Integrals of the form $\int \frac{dx}{a + b\cos x}$ and $\int \frac{dx}{a + b\sin x}$.

To evaluate such form of integrals, proceed as follows:

(1) Put $\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$ and $\sin x = \frac{2\tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$.

(2) Replace $1 + \tan^2 \frac{x}{2}$ in the numerator by $\sec^2 \frac{x}{2}$.

(3) Put $\tan \frac{x}{2} = t$ so that $\frac{1}{2}\sec^2 \frac{x}{2} dx = dt$.

(4) Now, evaluate the integral obtained which will be of the form $\int \frac{dt}{at^2 + bt + c}$ by the method discussed earlier.

(i) $\int \frac{dx}{a + b\cos x}$

Case I: When $a > b$, then $\int \frac{dx}{a + b\cos x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2}\right) + c$

Case II: When $a < b$, then $\int \frac{dx}{a + b\cos x} = \frac{1}{\sqrt{b^2 - a^2}} \log\left|\frac{\sqrt{b-a}\tan \frac{x}{2} + \sqrt{b+a}}{\sqrt{b-a}\tan \frac{x}{2} - \sqrt{b+a}}\right| + c$

Case III: When $a = b$, then $\int \frac{dx}{a + b\cos x} = \frac{1}{a}\tan \frac{x}{2} + c$.

(ii) $\int \frac{dx}{a + b\sin x}$

Case I: When $a^2 > b^2$ or $a > 0$ and $a > b$, then $\int \frac{dx}{a + b\sin x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left[\frac{a\tan\frac{x}{2} + b}{\sqrt{a^2 - b^2}}\right] + c$

Case II: When $a^2 < b^2$, then $\int \frac{dx}{a + b\sin x} = \frac{1}{\sqrt{b^2 - a^2}} \log\left|\frac{(a\tan\frac{x}{2} + b) - (\sqrt{b^2 - a^2})}{(a\tan\frac{x}{2} + b) + \sqrt{b^2 - a^2}}\right| + c$

Case III: When $a^2 = b^2$

In this case, either $b = a$ or $b = -a$

(a) When $b = a$, then $\int \frac{dx}{a + b\sin x} = \frac{-1}{a}\cot\left(\frac{\pi}{4} + \frac{x}{2}\right) + c = \frac{1}{a}[\tan x - \sec x] + c$

(b) When $b = -a$, then $\int \frac{dx}{a + b\sin x} = \frac{1}{a}\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$.

1. Integrals of the form $\int \frac{dx}{a + b\cos x + c\sin x}, \int \frac{dx}{a\sin x + b\cos x}$.

(1) **Integral of the form** $\int \frac{dx}{a + b\cos x + c\sin x}$ **:** To evaluate such integrals, we put $b = r\cos\alpha$ and $c = r\sin\alpha$.

So that, $r^2 = b^2 + c^2$ and $\alpha = \tan^{-1}\frac{c}{b}$. $\therefore$ $I = \int \frac{dx}{a + r(\cos\alpha\cos x + \sin\alpha\sin x)} = \int \frac{dx}{a + r\cos(x - \alpha)}$

Again, Put $x-\alpha=t \Rightarrow dx=dt$, we have $I=\int \frac{dt}{a+r\cos t}$

Which can be evaluated by the method discussed earlier.

(2) **Integral of the form** $\int \frac{dx}{a\sin x+b\cos x}$ **:** To evaluate this type of integrals we substitute

$a=r\cos\theta,\ b=r\sin\theta$ and so $r=\sqrt{a^2+b^2},\ \alpha=\tan^{-1}\frac{b}{a}$

So, $\int \frac{dx}{a\sin x+b\cos x}=\frac{1}{r}\int \frac{dx}{\sin(x+\alpha)}=\frac{1}{r}\int \operatorname{cosec}(x+\alpha)dx$

$$=\frac{1}{r}\log\left|\tan\left(\frac{x}{2}+\frac{\alpha}{2}\right)\right|=\frac{1}{\sqrt{a^2+b^2}}\log\left|\tan\left(\frac{x}{2}+\frac{1}{2}\tan^{-1}\frac{b}{a}\right)\right|+c$$

Note: The integral of the above form can be evaluated by using $\cos x=\frac{1-\tan^2 x/2}{1+\tan^2 x/2}$ and $\sin x=\frac{2\tan x/2}{1+\tan^2 x/2}$.

Important Tips

- *If* $a>b,\ a^2>b^2+c^2$, *then* $\int \frac{dx}{a+b\cos x+c\sin x}=\frac{2}{\sqrt{a^2-b^2-c^2}}\tan^{-1}\left[\frac{(a-b)\tan x/2+c}{\sqrt{a^2-b^2-c^2}}\right]+k$
- *If* $a>b,\ a^2<b^2+c^2$, *then*

$$\int \frac{dx}{a+b\cos x+c\sin x}=\frac{1}{\sqrt{b^2+c^2-a^2}}\log\left[\frac{(a-b)\tan x/2+c-\sqrt{b^2+c^2-a^2}}{(a-b)\tan x/2+c+\sqrt{b^2+c^2-a^2}}\right]+k$$

- *If* $a<b$, $\int \frac{dx}{a+b\cos x+c\sin x}=\frac{-1}{\sqrt{b^2+c^2-a^2}}\log\left[\frac{(b-a)\tan x/2-c-\sqrt{b^2+c^2-a^2}}{(b-a)\tan x/2-c+\sqrt{b^2+c^2-a^2}}\right]+k.$

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2. Integrals of the form

$$\int \frac{dx}{a+b\cos^2 x},\ \int \frac{dx}{a+b\sin^2 x},\ \int \frac{dx}{a\sin^2 x+b\cos^2 x},\ \int \frac{dx}{(a\sin x+b\cos x)^2},\ \int \frac{dx}{a+b\sin^2 x+c\cos^2 x}$$

To evaluate the above forms of integrals proceed as follows:

(1) Divide both the numerator and denominator by $\cos^2 x$.

(2) Replace $\sec^2 x$ in the denominator, if any by $(1+\tan^2 x)$.

(3) Put $\tan x = t \Rightarrow \sec^2 x dx = dt$.

(4) Now, evaluate the integral thus obtained, by the method discussed earlier.

3. Integrals of the form $\int \frac{a \sin x + b \cos x}{c \sin x + d \cos x}$ and $\int \frac{a \sin x + b \cos x + q}{c \sin x + d \cos x + r}$

(1) **Integrals of the form** $\int \frac{a \sin x + b \cos x}{c \sin x + d \cos x} dx$ **:** Such rational functions of **sin *x*** and **cos *x*** may be integrated by expressing the numerator of the integrand as follows:

Numerator = *M* (Diff. of denominator) + *N* (Denominator)

i.*e*., $a \sin x + b \cos x = M \frac{d}{dx}(c \sin x + d \cos x) + N(c \sin x + d \cos x)$

The arbitary constants *M* and *N* are determined by comparing the coefficients of $\sin x$ and $\cos x$ from two sides of the above identity. Then, the given integral is

$$I = \int \frac{a \sin x + b \cos x}{c \sin x + d \cos x} dx = \int \frac{M(c \cos x - d \sin x) + N(c \sin x + d \cos x)}{c \sin x + d \cos x} dx$$

$$= M \int \frac{c \cos x - d \sin x}{c \sin x + d \cos x} dx + N \int 1 dx$$

$$= M \log | c \sin x + d \cos x | + Nx + c.$$

(2) **Integrals of the form** $\int \frac{a \sin x + b \cos x + q}{c \sin x + d \cos x + r} dx$ **:** To evaluate this type of integrals, we express the numerator as follows: Numerator

$= M(\text{Denominat or}) + N(\text{Differentiation of denominato r}) + P$

i.e., $(c \sin x + b \cos x + q) = M(c \sin x + d \cos x + r) + N(c \cos x - d \sin x) + P.$

where *M, N, P* are constants to be determined by comparing the coefficients of $\sin x, \cos x$ and constant term on both sides.

$$\therefore \int \frac{a \sin x + b \cos x + q}{c \sin x + d \cos x + r} dx = \int M\, dx + N \int \frac{\text{Diff.of denominato r}}{\text{Denominato r}} dx + \int \frac{dx}{c \sin x - d \cos x + r}$$

$$= Mx + N \log|\text{ Denominato r}| + P \int \frac{dx}{c \sin x + d \cos x + r}.$$

Important Tips

☞ $\int \frac{a \cos x + b \sin x}{c \cos x + d \sin x} dx = \frac{ac + bd}{c^2 + d^2} x + \frac{ad - bc}{c^2 + d^2} \log|c \cos x + d \sin x| + c$.