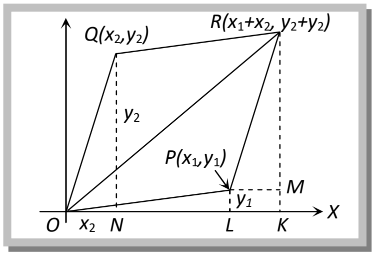
Geometry of Complex Numbers.

(1) **Geometrical representation of algebraic operations on complex numbers**

(i) **Sum:** Let the complex numbers $z_1 = x_1 + iy_1 = (x_1, y_1)$ and $z_2 = x_2 + iy_2 = (x_2, y_2)$ be represented by the points P and Q on the argand plane.



Then sum of z_1 and z_2 i.e., $z_1 + z_2$ is represented by the point R. Complex number z can be represented by $\overrightarrow{OR}$.

$= (x_1 + x_2) + i(y_1 + y_2) = (x_1 + iy_1) + (x_2 + iy_2) = (z_1 + z_2) = (x_1, y_1) + (x_2, y_2)$

In vector notation, we have $z_1 + z_2 = \overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR}$

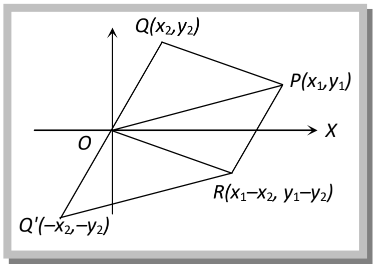
(ii) **Difference:** We first represent $-z_2$ by Q', so that QQ' is bisected at O.

The point R represents the difference $z_1 - z_2$.

In parallelogram ORPQ, $\overrightarrow{OR} = \overrightarrow{QP}$

We have in vectorial notation $z_1 - z_2 = \overrightarrow{OP} - \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{QO}$

$= \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR} = \overrightarrow{QP}$.



(iii) **Product:** Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1}$

$\therefore |z_1| = r_1$ and arg $(z_1) = \theta_1$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2 e^{i\theta_2}$

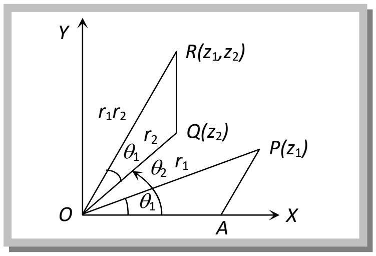
$\therefore |z_2| = r_2$ and arg $(z_2) = \theta_2$

Then, $z_1 z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2 \{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$

$\therefore |z_1 z_2| = r_1 r_2$ and arg $(z_1 z_2) = \theta_1 + \theta_2$

R is the point representing product of complex numbers z_1 and z_2.



Important Tips

☞Multiplication of i : Since $z = r(\cos\theta + i\sin\theta)$ and $i = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ then

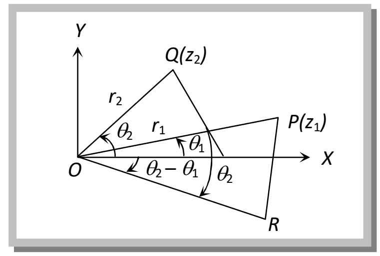
$$iz = \left[\cos\left(\frac{\pi}{2} + \theta\right) + i\sin\left(\frac{\pi}{2} + \theta\right)\right]$$

Hence, multiplication of z with i then vector for z rotates a right angle in the positive sense.

i.e., To multiply a vector by –1 is to turn it through two right angles.

i.e., To multiply a vector by $(\cos\theta + i\sin\theta)$ is to turn it through the angle θ in the positive sense.

(iv) **Division:** Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1}$



$\therefore |z_1| = r_1$ and $\arg(z_1) = \theta_1$

and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2 e^{i\theta_2}$

$\therefore |z_2| = r_2$ and $\arg(z_2) = \theta_2$

Then $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\dfrac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)}\ (z_2 \neq 0,\ r_2 \neq 0)$

$$\Rightarrow \frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

$$\therefore \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2}, arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$$

Note: The vertical angle R is $-(\theta_2 - \theta_1)$ i.e., $\theta_1 - \theta_2$.

If θ_1 and θ_2 are the principal values of z_1 and z_2 then $\theta_1 + \theta_2$ and $\theta_1 - \theta_2$ are not necessarily the principal value of $\arg(z_1 z_2)$ and $\arg(z_1 / z_2)$.