

Some Important Results from Exponential Series.

We have the exponential series

$$(1) e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots \infty = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \dots(i)$$

$$(2) \text{ Replacing } x \text{ by } -x \text{ in (i), we obtain } e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots \infty = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} \quad \dots(ii)$$

$$(3) \text{ Putting } x = 1 \text{ in (i) and (ii), we get, } e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \infty = \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots \infty = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

$$(4) \text{ From (i) and (ii), we obtain } \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots \infty = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \infty = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$(5) \frac{e + e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + \dots \infty = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \quad , \quad \frac{e - e^{-1}}{2} = 1 + \frac{1}{3!} + \frac{1}{5!} + \dots \infty = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$$

$$\text{Note: } e - 1 = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots \infty = \sum_{r=1}^{\infty} \frac{1}{r!}$$

$$\square e - 2 = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots \infty = \sum_{r=2}^{\infty} \frac{1}{r!}$$