Integration of Trigonometric Functions.

(1) **Integral of the form $\int \sin^m x \cos^n x\, dx$:** (i) To evaluate the integrals of the form $I = \int \sin^m x \cos^n x\, dx$, where *m* and *n* are rational numbers.

(a) Substitute $\sin x = t$, if *n* is odd;

(b) Substitute cos *x* = *t*, if *m* is odd;

(c) Substitute $\tan x = t$, if $m + n$ is a negative even integer; and

(d) Substitute $\cot x = t$, if $\frac{1}{2}(n-1)$ is an integer.

(e) If *m* and *n* are rational numbers and $\left(\frac{m+n-2}{2}\right)$ is a negative integer, then substitution $\cos x = t$ or $\tan x = t$ is found suitable.

(ii) Integrals of the form $\int R(\sin x, \cos x) dx$, where *R* is a rational function of $\sin x$ and $\cos x$, are transformed into integrals of a rational function by the substitution $\tan \frac{x}{2} = t$, where $-\pi < x < \pi$. This is

the so called universal substitution. Sometimes it is more convenient to make the substitution $\cot \frac{x}{2} = t$ for $0 < x < 2\pi$.

The above substitution enables us to integrate any function of the form $R(\sin x, \cos x)$. However, in practice, it sometimes leads to extremely complex rational function. In some cases, the integral can be simplified by:

(a) Substituting $\sin x = t$, if the integral is of the form $\int R(\sin x) \cos x\, dx$.

(b) Substituting $\cos x = t$, if the integral is of the form $\int R(\cos x) \sin x\, dx$.

(c) Substituting tan *x* = *t*, *i.e.*, $dx = \frac{dt}{1+t^2}$, if the integral is dependent only on $\tan x$.

(d) Substituting $\cos x = t$, if $R(-\sin x, \cos x) = -R(\sin x, \cos x)$

(e) Substituting $\sin x = t$, if $R(\sin x, -\cos x) = -R(\sin x, \cos x)$

(f) Substituting $\tan x = t$, if $R(-\sin x, -\cos x) = -R(\sin x, \cos x)$

Important Tips

☞ *To evaluate integrals of the form* $\int \sin mx \cos nx \, dx$, $\int \sin mx \, . \sin nx \, dx$, $\int \cos mx \, . \cos nx \, dx$ *and* $\int \cos mx \, . \sin nx \, dx$, *we use the following trigonometrical identities.*

$$\sin mx \, . \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x] \Rightarrow \cos mx \, . \sin nx = \frac{1}{2}[\sin(m+n)x - \sin(m-n)x]$$

$$\sin mx \, . \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x] \Rightarrow \cos mx \, . \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$$

(2) **Reduction formulae for special cases**

(i) $\int \sin^n x \, dx = \frac{-\cos x \, . \sin^{n-1} x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx$

(ii) $\int \cos^n x \, dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx$

(iii) $\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$

(iv) $\int \cot^n x \, dx = \frac{-1}{n-1} \cot^{n-1} x - \int \cot^{n-2} x \, dx$

(v) $\int \sec^n x \, dx = \frac{1}{(n-1)} \left[\sec^{n-2} x . \tan x + (n-2) \int \sec^{n-2} x \, dx \right]$

(vi) $\int \operatorname{cosec}^n x\, dx = \frac{1}{(n-1)}\left[-\operatorname{cos} ec^{n-2} x \,.\cot x + (n-2)\int \operatorname{cosec}^{n-2} x\, dx\right]$

(vii) $\int \sin^p x \cos^q x\, dx = -\frac{\sin^{q+1} x . \cos^{p-1} x}{p+q} + \frac{p-1}{p+q}\int \sin^{p-2} x . \cos^q x\, dx$

(viii) $\int \sin^p x \cos^q x\, dx = \frac{\sin^{p+1} x \,. \cos^{q-1} x}{p+q} + \frac{p-1}{p+q}\int \sin^p x . \cos^{q-2} x\, dx$

(ix) $\int \frac{dx}{(x^2+k)^n} = \frac{x}{k(2n-2)(x^2+k)^{n-1}} + \frac{(2n-3)}{k(2n-2)}\int \frac{dx}{(x^2+k)^{n-1}}$

Important Tips

☞ *Reduction formulae for* $I_{(n,m)} = \int \frac{\sin^n x}{\cos^m x} dx$ *is* $I_{(n,m)} = \frac{1}{m-1} . \frac{\sin^{n-1} x}{\cos^{m-1} x} - \frac{(n-1)}{(m-1)} . I_{(n-2,m-2)}$