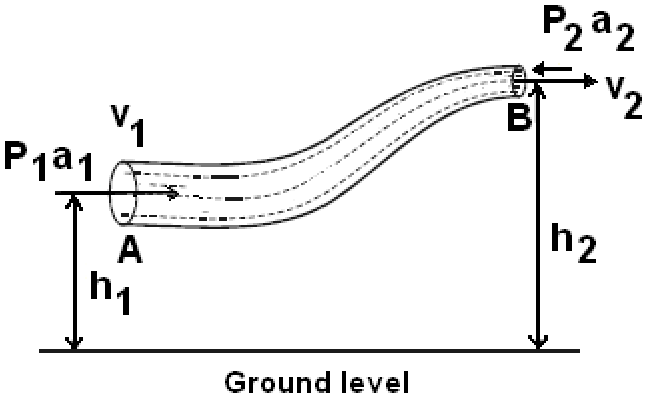


∴ Volume of liquid entering per second at A = a_1v_1.
If ρ is the density of the liquid, then mass of liquid entering per second at A = $a_1v_1\rho$.
Similarly, mass of liquid leaving per second at B = $a_2v_2\rho$
If there is no loss of liquid in the tube and the flow is steady, then mass of liquid entering per second at A = mass of liquid leaving per second at B
(i.e) $a_1v_1\rho = a_2v_2\rho$ or $a_1v_1 = a_2v_2$

i.e. av = constant
This is called as the equation of continuity. From this equation v is inversely proportional to area of cross-section along a tube of flow
i.e. the larger the area of cross section the smaller will be the velocity of flow of liquid and vice-versa.

Bernoulli's Equation

The theorem states that the work done by all forces acting on a system is equal to the change in kinetic energy of the system



Consider streamline flow of a liquid of density ρ through a pipe AB of varying cross section.
Let P_1 and P_2 be the pressures and a_1 and a_2, the cross sectional areas at A and B respectively. The liquid enters A normally with a velocity v_1 and leaves B normally with a velocity v_2. The liquid is accelerated against the force of gravity while flowing from A to B, because the height of B is greater than that of A from the ground level. Therefore P_1 is greater than P_2. This is maintained by an external force.
The mass m of the liquid crossing per second through any section of the tube in accordance with the equation of continuity is $a_1v_1\rho = a_2v_2\rho = m$
Or

$$a_1v_1 = a_2v_2 = \frac{m}{\rho}$$

As $a_1 > a_2$, $v_1 < v_2$
The force acting on the liquid at A = P_1a_1
The force acting on the liquid at B = $P_2\,a_2$
Work done per second on the liquid at A = $P_1a_1 \times v_1 = P_1V$
Work done by the liquid at B = $P_2a_2 \times v_2 = P_2V$
∴ Net work done per second on the liquid by the pressure energy in moving the liquid from A to B is = $P_1V - P_2V$
If the mass of the liquid flowing in one second from A to B is m, then increase in potential energy per second of liquid from A to B is = $mgh_2 - mgh_1$
Increase in kinetic energy per second of the liquid.

$$\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

According to work-energy principle,
work done per second by the
pressure energy = (Increase in potential energy + Increase in kinetic energy) per second

$$P_1V - P_2V = (mgh_2 - mgh_1) + \left(\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2\right)$$

$$P_1V + mgh_1 + \frac{1}{2}mv_1^2 = P_2V + mgh_2 + \frac{1}{2}mv_2^2$$

$$\frac{P_1V}{V} + \frac{m}{V}gh_1 + \frac{1}{2}\frac{m}{V}v_1^2 = \frac{P_2V}{V} + \frac{m}{V}gh_2 + \frac{1}{2}\frac{m}{V}v_2^2$$

$$P_1 + \rho gh_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho gh_2 + \frac{1}{2}\rho v_2^2$$

Since subscripts 1 and 2 refer to any location on the pipeline, we can write in general

$$P + \rho gh + \frac{1}{2}mv^2 = constant$$

The above equation is called Bernoulli's equation for steady non-viscous incompressible flow. Dividing the above equation by gh we can rewrite the above equation as

$$h + \frac{v^2}{2g} + \frac{P}{\rho g} = constnat, which\ is\ called\ total\ head$$

Term h is called elevation head or gravitational head

$$\frac{v^2}{2g}\ is\ called\ velocity\ head$$

$$\frac{P}{\rho g}\ is\ called\ pressure\ head$$

Above equation indicates for **ideal liquid velocity increases when pressure decreases and vice-versa**

Q) A vertical tube of diameter 4mm at the bottom has a water passing through it. If the pressure be atmospheric at the bottom where the water emerges at the rate of 800gm per minute, what is the pressure at a point in the tube 5cm above the bottom where the diameter is 3mm

Solution

Rate of flow of water = 800 gm/min = (40/3)gm/sec

Now mass of water per sec = velocity ×area × density

$40/3 = V_1 \times[\pi\ (0.2)^2]\times1$

$V_1 = (333.33/\pi)$ cm/sec

Now $A_1V_1 = A_2V_2$ Thus

$V_2 = (4/3)V_1$

$V_2 = (444.44/\pi)$ cm/sec is the velocity at height 25cm

Now P_1 = atmospheric pressure = 1.01×10^7 dyne

Now from Bernoulli's equation

$$P_1 + \rho gh_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho gh_2 + \frac{1}{2}\rho v_2^2$$

$$1.01 \times 10^6 + 0 + \frac{1}{2}\left(\frac{333.33}{\pi}\right)^2 = P_2 + 1 \times 981 \times 25 + \frac{1}{2}\left(\frac{444.44}{\pi}\right)^2$$

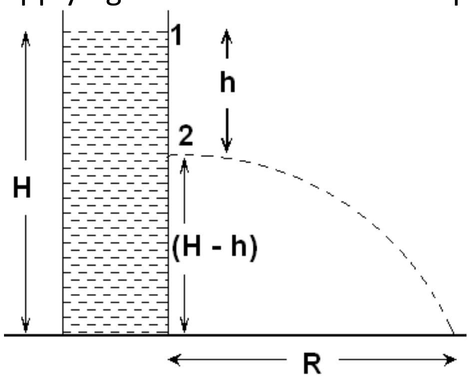
On solving

$P_2 = 0.98\times10^6$ dyne

Now pressure = $h\rho_0 g$ here ρ_0 is density of mercury = 13.6 in cgs system

$0.98\times10^6 = h\times980\times13.6$
H = 73.5 cm of Hg

Q) Water stands at a depth H in a tank whose side walls are vertical. A hole is made at one of the walls at depth h below the water surface. Find at what distance from the foot of the wall does the emerging stream of water strike the flower. What is the maximum possible range?
Solution
Applying Bernoulli's theorem at point 1 and 2



$$P_1 + \rho g h_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho g h_2 + \frac{1}{2}\rho v_2^2$$

$P_1 = P_2 = P$ (atmospheric pressure)
$V_1 = 0$, $h_2 = H-h$ and $h_1 = H$

$$\rho g H = \rho g(H-h) + \frac{1}{2}\rho v_2^2$$
$$v_2^2 = 2gh$$

The vertical component of velocity of water emerging from hole at 2 is zero. Therefore time taken (t) by the water to fall through a distance (H-h) is given bu

$$H - h = \frac{1}{2}gt^2$$
$$t = \sqrt{\frac{2(H-h)}{g}}$$

Required horizontal range $R = v_2 t$

$$R = \sqrt{2gh}\sqrt{\frac{2(H-h)}{g}}$$
$$R = 2\sqrt{h(H-h)}$$

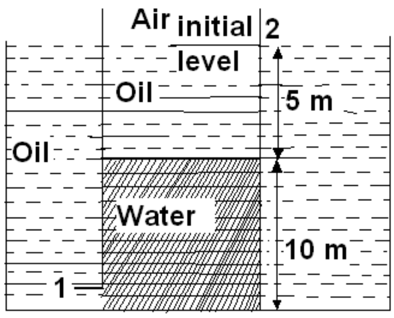
the range is maximu when dR/dh = 0

$$2\times\frac{1}{2}(Hh-h^2)^{\frac{-1}{2}}(H-2h) = 0$$

This gives h = H/2
Therefore Maximu range =

$$R = 2\sqrt{\frac{H}{2}\left(H-\frac{H}{2}\right)} = H$$

Q) A tank with a small circular hole contains oil on top of water. It is immersed in a large tank of same oil. Water flows through the hole. What is the velocity of the flow initially? When the flow stops, what would be the position of the oil-water interface in the tank? The ratio of the cross-section area tank to the that of hole is 50, determine the time at which the flow stops, density of oil = 800 kg/m^3



Solution:

Pressure at hole and pressure at point on the bottom of water is different thus water flows through the hole

Pressure at point 1 $P_1 = P_0 + h\rho_0 g$ here h = 15m and ρ_0 = 800 kg/m^3

Pressure at point 2 is $P_2 = P_0$

And potential = $5\rho_0 g + 10\rho g$ here ρ is density of water

$$P_1 + \rho g h_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho g h_2 + \frac{1}{2}\rho v_2^2$$

$$P_0 + h\rho_0 g + \frac{1}{2}\rho v_1^2 = P_0 + 5\rho_0 g + 10\rho g + \frac{1}{2}\rho v_2^2$$

For continuity equation $A_1V_1 = A_2 V_2$

$$V_2 = \frac{A_1}{A_2}V_1 = \frac{1}{50}V_1$$

$$15 \times 800 \times 10 + \frac{1}{2}1000v_1^2 = 5 \times 800 \times 10 + 10 \times 1000 \times 10 + \frac{1}{2}1000 \times \left(\frac{v_1}{50}\right)^2$$

$$120000 + 500v_1^2 = 140000 + 500 \times \left(\frac{v_1}{50}\right)^2$$

$$v_1^2\left(500 - \frac{1}{2500}\right) = 20000$$

$$v_1^2(500) = 20000$$

$$V_1 = 6.32 \text{ m/s}$$

Let x be the height of water column when flow of water is stopped

Applying Bernoull's equation between point a and x we het

$$P_0 + 15\rho_0 g + \frac{1}{2}\rho v_1^2 = P_0 + 5\rho_0 g + x\rho g + \frac{1}{2}\rho v_2^2$$

Since velocities are zero

$$15\rho_0 g = 5\rho_0 g + x\rho g$$

$$15 \times 800 = 5 \times 800 + x \times 1000$$

$$X = 8 \text{ m}$$

Let at any moment of time height of water column be y then level of oil in samll tank is (15-y) accoding to bernolli's equation

$$P_0 + 15\rho_0 g + \frac{1}{2}\rho v_1^2 = P_0 + (5)\rho_0 g + y\rho g + \frac{1}{2}\rho v_2^2$$

$$\frac{1}{2}\rho v_1^2 = (-10)\rho_0 g + y\rho g + \frac{1}{2}\rho v_2^2$$

$av_1 = Av_2$

$$v_2 = \frac{a}{A}v_1 = \frac{v_1}{50}$$

$$\frac{1}{2}\rho v_1^2 = (-10)\rho_0 g + y\rho g + \frac{1}{2}\rho\left(\frac{v_1}{50}\right)^2$$

Neglecting term

$$\frac{1}{2}\rho\left(\frac{v_1}{50}\right)^2$$

$$\frac{1}{2}\rho v_1^2 = (-10)\rho_0 g + y\rho g$$

$$\frac{1}{2}1000v_1^2 = (-10)\times 800\times 10 + y\times 1000\times 10$$

$$v_1^2 = -160 + 20y$$

Differentiating

$$2v_1\frac{dv_1}{dt} = 20\frac{dy}{dt}$$

But

$$\frac{dy}{dt} = v_2 = -\frac{v_1}{50}$$

Negative sign since vleocity is decreasing

$$v_1\frac{dv_1}{dt} = 10\times\frac{-v_1}{50}$$

$$\frac{dv_1}{dt} = \frac{-1}{5}$$

$$dv_1 = \frac{-1}{5}dt$$

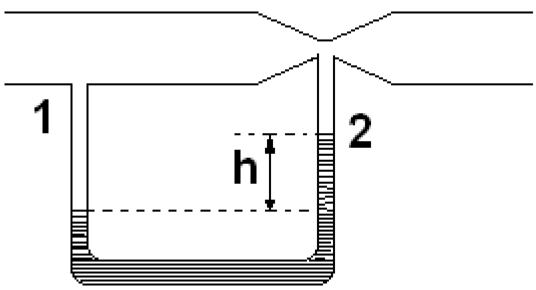
Integrating

$$\int_{6.32}^{0} dv_1 = \frac{-1}{5}\int_0^t dt$$

$$-6.23 = \frac{-t}{5}$$

t = 31.15 sec

Venturimeter:



This is a device based on Bernoulli's principle used for measuring the flow of a liquid in pipes. A liquid of density ρ flows through a pipe of cross-sectional area A. Let the constricted part of the cross-sectional area be 'a'. A manometer tube with a liquid say mercury having a density ρ_0 is attached to the tube as shown in figure

If P_1 is the pressure at point 1 and P_2 the pressure at point 2, we have

$$P_1 + \frac{1}{2}mv_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

Where v_1 and v_2 are the velocities at these points respectively

$$\frac{1}{2}v_1^2 - \frac{1}{2}v_2^2 = \frac{P_2}{\rho} - \frac{P_1}{\rho}$$

We have $Av_1 = av_2$

$$v_2 = \frac{A}{a}v_1$$

$$\frac{1}{2}v_1^2 - \frac{1}{2}\left(\frac{A}{a}\right)^2 v_1^2 = \frac{P_2 - P_1}{\rho}$$

$$v_1^2 - \left(\frac{A}{a}\right)^2 v_1^2 = \frac{2(P_2 - P_1)}{\rho}$$

$$v_1^2\left(1 - \frac{A^2}{a^2}\right) = \frac{2(P_2 - P_1)}{\rho}$$

$$v_1^2 = \frac{\frac{2(P_2 - P_1)}{\rho}}{1 - \frac{A^2}{a^2}} = \frac{2a^2(P_2 - P_1)}{(a^2 - A^2)\rho}$$

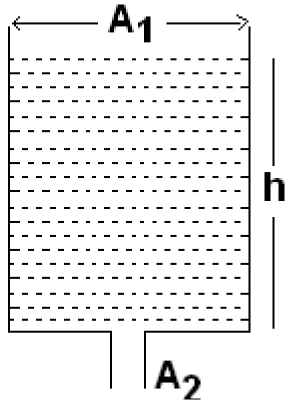
$$v_1 = \sqrt{\frac{2a^2(P_2 - P_1)}{(a^2 - A^2)\rho}}$$

Volume of liquid flowing through the pipe per second $Q = Av_1$

$$Q = Aa\sqrt{\frac{2(P_2 - P_1)}{(a^2 - A^2)\rho}}$$

Speed of Efflux

As shown in figure a tank of cross-sectional area A, filled to a depth h with a liquid of density ρ. There is a hole of cross-section area A_2 at the bottom and the liquid flows out of the tank through the hole $A_2 << A_1$



Let v_1 and v_2 be the speeds of the liquid at A_1 and A_2. As both the cross sections are opened to the atmosphere, the pressure there equals to atmospheric pressure P_o. If the height of the free surface above the hole is h_1

Bernoulli's equation gives

$$P_0 + \frac{1}{2}\rho v_1^2 + \rho gh = P_0 + \frac{1}{2}\rho v_2^2$$

By the equation of continuity,

$A_1v_1 = A_2 v_2$

$$P_0 + \frac{1}{2}\rho\left(\frac{A_2}{A_1}\right)^2 v_2^2 + \rho g h = P_0 + \frac{1}{2}\rho v_2^2$$

$$\left[1 - \left(\frac{A_2}{A_1}\right)^2\right] v_2^2 = 2gh$$

$$v_2 = \sqrt{\frac{2gh}{1 - \left(\frac{A_2}{A_1}\right)^2}}$$

If $A_2 <<< A_1$, the equation reduces to $v_2 = \sqrt{(2gh)}$

The speed of efflux is the same as the speed a body that would acquire in falling freely through a height h. This is known as Torricelli's theorem.