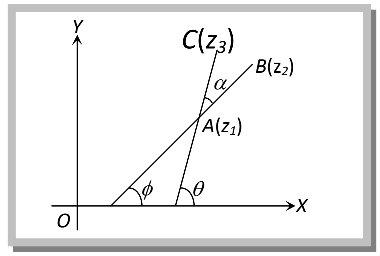
Rotation Theorem.

Rotational theorem i.e., angle between two intersecting lines. This is also known as coni method. Let z_1, z_2 and z_3 be the affixes of three points A, B and C respectively taken on argand plane.



Then we have $\overrightarrow{AC} = z_3 - z_1$ and $\overrightarrow{AB} = z_2 - z_1$

and let arg $\overrightarrow{AC} = \arg(z_3 - z_1) = \theta$ and $\overrightarrow{AB} = \arg(z_2 - z_1) = \phi$

Let $\angle CAB = \alpha$, $\because \angle CAB = \alpha = \theta - \phi$

$$= \arg \overrightarrow{AC} - \arg \overrightarrow{AB} = \arg(z_3 - z_1) - \arg(z_2 - z_1) = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$$

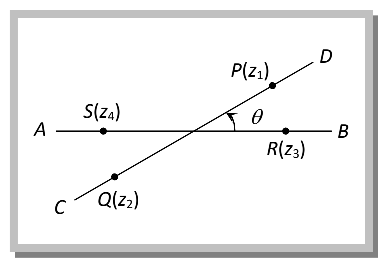
or angle between AC and AB = $\arg\left(\frac{\text{affix } of\ C - \text{affix } of\ A}{\text{affix } of\ B - \text{affix } of\ A}\right)$

For any complex number z we have $z = |z|\, e^{i(\arg z)}$

Similarly, $\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \left|\left(\frac{z_3 - z_1}{z_2 - z_1}\right)\right| e^{i\left(\arg\frac{z_3 - z_1}{z_2 - z_1}\right)}$ or $\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} e^{i(\angle CAB)} = \frac{AC}{AB} e^{i\alpha}$

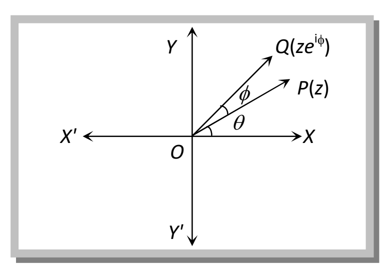
Note: Here only principal values of the arguments are considered.

$\arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = \theta$, if AB coincides with CD, then $\arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = 0$ or $\pm\pi$, so that $\frac{z_1 - z_2}{z_3 - z_4}$ is real. It follows that if $\frac{z_1 - z_2}{z_3 - z_4}$ is real, then the points A, B, C, D are collinear.



If AB is perpendicular to CD, then $\arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = \pm\pi/2$, so $\frac{z_1 - z_2}{z_3 - z_4}$ is purely imaginary. It follows that if $z_1 - z_2 = \pm k(z_3 - z_4)$, where k purely imaginary number, then AB and CD are perpendicular to each other.

(1) **Complex number as a rotating arrow in the argand plane:** Let $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ .....(i) $r.e^{i\theta}$ be a complex number representing a point P in the argand plane.



Then $OP = |z| = r$ and $\angle POX = \theta$

Now consider complex number $z_1 = ze^{i\phi}$

or $z_1 = re^{i\theta}.e^{i\phi} = re^{i(\theta+\phi)}$ {from (i)}

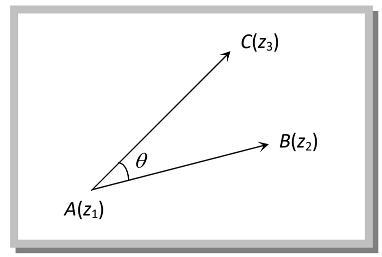
Clearly the complex number z_1 represents a point Q in the argand plane, when $OQ = r$ and $\angle QOX = \theta + \phi$.

Clearly multiplication of z with $e^{i\phi}$ rotates the vector $\overrightarrow{OP}$ through angle ϕ in anticlockwise sense.

Similarly multiplication of z with $e^{-i\phi}$ will rotate the vector $\overrightarrow{OP}$ in clockwise sense.

Note: If z_1, z_2 and z_3 are the affixes of the points A,B and C such that $AC = AB$ and $\angle CAB = \theta$.

Therefore, $\overrightarrow{AB} = z_2 - z_1$, $\overrightarrow{AC} = z_3 - z_1$.



Then $\overrightarrow{AC}$ will be obtained by rotating $\overrightarrow{AB}$ through an angle θ in anticlockwise sense, and therefore,

$\overrightarrow{AC} = \overrightarrow{AB}\, e^{i\theta}$ or $(z_3 - z_1) = (z_2 - z_1)e^{i\theta}$ or $\dfrac{z_3 - z_1}{z_2 - z_1} = e^{i\theta}$

If A, B and C are three points in argand plane such that $AC = AB$ and $\angle CAB = \theta$ then use the rotation about A to find $e^{i\theta}$, but if $AC \neq AB$ use coni method.

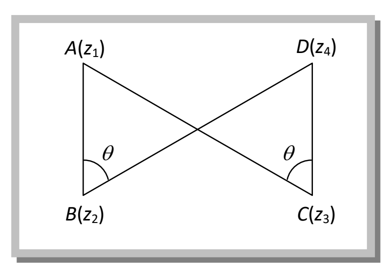
Let z_1 and z_2 be two complex numbers represented by point P and Q in the argand plane such that $\angle POQ = \theta$. Then, $z_1 e^{i\theta}$ is a vector of magnitude $|z_1| = OP$ along $\overrightarrow{OQ}$ and $\dfrac{z_1 e^{i\theta}}{|z_1|}$ is a unit vector along $\overrightarrow{OQ}$. Consequently, $|z_2| \cdot \dfrac{z_1 e^{i\theta}}{|z_1|}$ is a vector of magnitude $|z_2| = OQ$ along OQ i.e.,

$$z_2 = \frac{|z_2|}{|z_1|} . z_1 e^{i\theta} = z_2 = \left|\frac{z_2}{z_1}\right| .$$

(2) **Condition for four points to be noncyclic:**If points A,B,C and D are concyclic

$\angle ABD = \angle ACD$

Using rotation theorem



In ΔABD $\dfrac{(z_1 - z_2)}{|z_1 - z_2|} = \dfrac{z_4 - z_2}{|z_4 - z_2|} e^{i\theta}$ .....(i)

In ΔACD $\dfrac{(z_1 - z_3)}{|z_1 - z_3|} = \dfrac{z_4 - z_3}{|z_4 - z_3|} e^{i\theta}$ .....(ii)

From (i) and (ii)

$$\frac{(z_1 - z_2)}{|z_1 - z_3|}\frac{(z_4 - z_3)}{|z_4 - z_2|} = \left|\frac{(z_1 - z_2)(z_4 - z_3)}{(z_1 - z_3)(z_4 - z_2)}\right| = \text{Real}$$

So if z_1, z_2, z_3 and z_4 are such that $\frac{(z_1 - z_2)(z_4 - z_3)}{(z_1 - z_3)(z_4 - z_2)}$ is real, then these four points are concyclic.