## Homogeneous Differential Equation.

(1) **Homogeneous differential equation:** A function $f(x, y)$ is called a homogeneous function of degree $n$ if $f(\lambda x, \lambda y) = \lambda^n f(x, y)$ .

For example, $f(x,y) = x^2 - y^2 + 3xy$ is a homogeneous function of degree 2, because $f(\lambda x, \lambda y) = \lambda^2 x^2 - \lambda^2 y^2 + 3\lambda x . \lambda y = \lambda^2 f(x,y)$. A homogeneous function $f(x, y)$ of degree $n$ can always be written as $f(x,y) = x^n f\left(\frac{y}{x}\right)$ or $f(x,y) = y^n f\left(\frac{x}{y}\right)$. If a first-order first degree differential equation is expressible in the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ where $f(x, y)$ and $g(x, y)$ are homogeneous functions of the same degree, then it is called a homogeneous differential equation. Such type of equations can be reduced to variable separable form by the substitution $y = vx$. The given differential equation can be written as $\frac{dy}{dx} = \frac{x^n f(y/x)}{x^n g(y/x)} = \frac{f(y/x)}{g(y/x)} = F\left(\frac{y}{x}\right)$. If $y = vx$ , then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Substituting the value of $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$, we get $v + x\frac{dv}{dx} = F(v) \Rightarrow \frac{dv}{F(v) - v} = \frac{dx}{x}$. On integration, $\int \frac{1}{F(v) - v} dv = \int \frac{dx}{x} + C$ where $C$ is an arbitrary constant of integration. After integration, $v$ will be replaced by $\frac{y}{x}$ in complete solution.

(2) **Algorithm for solving homogeneous differential equation**

**Step** (i)**:** Put the differential equation in the form $\frac{dy}{dx} = \frac{\phi(x,y)}{\psi(x,y)}$

**Step** (ii): Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in the equation in step (i) and cancel out $x$ from the right hand side. The equation reduces to the form $v + x\frac{dv}{dx} = F(v)$.

**Step** (iii)**:** Shift $v$ on RHS and separate the variables $v$ and $x$

**Step** (iv)**:** Integrate both sides to obtain the solution in terms of $v$ and $x$.

**Step** (v): Replace $v$ by $\frac{y}{x}$ in the solution obtained in step (iv) to obtain the solution in terms of $x$ and $y$.

(3) **Equation reducible to homogeneous form**

A first order, first degree differential equation of the form

$$\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}, \text{ where } \frac{a_1}{a_2} \neq \frac{b_1}{b_2} \quad .....(i)$$

This is non-homogeneous.

It can be reduced to homogeneous form by certain substitutions. Put $x = X + h, y = Y + k$

Where *h* and *k* are constants, which are to be determined.

$$\therefore \frac{dy}{dx} = \frac{dy}{dY} \cdot \frac{dY}{dX} \cdot \frac{dX}{dx} = \frac{dY}{dX}$$

Substituting these values in (i), we have $\dfrac{dY}{dX} = \dfrac{(a_1 X + b_1 Y) + a_1 h + b_1 k + c_1}{(a_2 X + b_2 Y) + a_2 h + b_2 k + c_2}$

.....(ii)

Now *h*, *k* will be chosen such that $\left.\begin{array}{l} a_1 h + b_1 k + c_1 = 0 \\ a_2 h + b_2 k + c_2 = 0 \end{array}\right]$

.....(iii)

*i.e.* $\dfrac{h}{b_1 c_2 - b_2 c_1} = \dfrac{k}{c_1 a_2 - c_2 a_1} = \dfrac{1}{a_1 b_2 - a_2 b_1}$ .....(iv)

For these values of *h* and *k* the equation (ii) reduces to $\dfrac{dY}{dX} = \dfrac{a_1 X + b_1 Y}{a_2 X + b_2 Y}$ which is a homogeneous differential equation and can be solved by the substitution $Y = vX$. Replacing $X$ and $Y$ in the solution so obtained by $x - h$ and $y - k$ respectively, we can obtain the required solution in terms of $x$ and $y$.