Integration of Rational Functions by using Partial Fractions.

(1) **Proper rational functions:** Functions of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial and $g(x) \neq 0$, are called rational functions of *x*.

If degree of $f(x)$ is less than degree of $g(x)$, then $\frac{f(x)}{g(x)}$ is called a proper rational function.

(2) **Improper rational function:** If degree of $f(x)$ is greater than or equal to degree of $g(x)$, then $\frac{f(x)}{g(x)}$, is called an improper rational function and every improper rational function can be transformed to a proper rational function by dividing the numerator by the denominator.

For example, $\frac{x^3}{x^2 - 5x + 6}$ is an improper rational function and can be expressed as $(x+5) + \frac{19x - 30}{x^2 - 5x + 6}$, which is the sum of a polynomial $(x+5)$ and a proper function $\frac{19x - 30}{x^2 - 5x + 6}$.

(3) **Partial fractions:** Any proper rational function can be broken up into a group of different rational fractions, each having a simple factor of the denominator of the original rational function. Each such fraction is called a partial fraction.

If by some process, we can break a given rational function $\frac{f(x)}{g(x)}$ into different fractions, whose denominators are the factors of $g(x)$, then the process of obtaining them is called the resolution or decomposition of $\frac{f(x)}{g(x)}$ into its partial fractions.

Depending on the nature of the factors of the denominator, the following cases arise.

Case I: When the denominator consists of non-repeated linear factors: To each linear factor $(x - a)$ occurring once in the denominator of a proper fraction, there corresponds a single partial fraction of the form $\frac{A}{x - a}$, where *A* is a constant to be determined.

Case II: When the denominator consists of linear factors, some repeated: To each linear factor $(x - a)$ occurring *r* times in the denominator of a proper rational function, there corresponds a sum of *r* partial fractions of the form.

$$\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\ldots\ldots\ldots.+\frac{A_r}{(x-a)^r}$$

Where $A's$ are constants to be determined. Of course, A_r is not equal to zero.

Case III: When the denominator consists of quadratic factors: To each irreducible non repeated quadratic factor ax^2+bx+c, there corresponds a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be determined.

To each irreducible quadratic factor ax^2+bx+c occurring r times in the denominator of a proper rational fraction there corresponds a sum of r partial fractions of the form

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots\ldots\ldots\ \ldots\ldots+\frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

Where, A's and B's are constants to be determined.

(4) **General methods of finding the constants**

(i) In the given proper fraction, first of all factorize the denominator.

(ii) Express the given proper fraction into its partial fractions according to rules given above and multiply both the sides by the denominator of the given fraction.

(iii) Equate the coefficients of like powers of x in the resulting identity and solve the equations so obtained simultaneously to find the various constant is short method. Sometimes, we substitute particular values of the variable x in the identity obtained after clearing of fractions to find some or all the constants. For non-repeated linear factors, the values of x used as those for which the denominator of the corresponding partial fractions become zero.

Note: If the given fraction is improper, then before finding partial fractions, the given fraction must be expressed as sum of a polynomial and a proper fraction by division.

(5) **Special cases:** Some times a suitable substitution transform the given function to a rational fraction which can be integrated by breaking it into partial fractions.