Roots of a Complex Number.

(1) **n^th roots of complex number** ($z^{1/n}$) **:** Let $z = r(\cos + i\sin\theta)$ be a complex number. To find the roots of a complex number, first we express it in polar form with the general value of its amplitude and use the De' Moivre’s theorem. By using De'moivre's theorem n^{th} roots having n distinct values of such a complex number are given by

$$z^{1/n} = r^{1/n}\left[\cos\frac{2m\pi+\theta}{n} + i\sin\frac{2m\pi+\theta}{n}\right], \text{where } m = 0, 1, 2, \ldots\ldots, (n-1).$$

Properties of the roots of $z^{1/n}$:

(i) All roots of $z^{1/n}$ are in geometrical progression with common ratio $e^{2\pi i/n}$.

(ii) Sum of all roots of $z^{1/n}$ is always equal to zero.

(iii) Product of all roots of $z^{1/n} = (-1)^{n-1}z$.

(iv) Modulus of all roots of $z^{1/n}$ are equal and each equal to $r^{1/n}$ or $|z|^{1/n}$.

(v) Amplitude of all the roots of $z^{1/n}$ are in A.P. with common difference $\frac{2\pi}{n}$.

(vi) All roots of $z^{1/n}$ lies on the circumference of a circle whose center is origin and radius equal to $|z|^{1/n}$. Also these roots divides the circle into n equal parts and forms a polygon of n sides.

(2) **The n^{th} roots of unity:** The n^{th} roots of unity are given by the solution set of the equation

$$x^n = 1 = \cos 0 + i\sin 0 = \cos 2k\pi + i\sin 2k\pi$$

$$x = [\cos 2k\pi + i\sin 2k\pi]^{1/n}$$

$$x = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \text{where } k = 0, 1, 2, \ldots\ldots, (n-1).$$

Properties of n^{th} roots of unity

(i) Let $\alpha = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} = e^{i(2\pi/n)}$, the n^{th} roots of unity can be expressed in the form of a series i.e., $1, \alpha, \alpha^2, \ldots\ldots \alpha^{n-1}$. Clearly the series is G.P. with common difference α i.e., $e^{i(2\pi/n)}$.

(ii) The sum of all n roots of unity is zero i.e., $1 + \alpha + \alpha^2 + \ldots\ldots + \alpha^{n-1} = 0$.

(iii) Product of all n roots of unity is $(-1)^{n-1}$.

(iv) Sum of p^{th} power of n roots of unity

$$1+\alpha^{p}+\alpha^{2p}+.....+\alpha^{(n-1)p}=\begin{cases}0, \text{when } p \text{ is not multiple of } n\\ n, \text{when } p \text{ is a multiple of } n\end{cases}$$

(v) The n, n[th] roots of unity if represented on a complex plane locate their positions at the vertices of a regular plane polygon of n sides inscribed in a unit circle having centre at origin, one vertex on positive real axis.

Note: $x^{n}-1=(x-1)(x^{n-1}+x^{n-2}+.....+x+1)$

$(\sin\theta+i\cos\theta)=-i^{2}\sin\theta+i\cos\theta=i(\cos\theta-i\sin\theta)$

(3) **Cube roots of unity:**Cube roots of unity are the solution set of the equation $x^{3}-1=0\Rightarrow$ $x=(1)^{1/3}\Rightarrow x=(\cos 0+i\sin 0)^{1/3}\Rightarrow x=\cos\frac{2k\pi}{3}+i\sin\left(\frac{2k\pi}{3}\right)$, where $k=0,1,2$

Therefore roots are $1,\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3},\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$ or $1,e^{2\pi i/3},e^{4\pi i/3}$.

Alternative: $x=(1)^{1/3}\Rightarrow x^{3}-1=0\Rightarrow(x-1)(x^{2}+x+1)=0$

$x=1,\frac{-1+i\sqrt{3}}{2},\frac{-1-i\sqrt{3}}{2}$

If one of the complex roots is ω, then other root will be ω^{2} or vice-versa.

Properties of cube roots of unity

(i) $1+\omega+\omega^{2}=0$

(ii) $\omega^{3}=1$

(iii) $1+\omega^{r}+\omega^{2r}=\begin{cases}0, \text{if } r \text{ not a multiple of } 3\\ 3, \text{if } r \text{ is a multiple of } 3\end{cases}$

(iv) $\overline{\omega}=\omega^{2}$ and $(\overline{\omega})^{2}=\omega$ and $\omega.\overline{\omega}=\omega^{3}$.

(v) Cube roots of unity from a G.P.

(vi) Imaginary cube roots of unity are square of each other i.e., $(\omega)^{2}=\omega^{2}$ and $(\omega^{2})^{2}=\omega^{3}.\omega=\omega$.

(vii) Imaginary cube roots of unity are reciprocal to each other i.e., $\frac{1}{\omega}=\omega^{2}$ and $\frac{1}{\omega^{2}}=\omega$.

(viii) The cube roots of unity by, when represented on complex plane, lie on vertices of an equilateral triangle inscribed in a unit circle having centre at origin, one vertex being on positive real axis.

(ix) A complex number $a+ib$, for which $|a:b|=1:\sqrt{3}$ or $\sqrt{3}:1$, can always be expressed in terms of i, ω, ω^2.

Note: If $\omega = \frac{-1+i\sqrt{3}}{2} = e^{2\pi i/3}$, then $\omega^2 = \frac{-1-i\sqrt{3}}{2} = e^{-4\pi i/3} = e^{-2\pi i/3}$ or vice-versa $\omega . \overline{\omega} = \omega^3$.

$a + b\omega + c\omega^2 = 0 \Rightarrow a = b = c$, if a, b, c are real.

Cube root of – 1 are $-1, -\omega, -\omega^2$.

Important Tips

- ☞ $x^2 + x + 1 = (x-\omega)(x-\omega^2)$
- ☞ $x^2 - x + 1 = (x+\omega)(x+\omega^2)$
- ☞ $x^2 + xy + y^2 = (x - y\omega)(x - y\omega^2)$
- ☞ $x^2 - xy + y^2 = (x + y\omega)(x + y\omega^2)$
- ☞ $x^2 + y^2 = (x+iy)(x-iy)$
- ☞ $x^3 + y^3 = (x+y)(x+y\omega)(x+y\omega^2)$
- ☞ $x^3 - y^3 = (x-y)(x-y\omega)(x-y\omega^2)$
- ☞ $x^2 + y^2 + z^2 - xy - yz - zx = (x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$
- ☞ $x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$

Fourth roots of unity: The four, fourth roots of unity are given by the solution set of the equation $x^4 - 1 = 0. \Rightarrow (x^2-1)(x^2+1) = 0 \Rightarrow x = \pm 1, \ \pm i$

Note: Sum of roots = 0 and product of roots =–1.

Fourth roots of unity are vertices of a square which lies on coordinate axes.

Continued product of the roots

If $z = r(\cos\theta + i\sin\theta)$ i.e., $|z| = r$ and $amp\ (z) = \theta$ then continued product of roots of $z^{1/n}$ is

$= r(\cos\phi + i\sin\phi)$, where $\phi = \sum_{m=0}^{n-1} \frac{2m\pi + \theta}{n} = (n-1)\pi + \theta$.

Thus continued product of roots of

$$z^{1/n} = r[\cos\{(n-1)\pi + \theta\} + i\sin\{(n-1)\pi+\theta\}] = \begin{cases} z, \text{if } n \text{ is odd} \\ -z, \text{if } n \text{ is even} \end{cases}$$

Similarly, the continued product of values of $z^{m/n}$ is $= \begin{cases} z^m, & \text{if } n \text{ is odd} \\ (-z)^m, & \text{if } n \text{ is even} \end{cases}$

Important Tips

☞ If $x+\frac{1}{x}=2\cos\theta$ or $x-\frac{1}{x}=2i\sin\theta$ then

$x=\cos\theta+i\sin\theta, \frac{1}{x}=\cos\theta-i\sin\theta, x^n+\frac{1}{x^n}=2\cos n\theta, x^n-\frac{1}{x^n}=2i\sin n\theta.$

☞ If n be a positive integer then , $(1+i)^n+(1-i)^n=2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}$.

☞ If z is a complex number, then e^z is periodic.

☞ n[th] root of –1 are the solution of the equation $z^n+1=0$

$z^n-1=(z-1)(z-\alpha)(z-\alpha^2).....(z-\alpha^{n-1})$, where $\alpha=n^{th}$ root of unity

$z^n-1=(z-1)(z+1)\prod_{r=1}^{(n-2)/2}(z^2-2z\cos\frac{2r\pi}{n}+1)$, if n is even.

$$z^n+1=\begin{cases}\prod_{r=0}^{(n-2)/2}\left[z^2-2z\cos\left(\frac{(2r+1)\pi}{n}\right)+1\right], \text{if } n \text{ is even.}\\ (z+1)\prod_{r=0}^{(n-3)/2}\left[z^2-2z\cos\left(\frac{(2r+1)\pi}{n}\right)+1\right], \text{if } n \text{ is odd.}\end{cases}$$

☞ If $x=\cos\alpha+i\sin\alpha, y=\cos\beta+i\sin\beta, z=\cos\gamma+i\sin\gamma$ and given, $x+y+z=0$, then

(i) $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$ (ii) $yz+zx+xy=0$ (iii) $x^2+y^2+z^2=0$ (iv) $x^3+y^3+z^3=3xyz$

then, putting, values if x,y,z in these results

$x+y+z=0\Rightarrow\cos\alpha+\cos\beta+\cos\gamma=0=\sin\alpha+\sin\beta+\sin\gamma\Rightarrow yz+zx+xy=0\Rightarrow$

$$\begin{cases}\cos(\beta+\gamma)+\cos(\gamma+\alpha)+\cos(\alpha+\beta)=0\\ \sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)=0\end{cases}$$

$x^2+y^2+z^2=0\Rightarrow\begin{cases}\sum\cos 2\alpha=0\\ \sum\sin 2\alpha=0,\end{cases}$ the summation consists 3 terms

$x^3+y^3+z^3=3xyz$, gives similarly

$\sum\cos 3\alpha=3\cos(\alpha+\beta+\gamma)\Rightarrow\sum\sin 3\alpha=3\sin(\alpha+\beta+\gamma)$

If the condition given be $x+y+z=xyz$, then $\sum\cos\alpha=\cos(\alpha+\beta+\gamma)$ etc.