1. Types of Relations.

(1) **Reflexive relation:**A relation R on a set A is said to be reflexive if every element of A is related to itself.

Thus, R is reflexive ⇔ (a, a) ∈ R for all a∈ A.

A relation R on a set A is not reflexive if there exists an element a∈ A such that (a, a) ∉ R.

Example: Let A = {1, 2, 3} and R = {(1, 1); (1, 3)}

Then R is not reflexive since $3 \in A$ but (3, 3) ∉ R

Note: ❑ The identity relation on a non-void set A is always reflexive relation on A. However, a reflexive relation on A is not necessarily the identity relation on A.

❑ The universal relation on a non-void set A is reflexive.

(2) **Symmetric relation:**A relation R on a set A is said to be a symmetric relation iff

(a, b) ∈ R ⇒ (b, a) ∈ R for all a, b ∈ A

i.e. aRb⇒bRa for all a, b ∈ A.

It should be noted that R is symmetric iff $R^{-1} = R$

Note: The identity and the universal relations on a non-void set are symmetric relations.

A relation R on a set A is not a symmetric relation if there are at least two elements a, b ∈ A such that (a, b) ∈ R but (b, a) ∉ R.

A reflexive relation on a set A is not necessarily symmetric.

(3) **Anti-symmetric relation:** Let A be any set. A relation R on set A is said to be an anti-symmetric relation iff (a, b) ∈ R and (b, a) ∈ R ⇒ a = b for all a, b ∈ A.

Thus, if a ≠ b then a may be related to b or b may be related to a, but never both.

Example: Let N be the set of natural numbers. A relation $R \subseteq N \times N$ is defined by xRy iff x divides y(i.e., x/y).

Then $x\,R\,y,\ y\,R\,x \Rightarrow x$ divides y, y divides $x \Rightarrow x = y$

Note: The identity relation on a set A is an anti-symmetric relation.

The universal relation on a set A containing at least two elements is not anti-symmetric, because if a ≠ b are in A, then ais related to b and b is related to a under the universal relation will imply that a = b but a ≠ b.

The set $\{(a,a): a \in A\} = D$ is called the diagonal line of $A \times A$. Then "the relation R in A is antisymmetric iff $R \cap R^{-1} \subseteq D$".

(4) **Transitive relation:** Let A be any set. A relation R on set A is said to be a transitive relation iff (a, b) ∈ R and (b, c) ∈ R ⇒ (a, c) ∈ R for all a, b, c ∈ A i.e., aRb and bRc⇒aRc for all a, b, c ∈ A.

In other words, if a is related to b, b is related to c, then a is related to c.

Transitivity fails only when there exists a, b, c such that a R b, b R c but a $\not R$ c.

Example: Consider the set A = {1, 2, 3} and the relations

$R_1 = \{(1, 2),(1,3)\}$; R_2 = {(1, 2)}; R_3 = {(1, 1)}; R_4 = {(1, 2), (2, 1), (1, 1)}

Then R_1, R_2, R_3 are transitive while R_4 is not transitive since in $R_4, (2, 1) \in R_4; (1,2) \in R_4$ but $(2, 2) \notin R_4$.

Note: The identity and the universal relations on a non-void sets are transitive.

The relation 'is congruent to' on the set T of all triangles in a plane is a transitive relation.

(5) **Identity relation:** Let A be a set. Then the relation I_A = {(a, a) : a ∈ A} on A is called the identity relation on A.

In other words, a relation I_A on A is called the identity relation if every element of A is related to itself only. Every identity relation will be reflexive, symmetric and transitive.

Example: On the set = {1, 2, 3}, R = {(1, 1), (2, 2), (3, 3)} is the identity relation on A .

Note: It is interesting to note that every identity relation is reflexive but every reflexive relation need not be an identity relation.

Also, identity relation is reflexive, symmetric and transitive.

(6) **Equivalence relation:** A relation R on a set A is said to be an equivalence relation on A iff

(i) It is reflexive i.e. (a, a) ∈ R for all a ∈ A

(ii) It is symmetric i.e. (a, b) ∈ R ⇒ (b, a) ∈ R, for all a, b ∈ A

(iii) It is transitive i.e. (a, b) $\in$ R and (b, c) $\in$ R $\Rightarrow$ (a, c) $\in$ R for all a, b, c $\in$ A.

Note: **Congruence modulo (m):** Let m be an arbitrary but fixed integer. Two integers a and b are said to be congruence modulo m if $a-b$ is divisible by m and we write $a \equiv b$ (mod m).

Thus $a \equiv b$ (mod m) $\Leftrightarrow a-b$ is divisible by m. For example, $18 \equiv 3$ (mod 5) because 18 – 3 = 15 which is divisible by 5. Similarly, $3 \equiv 13$ (mod 2) because 3 – 13 = –10 which is divisible by 2. But 25 $\neq$ 2 (mod 4) because 4 is not a divisor of 25 – 3 = 22.

The relation "Congruence modulo m" is an equivalence relation.

Important Tips

- If R and S are two equivalence relations on a set A , then R $\cap$ S is also an equivalence relation on A.
- The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.
- The inverse of an equivalence relation is an equivalence relation.