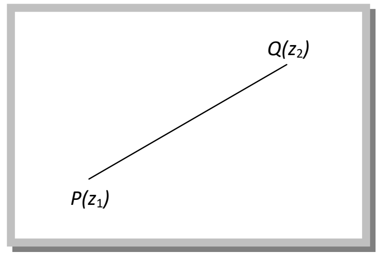
Use of Complex Numbers in Co-ordinate Geometry.

(1) **Distance formula:** The distance between two points $P(z_1)$ and $Q(z_2)$ is given by

$PQ = |z_2 - z_1|$ = |affix of Q – affix of P|



Note: The distance of point z from origin $|z - 0| = |z| = |z - (0 + i0)|$. Thus, modulus of a complex number z represented by a point in the argand plane is its distance from the origin.

Three points $A(z_1), B(z_2)$ and $C(z_3)$ are collinear then $AB + BC = AC$

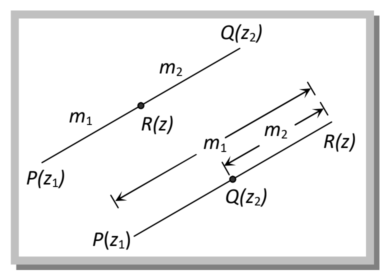
i.e., $|z_1 - z_2| + |z_2 - z_3| = |z_1 - z_3|$.

(2) **Section formula:** If R(z) divides the joining of $P(z_1)$ and $Q(z_2)$ in the ratio $m_1 : m_2 (m_1, m_2 > 0)$

(i) If R(z) divides the segment PQ internally in the ratio of $m_1 : m_2$ then $z = \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}$

(ii) If R(z) divides the segment PQ externally in the ratio of $m_1 : m_2$

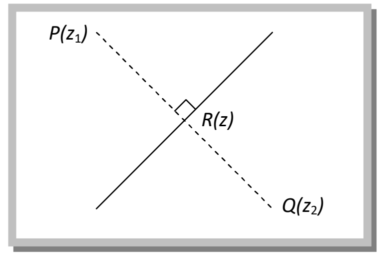
then $z = \frac{m_1 z_2 - m_2 z_1}{m_1 - m_2}$



Note: If R(z) is the midpoint of PQ then affix of R is $\frac{z_1 + z_2}{2}$

If z_1, z_2, z_3 are affixes of the vertices of a triangle, then affix of its centroid is $\frac{z_1 + z_2 + z_3}{3}$.

(3) **Equation of the perpendicular bisector:** If $P(z_1)$ and $Q(z_2)$ are two fixed points and $R(z)$ is moving point such that it is always at equal distance from $P(z_1)$ and $Q(z_2)$



i.e., PR = or $|z - z_1| = |z - z_2|$

$\Rightarrow |z - z_1|^2 = |z - z_2|^2$

$\Rightarrow (z - z_1)\overline{(z - z_1)} = (z - z_2)\overline{(z - z_2)}$

$\Rightarrow (z - z_1)(\bar{z} - \bar{z}_1) = (z - z_2)(\bar{z} - \bar{z}_2)$

$\Rightarrow \bar{z}\; z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_1 - z_2) = z_1\bar{z}_1 - z_2\bar{z}_2$ $\qquad \Rightarrow z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_1 - z_2) = |z_1|^2 - |\bar{z}_2|^2$

Hence, z lies on the perpendicular bisector of z_1 and z_2.

(4) **Equation of a straight line**

(i) **Parametric form:** Equation of a straight line joining the point having affixes z_1 and z_2 is

$z = t z_1 + (1 - t)z_2$, when $t \in R$

(ii) **Non parametric form:** Equation of a straight line joining the points having affixes z_1 and z_2 is $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0 \Rightarrow z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - z_2\bar{z}_1 = 0$.

Note: Three points z_1, z_2 and z_3 are collinear $\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$

(iii) **General equation of a straight line:** The general equation of a straight line is of the form $\bar{a}z + a\bar{z} + b = 0$, where a is complex number and b is real number.

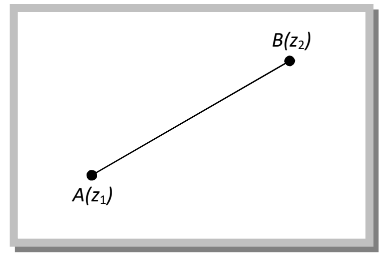
(iv) **Slope of a line:** The complex slope of the line $\bar{a}z + a\bar{z} + b = 0$ is $-\frac{a}{\bar{a}} = -\frac{\text{coeff. of } \bar{z}}{\text{coeff. of } z}$ and real slope of the line $\bar{a}z + a\bar{z} + b = 0$ is $-\frac{\text{Re}(a)}{\text{Im}(a)} = -i\frac{(a + \bar{a})}{(a - \bar{a})}$.

Note: If α_1 and α_2 are the are the complex slopes of two lines on the argand plane, then

(i) If lines are perpendicular then $\alpha_1 + \alpha_2 = 0$ (ii) If lines are parallel then $\alpha_1 = \alpha_2$

If lines $a\bar{z} + \bar{a}z + b = 0$ and $a_1\bar{z} + \bar{a}_1 z + b_1 = 0$ are the perpendicular or parallel, then $\left(\frac{-a}{a}\right) + \left(\frac{-a_1}{\bar{a}_1}\right) = 0$ or $\frac{-a}{\bar{a}} = \frac{-a_1}{\bar{a}_1} \Rightarrow a\bar{a}_1 + a_1\bar{a} = 0$ or $a\bar{a}_1 - \bar{a}a_1 = 0,$ where a, a_1 are the complex numbers and $b, b_1 \in R.$

(v) **Slope of the line segment joining two points:** If $A(z_1)$ and $B(z_2)$ represent two points in the argand plane then the complex slope of AB is defined by $\frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$.

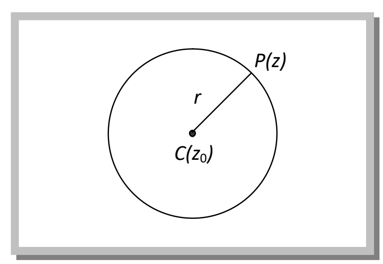


Note: If three points $A(z_1), B(z_2), C(z_3)$ are collinear then slope of AB = slope of BC = slope of AC

$$\frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} = \frac{z_2 - z_3}{\bar{z}_2 - \bar{z}_3} = \frac{z_1 - z_3}{\bar{z}_1 - \bar{z}_3}$$

(vi) **Length of perpendicular:** The length of perpendicular from a point z_1 to the line $\bar{a}z + a\bar{z} + b = 0$ is given by $\frac{|\bar{a}z_1 + a\bar{z}_1 + b|}{|a| + |\bar{a}|}$ or $\frac{|\bar{a}z_1 + a\bar{z}_1 + b|}{2|a|}$

(5) **Equation of a circle:** The equation of a circle whose centre is at point having affix z_o and radius r is $|z - z_o| = r$



Note: If the centre of the circle is at origin and radius r, then its equation is $|z| = r$. $|z - z_0| < r$ represents interior of a circle $|z - z_0| = r$ and $|z - z_0| > r$ represent exterior of the circle $|z - z_o| = r$. Similarly, $|z - z_0| > r$ is the set of all points lying outside the circle and $|z - z_0| \geq r$ is the set of all points lying outside and on the circle $|z - z_0| = r.$

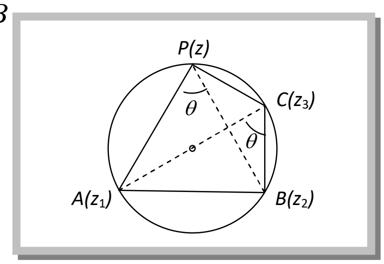
(i) **General equation of a circle:** The general equation of the circle is $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ where a is complex number and $b \in R$.

$\therefore$ Centre and radius are – a and $\sqrt{|a|^2 - b}$ respectively.

Note: Rule to find the center and radius of a circle whose equation is given:

- Make the coefficient of $z\bar{z}$ equal to 1 and right hand side equal to zero.
- The center of circle will be = – a $= -\text{coefficent of } \bar{z}$
- Radius $= \sqrt{|a|^2 - \text{constant term}}$

(ii) **Equation of circle through three non-collinear points:** Let $A(z_1), B(z_2), C(z_3)$ are three points on the circle and $P(z)$ be any point on the circle, then $\angle ACB = \angle APB$



Using coni method

In ΔACB, $\frac{z_2 - z_3}{z_1 - z_3} = \frac{BC}{CA} e^{i\theta}$ .....(i)

In ΔAPB, $\frac{z_2 - z}{z_1 - z} = \frac{BP}{AP} e^{i\theta}$ .....(ii)

From (i) and (ii) we get

$\frac{(z - z_1)(z_2 - z_3)}{(z - z_2)(z_1 - z_3)}$ = Real .....(iii)

(iii) **Equation of circle in diametric form**: If end points of diameter represented by $A(z_1)$ and $B(z_2)$ and $P(z)$ be any point on circle then, $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$

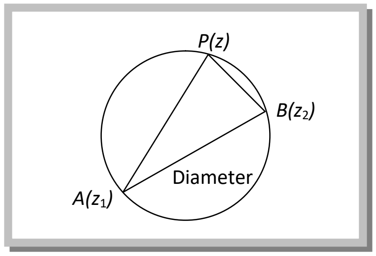
Which is required equation of circle in diametric form.

(iv) **Other forms of circle:**(a) Equation of all circle which are orthogonal to $|z - z_1| = r_1$ and $|z - z_2| = r_2$. Let the circle be $|z - \alpha|$ = r cut given circles orthogonally

$\Rightarrow r^2 + r_1^2 = |\alpha - z_1|^2$ ......(i)

and $r^2 + r_2^2 = |\alpha - z_2|^2$ .....(ii)

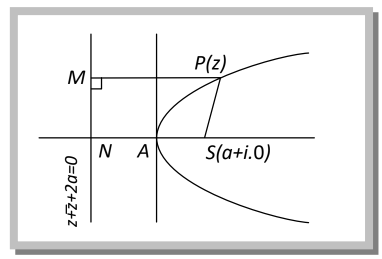
on solving $r_2^2 - r_1^2 = \alpha(\bar{z}_1 - \bar{z}_2) + \bar{\alpha}(z_1 - z_2) + |z_2|^2 - |z_1|^2$ and let $\alpha = a + ib$



(b) $\left|\frac{z-z_1}{z-z_2}\right|$ = k is a circle if $k \neq 1$ and a line if k = 1.

(c) The equation $|z-z_1|^2 + |z-z_2|^2 = k,$ will represent a circle if $k \geq \frac{1}{2}|z_1 - z_2|^2$

(6) **Equation of parabola:** Now for parabola $SP = PM$

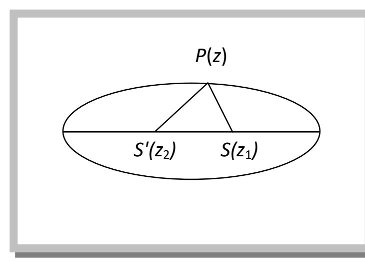


$|z-a| = \frac{|z+\bar{z}+2a|}{2}$

or $z\bar{z} - 4a(z+\bar{z}) = \frac{1}{2}\{z^2 + (\bar{z})^2\}$

Where $a \in R$ (focus)

Directrix is $z + \bar{z} + 2a = 0$



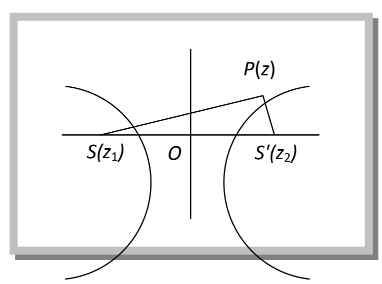
(7) **Equation of ellipse:** For ellipse $SP + S'P = 2a$

$\Rightarrow |z-z_1| + |z-z_2| = 2a$

Where $2a > |z_1 - z_2|$ (since eccentricity <1)

Then point z describes an ellipse having foci at z_1 and z_2 and $a \in R^+$.

(8) **Equation of hyperbola:** For hyperbola $SP - S'P = 2a$



$\Rightarrow |z-z_1| - |z-z_2| = 2a$

Where $2a < |z_1 - z_2|$ (since eccentricity >1)

Then point z describes a hyperbola having foci at z_1 and z_2 and $a \in R^+$