Exact Differential Equation.

(1) **Exact differential equation:** If M and N are functions of x and y, the equation $Mdx + Ndy = 0$ is called exact when there exists a function $f(x, y)$ of x and y such that

$d[f(x, y)] = Mdx + Ndy$ *i.e.*, $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy$

where $\frac{\partial f}{\partial x}$ = Partial derivative of $f(x, y)$ with respect to x (keeping y constant)

$\frac{\partial f}{\partial y}$ = Partial derivative of $f(x, y)$ with respect to y (treating x as constant)

Note: An exact differential equation can always be derived from its general solution directly by differentiation without any subsequent multiplication, elimination etc.

(2) **Theorem:** The necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ *i.e.*, partial derivative of $M(x, y)$ *w.r.t.* y = Partial derivative of $N(x, y)$ *w.r.t.* x

(3) **Integrating factor:** If an equation of the form $Mdx + Ndy = 0$ is not exact, it can always be made exact by multiplying by some function of x and y. Such a multiplier is called an integrating factor.

(4) **Working rule for solving an exact differential equation:**

Step (i): Compare the given equation with $Mdx + Ndy = 0$ and find out M and N. Then find out $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$. If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

Step (ii): Integrate M with respect to x treating y as a constant.

Step (iii): Integrate N with respect to y treating x as constant and omit those terms which have been already obtained by integrating M.

Step (iv): On adding the terms obtained in steps (ii) and (iii) and equating to an arbitrary constant, we get the required solution.

In other words, solution of an exact differential equation is $\underset{\text{Regarding } y \text{ as constant}}{\int Mdx} + \underset{\text{Only those terms not containing } x}{\int Ndy} = c$

(5) **Solution by inspection:**If we can write the differential equation in the form $f(f_1(x,y))d(f_1(x,y))+\phi(f_2(x,y))d(f_2(x,y))+......=0$, then each term can be easily integrated separately. For this the following results must be memorized.

(i) $d(x+y) = dx + dy$

(ii) $d(xy) = xdy + ydx$

(iii) $d\left(\frac{x}{y}\right)=\frac{ydx-xdy}{y^2}$

(iv) $d\left(\frac{y}{x}\right)=\frac{xdy-ydx}{x^2}$

(v) $d\left(\frac{x^2}{y}\right)=\frac{2xydx-x^2dy}{y^2}$

(vi) $d\left(\frac{y^2}{x}\right)=\frac{2xydy-y^2dx}{x^2}$

(vii) $d\left(\frac{x^2}{y^2}\right)=\frac{2xy^2dx-2x^2ydy}{y^4}$

(viii) $d\left(\frac{y^2}{x^2}\right)=\frac{2x^2ydy-2xy^2dx}{x^4}$

(ix) $d\left(\tan^{-1}\frac{x}{y}\right)=\frac{ydx-xdy}{x^2+y^2}$

(xi) $d[\ln(xy)]=\frac{xdy+ydx}{xy}$

(xii) $d\left(\ln\left(\frac{x}{y}\right)\right)=\frac{ydx-xdy}{xy}$

(xiii) $d\left[\frac{1}{2}\ln(x^2 + y^2)\right] = \frac{xdx + ydy}{x^2 + y^2}$

(xiv) $d\left[\ln\left(\frac{y}{x}\right)\right] = \frac{xdy - ydx}{xy}$

(xv) $d\left(-\frac{1}{xy}\right) = \frac{xdy + ydx}{x^2 y^2}$

(xvi) $d\left(\frac{e^x}{y}\right) = \frac{ye^x dx - e^x dy}{y^2}$

(xvii) $d\left(\frac{e^y}{x}\right) = \frac{xe^y dy - e^y dx}{x^2}$

(xviii) $d(x^m y^n) = x^{m-1} y^{n-1}(mydx + nxdy)$

(xix) $d\left(\sqrt{x^2 + y^2}\right) = \frac{xdx + y\,dy}{\sqrt{x^2 + y^2}}$

(xx) $d\left(\frac{1}{2}\log\frac{x+y}{x-y}\right) = \frac{x\,dy - y\,dx}{x^2 - y^2}$

(xxi) $\frac{d[f(x,y)]^{1-n}}{1-n} = \frac{f'(x,y)}{(f(x,y))^n}$