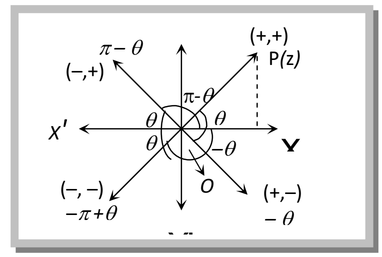
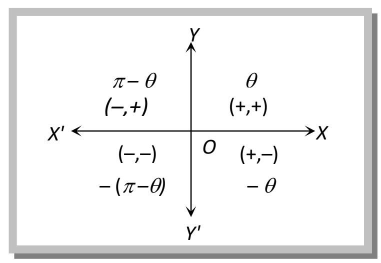
Argument of a Complex Number.

Let $z = a + ib$ be any complex number. If this complex number is represented geometrically by a point P, then the angle made by the line OP with real axis is known as argument or amplitude of z and is expressed as

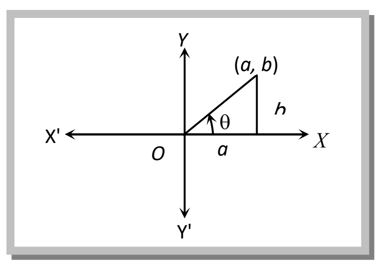


$\arg(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right)$, $\theta = \angle POM$. Also, argument of a complex number is not unique, since if θ be a value of the argument, so also is $2n\pi + \theta$, where $n \in I$.

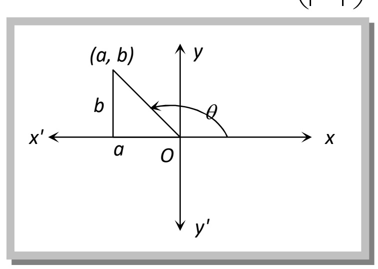
(1) Principal value of arg (z): The value θ of the argument, which satisfies the inequality $-\pi < \theta \le \pi$ is called the principal value of argument. Principal values of argument z will be $\theta, \pi - \theta, -\pi + \theta$ and $-\theta$ according as the point z lies in the 1st, 2nd, 3rd and 4th quadrants respectively, where $\theta = \tan^{-1}\left|\frac{b}{a}\right| = \alpha$ (acute angle). Principal value of argument of any complex number lies between $-\pi < \theta \le \pi$.



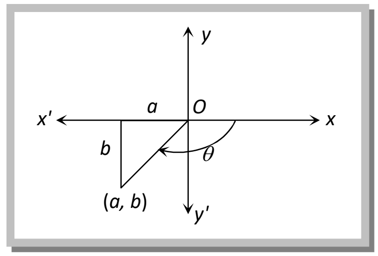
(i) $a, b \in$ First quadrant $a > 0, b > 0$. $\arg(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right)$. It is an acute angle and positive.



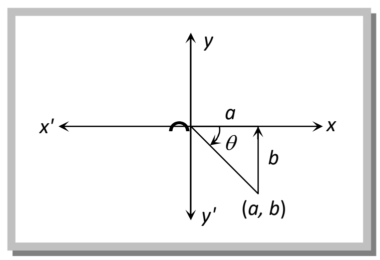
(ii) $(a, b) \in$ Second quadrant, $a < 0, b > 0$, $\arg(z) = \theta = \pi - \tan^{-1}\left(\frac{b}{|a|}\right)$. It is an obtuse angle and positive.



(iii) $(a,b) \in$ Third quadrant $a<0, b<0$, $\arg(z)=\theta=-\pi+\tan^{-1}\left(\frac{b}{a}\right)$. It is an obtuse angle and negative.



(iv) $(a,b) \in$ Fourth quadrant $a>0, b<0$, $\arg(z)=\theta=-\tan^{-1}\left(\frac{|b|}{a}\right)$. It is an acute angle and negative.



Quadrant	x	y	arg(z)	Interval of θ
I	+	+	θ	$0<\theta<\pi/2$
II	–	+	$\pi-\theta$	$\pi/2<\theta<\pi$
III	–	–	$-(\pi-\theta)$	$-\pi<\theta<-\pi/2$
IV	+	–	$-\theta$	$-\pi/2<\theta<0$

Note: Argument of the complex number 0 is not defined.

Principal value of argument of a purely real number is 0 if the real number is positive and is π if the real number is negative.

Principal value of argument of a purely imaginary number is $\pi/2$ if the imaginary part is positive and is $-\pi/2$ if the imaginary part is negative.

(2) **Properties of arguments**

(i) $arg\ (z_1 z_2) = arg(z_1) + arg(z_2) + 2k\pi,\ \ (k = 0$ or 1 or – 1)

In general

$arg(z_1 z_2 z_3 \ldots\ldots\ldots z_n) = arg(z_1) + arg(z_2) + arg(z_3) + \ldots\ldots\ldots + arg(z_n) + 2k\pi,\ \ (k = 0\ \text{or}\ 1\ \text{or}\ -1)$

(ii) $arg(z_1 \bar{z}_2) = arg(z_1) - arg(z_2)$

(iii) $arg\left(\frac{z_1}{z_2}\right) = arg\, z_1 - arg\, z_2 + 2k\pi,\ (k = 0$ or 1 or – 1)

(iv) $arg\left(\frac{z}{\bar{z}}\right) = 2arg\, z + 2k\pi,\ (k = 0$ or 1 or – 1)

(v) $arg(z^n) = n\, arg\, z + 2k\pi,\ (k = 0$ or 1 or – 1)

(vi) If $arg\left(\frac{z_2}{z_1}\right) = \theta,$ then $arg\left(\frac{z_1}{z_2}\right) = 2k\pi - \theta$, where $k \in I$

(vii) $arg\,\bar{z} = -arg\, z = arg\frac{1}{z}$

(viii) $arg(z - \bar{z}) = \pm\pi/2$

(ix) $arg(-z) = arg(z) \pm \pi$

(x) $arg(z) + arg(\bar{z}) = 0$ or $arg(z) = -arg(\bar{z})$

(xi) $arg(z) - arg(\bar{z}) = \pm\pi$

(xii) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2|z_1||z_2|\cos(\theta_1 - \theta_2)$, where $\theta_1 = arg(z_1)$ and $\theta_2 = arg(z_2)$

Note: Proper value of k must be chosen so that R.H.S. of (i), (ii), (iii) and (iv) lies in $(-\pi, \pi)$

The property of argument is same as the property of logarithm.

If arg (z) lies between $-\pi$ and $\pi(\pi$ inclusive), then this value itself is the principal value of arg (z). If not, see whether arg (z) $> \pi$ or $\le -\pi$. If $arg(z) > \pi,$ go on subtracting 2π until it lies between $-\pi$ and $\pi(\pi$ inclusive). The value thus obtained will be the principal value of arg (z).

The general value of $arg(\bar{z})$ is $2n\pi - arg(z)$.

Important Tips

- If $z_1 = z_2 \Leftrightarrow |z_1| = |z_2|$ and arg z_1 = arg z_2.
- $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow$ arg $(z_1) =$ arg (z_2) i.e., z_1 and z_2 are parallel.
- $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow$ arg $(z_1) -$ arg $(z_2) = 2n\pi,$ where n is some integer.
- $|z_1 - z_2| = ||z_1| - |z_2|| \Leftrightarrow arg(z_1) - arg(z_2) = 2n\pi$, where n is some integer.
- $|z_1 + z_2| = |z_1 - z_2| \Leftrightarrow$ arg $(z_1) -$ arg $(z_2) = \pi/2$.
- If $|z_1| \le 1, |z_2| \le 1$ then (i) $|z_1 + z_2|^2 \le (|z_1| - |z_2|)^2 + ($arg $(z_1) -$ arg $(z_2))^2$ (ii) $|z_1 + z_2|^2 \ge (|z_1| + |z_2|)^2 - (arg(z_1) - arg(z_2))^2$
- $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$.
- $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$.
- If $|z_1| = |z_2|$ and $amp(z_1) + amp(z_2) = 0,$ then $z_1 + z_2$ are conjugate complex numbers of each other.
- $z \ne 0,\ amp(z + \bar{z}) = 0$ or $\pi; amp(z\bar{z}) = 0; amp(z - \bar{z}) = \pm\pi/2.$
- arg $(1) = 0,$ arg $(-1) = \pi;$ arg $(i) = \pi/2,$ arg $(-i) = -\pi/2.$
- arg $(z) = \frac{\pi}{4} \Rightarrow \text{Re}(z) = \text{Im}(z).$
- Amplitude of complex number in I and II quadrant is always positive and in III[rd] and IV[th] quadrant is always negative.
- If a complex number multiplied by i (Iota) its amplitude will be increased by $\pi/2$ and will be decreased by $\pi/2$, if multiplied by –i, i.e. $arg(iz) = \frac{\pi}{2} + arg(z)$ and $arg(-iz) = arg(z) - \frac{\pi}{2}$.

Complex number	**Value of argument**
+ve Re (z)	0
–ve Re (z)	π
+ve Im (z)	$\pi / 2$
–ve Im (z)	$3\pi / 2 \ or - \pi / 2$
– (z)	$\lvert \theta \pm \pi \rvert, \ if \ \theta \ is \ -ve \ and \ +ve$ respectively
(iz)	$\left\{ \frac{\pi}{2} + arg(z) \right\}$
–(iz)	$\left\{ arg(z) - \frac{\pi}{2} \right\}$
(z^n)	n. arg (z)
$(z_1.z_2)$	arg (z_1) + arg (z_2)
$\left(\frac{z_1}{z_2} \right)$	arg (z_1) – arg (z_2)