Linear Differential Equation.

(1) **Linear and non-linear differential equations**: A differential equation is a linear differential equation if it is expressible in the form $P_o \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + ... + P_{n-1} \frac{dy}{dx} + P_n y = Q$ where $P_0, P_1, P_2, ..., P_{n-1}, P_n$ and Q are either constants or functions of independent variable x .

Thus, if a differential equation when expressed in the form of a polynomial involves the derivatives and dependent variable in the first power and there are no product of these, and also the coefficient of the various terms are either constants or functions of the independent variable, then it is said to be linear differential equation. Otherwise, it is a nonlinear differential equation.

It follows from the above definition that a differential equation will be non-linear differential equation if (i) its degree is more than one
(ii) Any of the differential coefficient has exponent more than one.
(iii) Exponentof the dependent variable is more than one.
(iv) Products containing dependent variable and its differential coefficients are present.

(2) **Linear differential equation of first order:**The general form of *a* linear differential equation of first order is $\frac{dy}{dx} + Py = Q$ .....(i)

Where *P* and *Q* are functions of *x* (or constants)

For example, $\frac{dy}{dx} + xy = x^3$, $x\frac{dy}{dx} + 2y = x^3$, $\frac{dy}{dx} + 2y = \sin x$ etc. are linear differential equations.
This type of differential equations are solved when they are multiplied by a factor, which is called integrating

factor, because by multiplication of this factor the left hand side of the differential equation (i) becomes exact differential of some function.
Multiplying both sides of (i) by $e^{\int P dx}$, we get $e^{\int P dx}\left(\frac{dy}{dx} + Py\right) = Q\, e^{\int P dx} \Rightarrow \frac{d}{dx}\left\{y\, e^{\int P dx}\right\} = Q\, e^{\int P dx}$

On integrating both sides *w. r. t. x*, we get; $y\, e^{\int P dx} = \int Q\, e^{\int P dx} dx + C$ ....(ii)

Which is the required solution, where C is the constant of integration. $e^{\int P dx}$ is called the integrating factor. The solution (ii) in short may also be written as $y.(I.F.) = \int Q.(I.F.)\, dx + C$

(3) **Algorithm for solving a linear differential equation:**

Step (i): Write the differential equation in the form $\frac{dy}{dx} + Py = Q$ and obtain P and Q.

Step (ii):Find integrating factor (I.F.) given by $I.F. = e^{\int P dx}$.

Step (iii):Multiply both sides of equation in step (i) by I.F.

Step (iv):Integrate both sides of the equation obtained in step (iii) *w. r. t.* x to obtain

$y(I.F.) = \int Q(I.F.)\, dx + C$

This gives the required solution.

(4) **Linear differential equations of the form** $\frac{dx}{dy} + Rx = S$. Sometimes a linear differential equation can be put in the form $\frac{dx}{dy} + Rx = S$ where R and S are functions of y or constants. Note that y is independent variable and x is a dependent variable.

(5) **Algorithm for solving linear differential equations of the form** $\frac{dx}{dy} + Rx = S$

Step (i):Write the differential equation in the form $\frac{dx}{dy} + Rx = S$ and obtain R and S.

Step (ii):Find I.F. by using $I.F. = e^{\int R\, dy}$

Step (iii):Multiply both sides of the differential equation in step (i) by *I.F.*

Step (iv):Integrate both sides of the equation obtained in step (iii) *w. r. t.* y to obtain the solution given by $x(I.F.) = \int S\,(I.F.)\, dy + C$ Where C is the constant of integration.

(6) **Equations reducible to linear form (Bernoulli's differential equation) :**The differential equation of type $\frac{dy}{dx} + Py = Qy^n$ ......(i)

Where P and Q are constants or functions of x alone and n is a constant other than zero or unity, can be reduced to the linear form by dividing by y^n and then putting $y^{-n+1} = v$, as explained below.

Dividing both sides of (i) by y^n, we get $y^{-n}\frac{dy}{dx} + Py^{-n+1} = Q$

Putting $y^{-n+1} = v$ so that $(-n+1)y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$, we get

$\frac{1}{-n+1}\frac{dv}{dx} + Pv = Q \Rightarrow \frac{dv}{dx} + (1-n)Pv = (1-n)Q$ which is a linear differential equation.

Remark: If $n = 1$, then we find that the variables in equation (i) are separable and it can be easily integrated by the method discussed in variable separable from.

(7) **Differential equation of the form:** $\frac{dy}{dx} + P\phi(y) = Q\psi(y)$

Where P and Q are functions of x alone or constants.

Dividing by $\psi(y)$, we get $\frac{1}{\psi(y)}\frac{dy}{dx} + \frac{\phi(y)}{\psi(y)}P = Q$

Now put $\frac{\phi(y)}{\psi(y)} = v$, so that $\frac{d}{dx}\left\{\frac{\phi(y)}{\psi(y)}\right\} = \frac{dv}{dx}$ or $\frac{dv}{dx} = k \cdot \frac{1}{\psi(y)}\frac{dy}{dx}$, where k is constant

We get $\frac{dv}{dx} + kP\,v = kQ$

Which is linear differential equation.