Binomial Theorem for any Index.

Statement:

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)}{3!}x^3 + ... + \frac{n(n-1)......(n-r+1)}{r!}x^r + ...\text{terms up to } \infty$$

When n is a negative integer or a fraction, where $-1 < x < 1$, otherwise expansion will not be possible.

If $x < 1$, the terms of the above expansion go on decreasing and if x be very small a stage may be reached when we may neglect the terms containing higher power of x in the expansion, then $(1+x)^n = 1 + nx$.

Important Tips

- ☞ Expansion is valid only when $-1 < x < 1$.
- ☞ ${}^nC_r$ cannot be used because it is defined only for natural number, so ${}^nC_r$ will be written as $\frac{(n)(n-1)......(n-r+1)}{r!}$
- ☞ The number of terms in the series is infinite.
- ☞ If first term is not 1, then make first term unity in the following way, $(x+y)^n = x^n\left[1+\frac{y}{x}\right]^n$, if $\left|\frac{y}{x}\right| < 1$.

General term: $T_{r+1} = \frac{n(n-1)(n-2)......(n-r+1)}{r!}x^r$

Some important expansions:

(i) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ... + \frac{n(n-1)(n-2)......(n-r+1)}{r!}x^r + ...$

(ii) $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - .... + \frac{n(n-1)(n-2).....(n-r+1)}{r!}(-x)^r + ...$

(iii) $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + ... + \frac{n(n+1)......(n+r-1)}{r!}x^r + ...$

(iv) $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + ... + \frac{n(n+1)......(n+r-1)}{r!}(-x)^r + ...$

(a) **Replace n by 1 in (iii):** $(1-x)^{-1} = 1 + x + x^2 + ... + x^r + ...\infty$, General term, $T_{r+1} = x^r$

(b) **Replace n by 1 in (iv):** $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots + (-x)^r + \ldots\ldots\infty$, General term, $T_{r+1} = (-x)^r$.

(c) **Replace n by 2 in (iii):** $(1-x)^{-2} = 1 + 2x + 3x^2 + \ldots\ldots + (r+1)x^r + \ldots\ldots\infty$, General term, $T_{r+1} = (r+1)x^r$.

(d) **Replace n by 2 in (iv):** $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots + (r+1)(-x)^r + \ldots\ldots\infty$

General term, $T_{r+1} = (r+1)(-x)^r$.

(e) **Replace n by 3 in (iii):** $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots\ldots + \frac{(r+1)(r+2)}{2!}x^r + \ldots\ldots\ldots\infty$

General term, $T_{r+1} = (r+1)(r+2)/2!.\, x^r$

(f) **Replace nby 3 in (iv):** $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \ldots\ldots + \frac{(r+1)(r+2)}{2!}(-x)^r + \ldots\ldots\infty$

General term, $T_{r+1} = \frac{(r+1)(r+2)}{2!}(-x)^r$