

## Integration by Parts.

(1) **When integrand involves more than one type of functions:** We may solve such integrals by a rule which is known as integration by parts. We know that,

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx} \Rightarrow d(uv) = u dv + v du \Rightarrow \int d(uv) = \int u dv + \int v du$$

If  $u$  and  $v$  are two functions of  $x$ , then  $\int u v dx = u \int v dx - \int \left\{ \frac{du}{dx} \cdot \int v dx \right\} dx$  i.e., the integral of the product of two functions = (First function)  $\times$  (Integral of second function) – Integral of {(Differentiation of first function)  $\times$  (Integral of second function)}

Integration with the help of above rule is called integration by parts. Before applying this rule proper choice of first and second function is necessary. Normally we use the following methods:

(i) In the product of two functions, one of the function is not directly integrable (i.e.,  $\log|x|$ ,  $\sin^{-1} x$ ,  $\cos^{-1} x$ ,  $\tan^{-1} x$  .....etc), then we take it as the first function and the remaining function is taken as the second function.

(ii) If there is no other function, then unity is taken as the second function e.g. In the integration of  $\int \sin^{-1} x dx$ ,  $\int \log x dx$ , 1 is taken as the second function.

(iii) If both of the function are directly integrable then the first function is chosen in such a way that the derivative of the function thus obtained under integral sign is easily integrable.

Usually, we use the following preference order for the first function. (Inverse, Logarithmic, Algebraic, Trigonometric, exponential). This rule is simply called as "I LATE".

### Important Tips

$$\Rightarrow \text{If } I_n = \int x^n \cdot e^{ax} dx, \text{ then } I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$$

$$\Rightarrow \text{If } I_n = \int (\log x) dx, \text{ then } I_n = x \log x - x$$

$$\Rightarrow \text{If } I_n = \int \frac{1}{\log x} dx, \text{ then } I_n = \log(\log x) + \log x + \frac{(\log x)^2}{2 \cdot (2!)} + \frac{(\log x)^3}{3(3!)} + \dots \dots \dots$$

$$\Rightarrow \text{If } I_n = \int (\log x)^n dx ; \text{ then } I_n = x(\log x)^n - n \cdot I_{n-1}$$

☞ *Successive integration by parts can be performed when one of the functions is  $x^n$  ( $n$  is positive integer) which will be successively differentiated and the other is either of the following  $\sin ax, \cos ax, e^{ax}, e^{-ax}, (x+a)^m$  which will be successively integrated.*

☞ **Chain rule :**  $\int u \cdot v dx = u v_1 - u' v_2 + u'' v_3 - u''' v_4 + \dots + (-1)^{n-1} u^{n-1} v_n + (-1)^n \int u^n \cdot v_n dx$

Where  $u^n$  stands for  $n^{\text{th}}$  differential coefficient of  $u$  and  $v_n$  stands for  $n^{\text{th}}$  integral of  $v$ .

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(2) **Integral is of the form**  $\int e^x \{f(x) + f'(x)\} dx$  : If the integral is of the form  $\int e^x \{f(x) + f'(x)\} dx$ , then by breaking this integral into two integrals integrate one integral by parts and keeping other integral as it is, by doing so, we get

(i)  $\int e^x [f(x) + f'(x)] dx = e^x f(x) + c$

(ii)  $\int e^{mx} [mf(x) + f'(x)] dx = e^{mx} f(x) + c$

(iii)  $\int e^{mx} \left[ f(x) + \frac{f'(x)}{m} \right] dx = \frac{e^{mx} f(x)}{m} + c$

(3) **Integral is of the form**  $\int [x f'(x) + f(x)] dx$  : If the integral is of the form  $\int [x f'(x) + f(x)] dx$  then by breaking this integral into two integrals, integrate one integral by parts and keeping other integral as it is, by doing so, we get,  $\int [x f'(x) + f(x)] dx = x f(x) + c$

(4) **Integrals of the form**  $\int e^{ax} \sin bx dx, \int e^{ax} \cos bx dx$  :

*Working rule* : To evaluate  $\int e^{ax} \sin bx dx$  or  $\int e^{ax} \cos bx dx$ , proceed as follows

- (i) Put the given integral equal to  $I$ .
- (ii) Integrate by parts, taking  $e^{ax}$  as the first function.
- (iii) Again, integrate by parts taking  $e^{ax}$  as the first function. This will involve  $I$ .
- (iv) Transpose and collect terms involving  $I$  and then obtain the value of  $I$ .

Let  $I = \int e^{ax} \sin bx dx$ . Then  $I = \int_I e^{ax} \cdot \sin bx dx = -e^{ax} \cdot \frac{\cos bx}{b} - \int ae^{ax} \cdot \left( \frac{-\cos bx}{b} \right) dx$

$$= \frac{-1}{b} e^{ax} \cdot \cos bx + \frac{a}{b} \int_I e^{ax} \cdot \cos bx dx = \frac{-1}{b} e^{ax} \cdot \cos bx + \frac{a}{b} \left[ \frac{e^{ax} \cdot \sin bx}{b} - \int ae^{ax} \cdot \frac{\sin bx}{b} dx \right]$$

$$= \frac{-1}{b} e^{ax} \cdot \cos bx + \frac{a}{b^2} e^{ax} \cdot \sin bx - \frac{a^2}{b^2} \int e^{ax} \cdot \sin bx \, dx = \frac{-1}{b} e^{ax} \cdot \cos bx + \frac{a}{b^2} e^{ax} \cdot \sin bx - \frac{a^2}{b^2} I$$

$$I + I \cdot \frac{a^2}{b^2} = \frac{e^{ax}}{b^2} (a \sin bx - b \cos bx) \Rightarrow I = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + c$$

$$\text{Thus, } \int e^{ax} \sin bx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + c = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \sin\left(bx - \tan^{-1} \frac{b}{a}\right) + c$$

$$\text{Similarly } \int e^{ax} \cdot \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + c = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \cos\left(bx - \tan^{-1} \frac{b}{a}\right) + c$$

**Note:**  $\int e^{ax} \cdot \sin(bx + c) \, dx = \frac{e^{ax}}{a^2 + b^2} [a \sin(bx + c) - b \cos(bx + c)] + k$

$$= \frac{e^{ax}}{\sqrt{a^2 + b^2}} \sin\left[\left(bx + c\right) - \tan^{-1}\left(\frac{b}{a}\right)\right] + k$$

$$\int e^{ax} \cdot \cos(bx + c) \, dx = \frac{e^{ax}}{a^2 + b^2} [a \cos(bx + c) + b \sin(bx + c)] + k$$

$$= \frac{e^{ax}}{\sqrt{a^2 + b^2}} \cos\left[\left(bx + c\right) - \tan^{-1}\left(\frac{b}{a}\right)\right] + k$$

### Important Tips

$$\int x e^{ax} \sin(bx + c) \, dx = \frac{x e^{ax}}{a^2 + b^2} [a \sin(bx + c) - b \cos(bx + c)] - \frac{e^{ax}}{(a^2 + b^2)^2} [(a^2 - b^2) \sin bx (bx + c) - 2ab \cos bx (bx + c)] + k$$

$$\int x \cdot e^{ax} \cos(bx + c) \, dx = \frac{x \cdot e^{ax}}{a^2 + b^2} [a \cos(bx + c) + b \sin(bx + c)] - \frac{e^{ax}}{(a^2 + b^2)^2} [(a^2 - b^2) \cos(bx + c) + 2ab \sin(bx + c)] + k$$

$$\int a^x \cdot \sin(bx + c) \, dx = \frac{a^x}{(\log a)^2 + b^2} [(\log a) \sin(bx + c) - b \cos(bx + c)] + k$$

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$$\int a^x \cdot \cos(bx + c) dx = \frac{a^x}{(\log a)^2 + b^2} [(\log a) \cos(bx + c) + b \sin(bx + c)] + k$$

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