Properties of Binomial Coefficients.

In the binomial expansion of $(1+x)^n$, $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + ... + {}^nC_r x^r + ... + {}^nC_n x^n$.
Where ${}^nC_0, {}^nC_1, {}^nC_2, ..., {}^nC_n$ are the coefficients of various powers of x and called binomial coefficients, and they are written as $C_0, C_1, C_2, ... C_n$.

Hence, $(1+x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_r x^r + ... + C_n x^n$ .....(i)

(1) The sum of binomial coefficients in the expansion of $(1+x)^n$ is 2^n.

Putting $x = 1$ in (i), we get $2^n = C_0 + C_1 + C_2 + ... + C_n$ .....(ii)

(2) Sum of binomial coefficients with alternate signs :Putting $x = -1$ in (i)

We get, $0 = C_0 - C_1 + C_2 - C_3 + ...$ .....(iii)

(3) Sum of the coefficients of the odd terms in the expansion of $(1+x)^n$ is equal to sum of the coefficients of even terms and each is equal to 2^{n-1}.

From (iii), we have $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ...$ ......(iv)

i.e., sum of coefficients of even and odd terms are equal.

From (ii) and (iv), $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$ ......(v)

(4) ${}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1} = \frac{n}{r}.\frac{n-1}{r-1}\,{}^{n-2}C_{r-2}$ and so on.

(5) Sum of product of coefficients: Replacing x by $\frac{1}{x}$ in (i) we get

$\left(1+\frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + ... \frac{C_n}{x^n} + ...$ (vi)

Multiplying (i) by (vi), we get $\frac{(1+x)^{2n}}{x^n} = (C_0 + C_1 x + C_2 x^2 + ...)\left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + ...\right)$

Now comparing coefficient of x^r on both sides.

We get, ${}^{2n}C_{n+r} = C_0C_r + C_1C_{r+1} + \ldots\ldots C_{n-r}.C_n$ .....(vii)

(6) Sum of squares of coefficients: Putting $r = 0$ in (vii), we get ${}^{2n}C_n = C_0^2 + C_1^2 + \ldots\ldots C_n^2$

(7) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$