Evaluation of the various forms of Integrals by use of Standard Results.

(1) Integral of the form $\int \frac{dx}{ax^2 + bx + c}$, where $ax^2 + bx + c$ can not be resolved into factors.

(2) Integral of the form $\int \frac{px + q}{ax^2 + bx + c} dx$.

(3) Integral of the form $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$.

(4) Integral of the form $\int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx$.

(5) Integral of the form $\int \frac{f(x)}{ax^2 + bx + c} dx$, where $f(x)$ is a polynomial of degree 2 or greater than 2.

(6) Integral of the form

(i) $\int \frac{x^2 + 1}{x^4 + kx^2 + 1} dx$,

(ii) $\int \frac{x^2 - 1}{x^4 + kx^2 + 1} dx$, Where k is any constant

(7) Integral of the form $\int \sqrt{ax^2 + bx + c}\, dx$

(8) Integral of the form $\int (px + q)\sqrt{ax^2 + bx + c}\, dx$

(9) Integral of the form $\int \frac{dx}{P\sqrt{Q}}$

(1) **Integrals of the form $\int \frac{dx}{ax^2+bx+c}$, where ax^2+bx+c cannot be resolved into factors.**

We have, $ax^2+bx+c = a\left(x^2+\frac{b}{a}.x+\frac{c}{a}\right) = a\left[\left(x+\frac{b}{2a}\right)^2 - \left(\frac{b^2}{4a^2}-\frac{c}{a}\right)\right]$

$$= a\left[\left(x+\frac{b}{2a}\right)^2 - \left(\frac{b^2-4ac}{4a^2}\right)\right]$$

Case (i): When $b^2-4ac>0$

$$\therefore \int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2}, \qquad \left[\text{form} \int \frac{dx}{x^2-a^2}\right]$$

$$= \frac{1}{a}.\frac{1}{2.\frac{\sqrt{b^2-4ac}}{2a}}\log\left|\frac{x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}}{x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}}\right|+c = \frac{1}{\sqrt{b^2-4ac}}\log\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|+c$$

Case (ii): When $b^2-4ac<0$

$$\int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2 + \left(\frac{\sqrt{4ac-b^2}}{2a}\right)^2}, \qquad \left[\text{form} \int \frac{dx}{x^2+a^2}\right]$$

$$= \frac{1}{a}.\frac{1}{\frac{\sqrt{4ac-b^2}}{2a}}\tan^{-1}\left[\frac{x+\frac{b}{2a}}{\frac{\sqrt{4ac-b^2}}{2a}}\right]+c = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\left[\frac{2ax+b}{\sqrt{4ac-b^2}}\right]+c$$

Working rule for evaluating $\int \frac{dx}{ax^2 + bx + c}$: To evaluate this form of integrals proceed as follows :

(i) Make the coefficient of x^2 unity by taking '*a*' common from $ax^2 + bx + c$.

(ii) Express the terms containing x^2 and x in the form of a perfect square by adding and subtracting the square of half of the coefficient of *x*.

(iii) Put the linear expression in *x* equal to *t* and express the integrals in terms of *t*.

(iv) The resultant integrand will be either in $\int \frac{dx}{x^2 + a^2}$ or $\int \frac{dx}{x^2 - a^2}$ or $\int \frac{dx}{a^2 - x^2}$ standard form.

After using the standard formulae, express the results in terms of *x*.

(2) **Integral of the form $\int \frac{px + q}{ax^2 + bx + c} dx$:** The integration of the function $\frac{px + q}{ax^2 + bx + c}$ is effected by breaking $px + q$ into two parts such that one part is the differential coefficient of the denominator and the other part is a constant.

If *M* and *N* are two constants, then we express $px + q$ as $px + q = M \frac{d}{dx}(ax^2 + bx + c) + N$

$= M.(2ax + b) + N = (2aM)x + Mb + N$.

Comparing the coefficients of *x* and constant terms on both sides, we have, $p = 2aM \Rightarrow M = \frac{p}{2a}$

and $q = Mb + N \Rightarrow N = q - Mb = q - \frac{p}{2a}b$.

Thus, *M* and *N* are known. Hence, the given integral is

$$\int \frac{px + q}{ax^2 + bx + c} dx = \int \frac{\frac{p}{2a}(2ax + b) + \left(q - \frac{p}{2a}b\right)}{ax^2 + bx + c} dx = \frac{p}{2a}\int \frac{2ax + b}{ax^2 + bx + c} dx + \left(q - \frac{p}{2a}b\right)\int \frac{dx}{ax^2 + bx + c}$$

$$= \frac{p}{2a}\log| ax^2 + bx + c| + \left(q - \frac{p}{2a}b\right)\int \frac{dx}{ax^2 + bx + c} + C$$

The integral on R.H.S. can be evaluated by the method discussed in previous section.

(i) If $b^2 - 4ac < 0$, then

$$\int \frac{px + q}{ax^2 + bx + c} dx = \frac{p}{2a}\log| ax^2 + bx + c| + \frac{(2aq - bp)}{a\sqrt{4ac - b^2}} \tan^{-1} \frac{2ax + b}{\sqrt{4ac - b^2}} + k$$

(ii) If $b^2 - 4ac > 0$, then

$$\int \frac{px + q}{ax^2 + bx + c} dx = \frac{p}{2a}\log| ax^2 + bx + c| + \frac{(2aq - bp)}{2a\sqrt{b^2 - 4ac}} \log\left| \frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}} \right| + k$$

(3) **Integral of the form** $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ **:** To evaluate this form of integrals proceed as follows :

(i) Make the coefficient of x^2 unity by taking $\sqrt{a}$ common from $\sqrt{ax^2+bx+c}$.

Then, $\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}} \int \frac{dx}{\sqrt{x^2+\frac{b}{a}x+\frac{c}{a}}}$.

(ii) Put $x^2+\frac{b}{a}x+\frac{c}{a}$, by the method of completing the square in the form, $\sqrt{A^2-X^2}$ or $\sqrt{X^2+A^2}$ or $\sqrt{X^2-A^2}$ where, X is a linear function of x and A is a constant.

(iii) After this, use any of the following standard formulae according to the case under consideration

$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)+c \Rightarrow \int \frac{dx}{\sqrt{x^2+a^2}} = \log|\,x+\sqrt{x^2+a^2}\,|+c$ and

$\int \frac{dx}{\sqrt{x^2-a^2}} = \log|\,x+\sqrt{x^2-a^2}\,|+c.$

Note: If $a<0,\ b^2-4ac>0,$ then $\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{-a}}\sin^{-1}\left(\frac{2ax+b}{\sqrt{b^2-4ac}}\right)+k.$

- If $a>0,\ b^2-4ac<0,$ then $\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\sinh^{-1}\left[\frac{2ax+b}{\sqrt{4ac-b^2}}\right]+k$

- If $a>0,\ b^2-4ac>0$ $\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\cosh^{-1}\frac{2ax+b}{\sqrt{b^2-4ac}}+k$

(4) **Integral of the form** $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$ **:** To evaluate this form of integrals, first we write,

$px+q = M\frac{d}{dx}(ax^2+bx+c)+N \Rightarrow\ px+q = M(2ax+b)+N$

Where M and N are constants.

By equating the coefficients of x and constant terms on both sides, we get

$p=2aM \Rightarrow M=\frac{p}{2a}$ and $q=bM+N \Rightarrow N=q-\frac{bp}{2a}$

In this way, the integral breaks up into two parts given by

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx = \frac{p}{2a}\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}dx + \left(q-\frac{bp}{2a}\right)\int \frac{dx}{\sqrt{ax^2+bx+c}} \qquad = I_1 + I_2, \text{ (say)}$$

Now, $I_1 = \frac{p}{2a}\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}dx$

Putting $ax^2+bx+c \Rightarrow (2ax+b)dx = dt$, we have,

$$I_1 = \frac{p}{2a}\int t^{-1/2}dt = \frac{p}{2a}.\frac{t^{1/2}}{\frac{1}{2}} + C_1 = \frac{p}{a}\sqrt{ax^2+bx+c} + C_1$$

and I_2 is calculated as in the previous section.

Note: $\frac{px+q}{\sqrt{ax^2+bx+c}}dx = \frac{p}{a}\sqrt{ax^2+bx+c} + \frac{2aq-bp}{2a}\int \frac{dx}{\sqrt{ax^2+bx+c}}$.

(5) **Integrals of the form $\int \frac{f(x)}{ax^2+bx+c}dx$, where $f(x)$ is a polynomial of degree 2 or greater than 2:**

To evaluate the integrals of the above form, divide the numerator by the denominator. Then, the integrals take the form given by $\frac{f(x)}{ax^2+bx+c} = Q(x) + \frac{R(x)}{ax^2+bx+c}dx$

where, $Q(x)$ is a polynomial and $R(x)$ is a linear polynomial in x.

Then, we have $\int \frac{f(x)}{ax^2+bx+c}dx = \int Q(x)dx + \int \frac{R(x)}{ax^2+bx+c}dx$

The integrals on R.H.S. can be obtained by the methods discussed earlier.

(6) **Integrals of the form $\int \frac{x^2+1}{x^4+kx^2+1}dx$ and $\int \frac{x^2-1}{x^4+kx^2+1}dx$:** To evaluate the integral of the form $I = \int \frac{x^2+1}{x^4+kx^2+1}dx$, proceed as follows

(i) Divide the numerator and denominator by x^2 to get $I = \int \frac{1+\frac{1}{x^2}}{x^2+k+\frac{1}{x^2}}\,dx$.

(ii) Put $x-\frac{1}{x}=t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$ and $x^2+\frac{1}{x^2}-2=t^2 \Rightarrow x^2+\frac{1}{x^2}=t^2+2$.

Then, the given integral reduces to the form $I=\int \frac{dt}{t^2+2+k}$, which can be integrand as usual.

(iii) To evaluate $I=\int \frac{x^2-1}{x^4+kx^2+1}dx$, we divide the numerator and denominator by x^2 and get

$$I=\int \frac{1-\frac{1}{x^2}}{x^2+k+\frac{1}{x^2}}dx$$

Then, we put $x+\frac{1}{x}=t \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt$ and $x^2+\frac{1}{x^2}+2=t^2 \Rightarrow x^2+\frac{1}{x^2}=t^2-2$.

Thus, we have $t=\int \frac{dt}{t^2-2+k}$, which can be evaluated as usual.

Important Tips

- ***Algebraic twins:*** $\int \frac{2x^2}{x^4+1}dx = \int \frac{x^2+1}{x^4+1}dx + \int \frac{x^2-1}{x^4+1}dx$

$\int \frac{2}{x^4+1}dx = \int \frac{x^2+1}{x^4+1}dx - \int \frac{x^2-1}{x^4+1}dx$, $\int \frac{2x^2}{x^4+1+kx^2}dx$, $\int \frac{2}{(x^4+1+kx^2)}dx$

We know the result of $I_1=\int \frac{x^2+1}{x^4+1}dx$ *and* $I_2=\int \frac{x^2-1}{x^4+1}dx$, *so for* $\int \frac{x^2}{x^4+1}dx$ *and for* $\int \frac{dx}{x^4+1}$, *we can use the result of* $\frac{I_1+I_2}{2}$ *and* $\frac{I_1-I_2}{2}$.

- ***Trigonometric twins:*** $\int \sqrt{\tan x}\,dx$, $\int \sqrt{\cot x}\,dx$, $\int \frac{dx}{(\sin^4 x+\cos^4 x)}$, $\int \frac{dx}{\sin^6 x+\cos^6 x}$,

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$\int \frac{\pm\sin x \pm \cos x}{a+b\sin x\cos}dx$

(7) **Integrals of the forms** $\int \sqrt{ax^2+bx+c}\,dx$ **:** To evaluate this form of integrals, express ax^2+bx+c in the form $a\left[(x+\alpha)^2+\beta^2\right]$ by the method of completing the square and apply the standard result discussed in the above section according to the case as may be.

Note: $\int \sqrt{ax^2+bx+c}\;dx = \frac{(2ax+b)\sqrt{ax^2+bx+c}}{4a}+\frac{4ac-b^2}{8a}\int\frac{dx}{\sqrt{ax^2+bx+c}}$

(8) **Integrals of the form** $\int (px+q)\sqrt{ax^2+bx+c}\,dx$ **:** To evaluate this form of integral, proceed as follows:

(i) First express $(px+q)$ as $px+q=M\frac{d}{dx}(ax^2+bx+c)+N \Rightarrow px+q=M(2ax+b)+N$

Where, M and N are constant.

(ii) Compare the coefficients of x and constant terms on both sides, will get

$p=2aM \Rightarrow M=\frac{p}{2a}$ and $q=Mb+N \Rightarrow N=q-Mb=q-\frac{p}{2a}b.$

(iii) Now, write the given integral as

$$\int (px+q)\sqrt{ax^2+bx+c}\,dx = \frac{p}{2a}\int (2ax+b)\sqrt{ax^2+bx+c}\;dx+\left(q-\frac{p}{2a}b\right)\int\sqrt{ax^2+bx+c}\;dx$$

$$=\frac{p}{2a}I_1+\left(q-\frac{p}{2a}b\right)I_2,\ \text{(say)}.$$

(iv) To evaluate $I_1,$ put $ax^2+bx+c=t$ and to evaluate $I_2,$ follows the method discussed in (7)

(9) **Integrals of the form $\int \frac{dx}{P\sqrt{Q}}$, (where P and Q and linear or quadratic expressions in x):**

To evaluate such types of integrals, we have following substitutions according to the nature of expressions of P and Q in x:

(i) When Q is linear and P is linear or quadratic, we put $Q = t^2$.

(ii) When P is linear and Q is quadratic, we put $P = \frac{1}{t}$.

(iii) When both P and Q are quadratic, we put $x = \frac{1}{t}$.